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An Aperiodic Pair of Tiles in $\mathbb{E}^n$ for All $n \geq 3$

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We give a set of only two tiles in $\mathbb{E}^n$ for each $n \geq 3$; these sets of tiles admit only non-periodic tilings in $\mathbb{E}^n$. The construction is based on similarities of the cubic lattice; a two-dimensional analogue of the construction can be found in Goodman-Strauss, *Europ. J. Combinatorics* (to appear).

1. INTRODUCTION

We give a relatively simple construction of an aperiodic pair of tiles in $\mathbb{E}^n$, $n \geq 3$. These tiles do tile $\mathbb{E}^n$, but must recreate a certain non-periodic structure based on similarities of the cubic lattice. These sets of tiles only admit **non-periodic** tilings, that is tilings that are invariant under no infinite-cyclic group of isometries, and thus we say these are **aperiodic** sets of tiles.

Additionally, the number of translation classes required is fairly small. The I tile occurs in only $n2^{n-1}$ orientations; the L tile in only 2^n orientations.

A related aperiodic pair of two-dimensional tiles is given in [5]. The constructions differ in some key ways, however: the construction in Section 2 certainly requires at least three coordinates. Here and in [5] the outlines of the proof of aperiodicity are roughly similar, but the two-dimensional tiles allow many more configurations than their more rigid higher dimensional analogues. Strangely then, the higher dimensional tiles are in some ways simpler to work with.

The tiles here recreate a particular substitution tiling in $\mathbb{E}^n$, a generalization of the well known L-tiling or 'chair' tiling, used to construct the 'trilobite' and 'cross' in [5].

Thus, in some ways, this result is superceded by [3], in which we show every substitution tiling in $\mathbb{E}^n$, $n \geq 2$ can be enforced by a matching rule tiling.

However, the following construction is vastly simpler than an application of the algorithm in [3]. In particular, we establish that as few as two tiles are sufficient to force aperiodicity in $\mathbb{E}^n$.

These aperiodic sets of tiles are among the first explicitly worked out examples in higher dimensions. Very recently collections of aperiodic tiles in $\mathbb{E}^3$ (which clearly generalize to $\mathbb{E}^n$) were given by Culik and Kari [1], and by Schmitt [8], each by very different methods.† Schmitt's construction produces an aperiodic set with as few as three tiles.

We should add that the author has circulated different versions of the following construction. In the (completely superceded and never to be published) 'An aperiodic tiling in $\mathbb{E}^n$ for each $n \geq 2$' (1995), we gave a set of roughly 4^n tiles; in the revised and renamed 'An aperiodic set of n tiles in $\mathbb{E}^n$ for all $n \geq 2$' (1997), we were able to reduce the number of tiles drastically, but the construction was still complex and hard to understand. (The original tiles made wonderful illustrations; alas, these had to be discarded in the final version!)

Some readers may object that one of our tiles has a disconnected interior. In Section 5 we discuss variations on the construction; in particular, in $\mathbb{E}^3$, two tiles with connected interior

† Schmitt has also produced a single tile that produces only *non-translational* tilings of $\mathbb{E}^3$; often it is said this is an aperiodic tile. However, this example and others like it demonstrate that non-periodicity really should be defined as not being invariant under any infinite cyclic group of isometries. We would prefer to call Schmitt's tile *weakly aperiodic* [2, 7].

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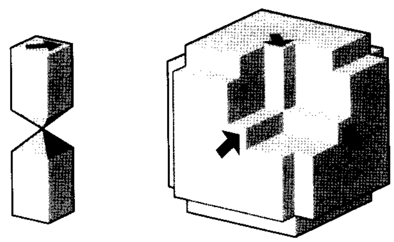

FIGURE 1. A pair of aperiodic tiles in $\mathbb{E}^3$.

will suffice. In $\mathbb{E}^n, n > 3$, we require three tiles with connected interior, or two tiles with connected interior but an unusual type of matching rule. In $\mathbb{E}^n$, $n > 3$, n odd, we suspect that two tiles with connected interiors and traditional matching rules will suffice. All of these problems are related to Lemma 4.3.

2. THE n-DIMENSIONAL L-TILINGS

Fix an integer $n > 2$; our setting is n-dimensional euclidean space $\mathbb{E}^n$ with some fixed orthonormal coordinate system. We will denote points as $\mathbf{x} = (x_1, \dots, x_n)$ or for $c \in \mathbb{R}$, $\mathbf{c} = (c, \dots, c)$. Let $\mathsf{H} = [-1, 1]^n$ be the n-dimensional hypercube.

Let $\mathcal{R} \subset \mathrm{Isom}(\mathbb{E}^n)$ be the set of reflections preserving the coordinate axes. That is, we might denote the elements of $\mathcal{R}$ as $\{\mathbf{r} = (r_1, r_2, \dots, r_n) \mid r_k \in \{+, -\}\}$ and let $\mathcal{R}$ act by $\mathbf{rx} = (r_1x_1, r_2x_2, \dots, r_nx_n)$ for $\mathbf{x} \in \mathbb{E}^n$.

Let $\mathcal{S}$ be the symmetric group acting on $\{1, \dots, n\}$. We can regard $\mathcal{S}$ as a subset of $\mathrm{Isom}(\mathbb{E}^n)$ by taking, for $\sigma \in \mathcal{S}$, $\sigma\mathbf{x} = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Note that $\mathcal{RS}$ is exactly the largest subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ that leaves H invariant.

Now we define our n-dimensional L-tile L. Take

$$\mathsf{L} := \bigcup_{\substack{\mathbf{r}\in\mathcal{R} \\ \mathbf{r}\neq(+,+\cdots+)}} \frac{1}{2}(\mathsf{H} + \mathbf{r1}).$$

A **configuration** of L-tiles is expressed as $\bigcup_{g\in\mathcal{G}} g\mathsf{L}$ where $\mathcal{G} \subset \mathrm{Isom}(\mathbb{E}^n)$ and for any pair of $g, g' \in \mathcal{G}$, we have $\mathrm{int}(g\mathsf{L}) \cap \mathrm{int}(g'\mathsf{L}) = \emptyset$ or $g\mathsf{L} = g'\mathsf{L}$. A **tiling** is a configuration covering all of $\mathbb{E}^n$. (Note that in this paper we have adopted the unfortunate convention of expressing a tiling as a particular union of tiles.)

We can now define the **substitution map** S on configurations of L-tiles as follows:

$$S(\mathsf{L}) := \mathsf{L} \bigcup \left(\bigcup_{\substack{\mathbf{r}\in\mathcal{R} \\ \mathbf{r}\neq(-,-\cdots-)}} \mathbf{r}(\mathsf{L} - \mathbf{1}) \right).$$

Note that the support of $S(\mathsf{L})$ is $2\mathsf{L}$. For $g \in \mathrm{Isom}(\mathbb{E}^n)$, there is a unique $g' \in \mathrm{Isom}(\mathbb{E}^n)$ satisfying $g'(2\mathbf{x}) = 2(g\mathbf{x})$. We define:

$$S(g\mathsf{L}) := g'S(\mathsf{L}),$$
$$S\Big(\bigcup_{g\in\mathcal{G}} g\mathsf{L}\Big) := \bigcup_{g\in\mathcal{G}} S(g\mathsf{L}).$$

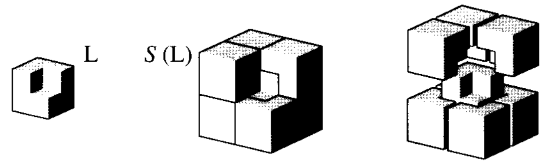

FIGURE 2. Constructing L_3.

Define, for all $k \in \{0, 1, \dots\}$, a k**-level supertile** to be any $gS^k(\mathsf{L})$, $g \in \mathrm{Isom}(\mathbb{E}^n)$.

We should illustrate the construction: the 2-dimensional L-tile can be viewed as a quartered square with one quadrant removed [5]. We might see the substitution map S on L as follows: each of the three remaining quadrants is quartered and its own central quadrant removed, forming three smaller, outer L-tiles. The three left over quarter quadrants form a fourth, central L-tile.

Consider the n-dimensional analogue to this substitution. The n-cube is halved on each edge, producing 2^n smaller n-cubes. We remove one and let the n-dimensional L-tile simply consist of $2^n - 1$ of these smaller cubes.

To divide the n-dimensional L-tile into $2^n - 1$ outer L-tiles and one central L-tile we halve each smaller cube on each of its edges to produce 2^n still smaller n-cubes. The $2^n - 1$ of these not incident to the centre of the original cube are fashioned into one outer L-tile.

This leaves one left over cube from each of the $2^n - 1$ cubes in the original L-tile. Together these form one central L-tile. The three-dimensional L-tile is illustrated in Figure 2.

We now define the species of L**-substitution tilings** $\Sigma(\mathsf{L}, S)$: a tiling τ is in $\Sigma(\mathsf{L}, S)$ if and only if for any bounded configuration $\bigcup_{g \in \mathcal{G}} g\mathsf{L}$ in τ there exists a $g' \in \mathrm{Isom}(\mathbb{E}^n)$,$k \in \mathbb{N}$ such that $g'\big(\bigcup_{g \in \mathcal{G}} g\mathsf{L}\big)$ is a configuration in $S^k(\mathsf{L})$. Note that $\Sigma(\mathsf{L}, S)$ is non-empty! (This is a consequence of existence theorems in [6], [4] and elsewhere.)

A useful interpretation is that a tiling τ is in $\Sigma(\mathsf{L}, S)$ if and only if it 'looks' like '$S^\infty(L_n)$' (though this expression is not well-defined!).

LEMMA 2.1. *For any tiling τ in $\Sigma(\mathsf{L}, S)$, for any isometry $h \in \mathrm{Isom}(\mathbb{E}^n)$ such that h generates an infinite cyclic group of isometries, $g\tau \neq \tau$.*

PROOF. First, as $\Sigma(\mathsf{L}, S) \neq \emptyset$, for every $\tau \in \Sigma(\mathsf{L}, S)$, $k \in \{0, 1, \dots\}$, there exists $\mathcal{G} \subset \mathrm{Isom}(\mathbb{E}^n)$ such that $\tau = \bigcup_{g \in \mathcal{G}} gS^k(\mathsf{L})$ and that for all $g, g' \in \mathcal{G}$, $\mathrm{int}(gS^k(\mathsf{L})) \cap \mathrm{int}(g'S^k(\mathsf{L})) = \emptyset$.

We will prove the following claim by induction on k. The lemma will follow quickly.

CLAIM. For any tiling $\tau \in \Sigma(\mathsf{L}, S)$, $k \in \{0, 1, \dots\}$, the set $\mathcal{G}\mathsf{L}$ is unique. That is, if $\tau = \cup_{g \in \mathcal{G}}(gS^k(\mathsf{L})) = \cup_{h \in \mathcal{H}}(hS^k(\mathsf{L}))$ then for all $g \in G$ there is an $h \in \mathcal{H}$ with $gS^k(\mathsf{L}) = hS^k(\mathsf{L})$.

Clearly the claim holds for $k = 0$, because τ is a tiling.

Assume the claim holds for $n = k$, and there exist $\mathcal{G}, \mathcal{H}$ such that $\tau = \cup_{g \in \mathcal{G}}(gS^{k+1}(\mathsf{L})) = \cup_{h \in \mathcal{H}}(hS^{k+1}(\mathsf{L}))$. Suppose there exists $g \in \mathcal{G}$ such that there is no $h \in \mathcal{H}$ with $gS^{k+1}(\mathsf{L}) = hS^{k+1}(\mathsf{L})$. There must be a $h \in \mathcal{H}$ such that $\mathrm{int}(gS^{k+1}(\mathsf{L})) \cap \mathrm{int}(hS^{k+1}(\mathsf{L})) \neq \emptyset$.

Let $g', h' \in \mathrm{Isom}(\mathbb{E}^n)$ satisfy, for all $\mathbf{x} \in \mathbb{E}^n$,$2^k g'\mathbf{x} = g2^k\mathbf{x}$.

Then, as the claim holds for $n = k$, we have that the configurations $S(\mathsf{L})$ and $(g')^{-1}h'S(\mathsf{L})$ are not the same yet both contain some tile $g_0\mathsf{L}$. This is a contradiction, as can easily be verified.

The claim proves the lemma: let $\tau \in \Sigma(\mathsf{L}, S)$, let $h \in \mathrm{Isom}(\mathbb{E}^n)$, $< h > \approx \mathbb{Z}$. Note that h has no fixed points; in particular, there is an $r \in \Re$ such that for all $\mathbf{x} \in \mathbb{E}^n$, $|\mathbf{x} - h\mathbf{x}| \geq r$ and such that there is an $\mathbf{x}_0 \in \mathbb{E}^n$ with $|\mathbf{x}_0 - h\mathbf{x}_0| = r$. Now there exists a $k \in \mathbb{N}$ such that $2^k\sqrt{n} > r$.

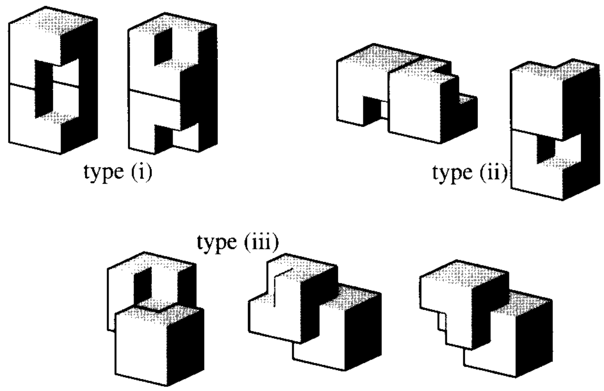

FIGURE 3. Typical pairs of adjacent L_3-tiles in $S^k(\mathsf{L}_3)$.

For any $gS^k\mathsf{L}$ in τ then $hgS^k\mathsf{L}$ does not coincide with with $gS^k\mathsf{L}$ but must intersect the interior of $gS^k\mathsf{L}$. By the claim, this configuration $hgS^k\mathsf{L}$ cannot lie in τ and τ is not invariant under h. □

The following lemma will prove useful. We will sketch the inductive proof, which is not hard to verify in detail.

LEMMA 2.2. *Let $hg\mathsf{L}, h\mathsf{L}$, $h, g \in \mathrm{Isom}(\mathbb{E}^n)$ be a distinct pair of tiles in $S^k(\mathsf{L})$. Then if $hg\mathsf{L}$ and $h\mathsf{L}$ meet on some $(n-1)$-dimensional set, $g\mathsf{L} = g'\mathsf{L}$ where g' is one of the following forms:*

(i) $g'(\mathbf{x}) = (\sigma(-, + \cdots +))\mathbf{x} + r\sigma(2, 0, \ldots, 0)$ *for some* $\sigma \in \mathcal{S}$, $r \in \{+1, -1\}$; *(that is g' is a reflection across one of the planes $x_i = r$).*
(ii) $g'(\mathbf{x}) = (\sigma(+, - \cdots -))\mathbf{x} + \sigma(2, 0, \ldots, 0)$ *(or is the inverse of such a map).*
(iii) $g'(\mathbf{x}) = \mathbf{r}\mathbf{x} + \mathbf{1}$ *for some* $\mathbf{r} \in \mathcal{R} - \{(- \cdots -)\}$ *(or is the inverse of such a map).*

The pairs of tiles in Figure 3 are all of the form $h\mathsf{L}$, $hg'\mathsf{L}$ where g' is of type (i), (ii) or (iii) as indicated.

PROOF. We show this by induction. If $hg\mathsf{L}, h\mathsf{L}$ are distinct tiles in $S(\mathsf{L})$ it is easy to verify that g' is of forms (i) or (iii). In the inductive step, let $hg\mathsf{L}, h\mathsf{L}$ be distinct tiles in $S^k(\mathsf{L})$. If there exists $h' \in \mathrm{Isom}(\mathbb{E}^n)$ such that $hg\mathsf{L}, h\mathsf{L}$ are distinct tiles in $h'S^{k-1}(\mathsf{L}) \subset S^k(\mathsf{L})$, we are done. If $hg\mathsf{L}, h\mathsf{L}$ do not intersect on an $(n-1)$-dimensional set, then we are done. The only remaining case is that there exist h_1, h_2 such that $h_1S(\mathsf{L}), h(\mathsf{L})$ are distinct supertiles in (distinct $k-1$-level supertiles in) $S^k(\mathsf{L})$ that intersect on an $(n-1)$-dimensional set with $hg\mathsf{L}$ a tile in $h_1S(\mathsf{L})$ and $g\mathsf{L}$ a tile in $h_2S(\mathsf{L})$.

Then by the inductive hypotheses, there exists a g_0' of one of the forms given in the lemma such that $h_1^{-1}h_2S(\mathsf{L}) = g_0'S(\mathsf{L})$. It is not hard to verify then that composing the possible g_0' with the maps given in the definition of $S(\mathsf{L})$ again produces a map of the appropriate form, and we are done. (More specifically, if g_0' is of type (i), g' will be of type (i); if g_0' is of type (iii), g' will be of type (ii) or (iii); and if g_0' is of type (ii), g' will be of type (i)). □

3. THE APERIODIC TILES

We now construct new tiles and markings that only admit tilings that replicate the structure of $\Sigma(\mathsf{L}, S)$.

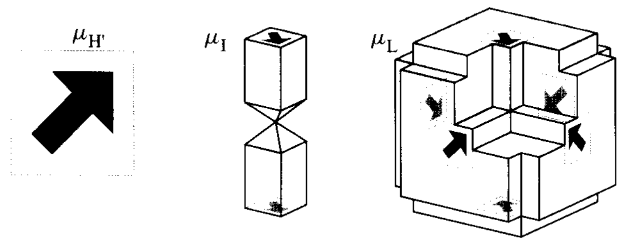

FIGURE 4. Constructing the markings on $\mathtt{I}$ and $\mathtt{L}$.

We first (Figure 5) let

$$\mathtt{I} := \left\{ \mathbf{x} \in \mathbb{E}^n \,\middle|\, |x_1| \leq 1, |x_i| \leq \min\left(\frac{1}{4}, x_1\right), 1 < i \leq n \right\}.$$

Let $\mathtt{X}$ be the support of

$$\mathtt{X} := \bigcup_{\sigma \in \mathcal{S}} \sigma \mathtt{I}$$

(that is, $\mathtt{X}$ is not a configuration of $\mathtt{I}$-tiles but simply a set of points. See Figure 8)

And let (Figure 6)

$$\mathtt{L} := \mathtt{X} \cup \mathsf{L} \; - \mathrm{int}\Big(\bigcup_{\mathbf{r} \in \mathcal{R}} (\mathtt{X} + \mathbf{r1}) \Big).$$

On any image $g\mathtt{L}$, $g \in \mathrm{Isom}(\mathbb{E}^n)$, we will denote the points $g\mathbf{r1}$, $\mathbf{r} \in \mathcal{R}$, $\mathbf{r} \neq (+ \cdots +)$ as **outside corners** of $g\mathtt{L}$, and the point $g\mathbf{0}$ as the **inside corner** of $g\mathtt{L}$. (Note that the outside corners of $\mathtt{L}$ are not actually points in $\mathtt{L}$! Moreover, the inside corner of $\mathtt{L}$ is actually in the interior of $\mathtt{L}$.)

We must mark or otherwise modify the $\mathtt{I}$ and $\mathtt{X}$ tiles to allow only certain local configurations. It is not difficult to modify our construction to use 'bumps' and 'nicks' instead of markings (and not have any matching rule beyond being required to fit together). But markings make better pictures and so we will describe a method for colouring certain points on the boundary of the tiles; in any tiling with these marked tiles, we will require that the colours match.

In the interest of precision, we will describe these markings through a series of maps μ from tiles and configurations to the colours black and white.

Let $\mathsf{H}' = \{0\} \times [-1, 1]^{n-1} = \{\mathbf{x} \mid x_1 = 0, |x_i| \leq 1, i \neq 1\}$ be the $(n-1)$-dimensional hypercube in $\mathbb{E}^n$.

Let $\mu_{\mathsf{H}'} : \mathsf{H}' \to \{\text{white}, \text{black}\}$ be any map such that (1) $\mu_{\mathsf{H}'} : \partial\mathsf{H}' = \text{white}$; (2) for any isometry g leaving H' invariant, $\mu_{\mathsf{H}'}(\mathsf{H}') = \mu_{\mathsf{H}'}(g\mathsf{H}')$ if and only if $g(\mathbf{1}) = \mathbf{1}$. Thus, by (2), the marking is symmetrical around the diagonal of H'. In Figure 4 the marking is drawn as an arrow.

We now mark $\mathtt{I}$; we define $\mu_{\mathtt{I}} : \mathtt{I} \to \{\text{white}, \text{black}\}$ as follows: let $\mathbf{x} \in \mathtt{I}$. If $\mathbf{x} \notin (\frac{1}{4}\mathsf{H}' + \mathbf{v})$, $\mathbf{v} = (\pm 1, 0, \ldots, 0)$ let $\mu_{\mathtt{I}}(\mathbf{x}) := \text{white}$. Otherwise let $\mu_{\mathtt{I}}(\mathbf{x}) := \mu_{\mathtt{I}}(4(\mathbf{x} - \mathbf{v}))$. (See Figure 4.)

And next, we define $\mu_{\mathtt{L}} : \mathtt{L} \to \{\text{white}, \text{black}\}$: let $\mathbf{x} \in \mathtt{L}$. If $\mathbf{x} \notin \mathcal{RS}(\frac{1}{4}\mathsf{H}' + (1, 0, \ldots, 0))$, let $\mu_{\mathtt{L}}(\mathbf{x}) := \text{white}$. Otherwise, we have $\mathbf{x} \in g(\frac{1}{4}\mathsf{H}' + (1, 0, \ldots, 0))$ with $g \in \mathcal{S}$ or $g = (-, - \cdots -)h$, $h \in \mathcal{S}$ (Note we are using the symmetry of H'). Then $\mu_{\mathtt{L}}(\mathbf{x}) := \mu_{\mathtt{L}}(4(g^{-1}\mathbf{x} - (1, 0, \ldots, 0)))$. Note that $\mu_{\mathtt{L}}$ is well defined. (See Figure 4).

Let $\mathcal{T} = \{\mathtt{I}, \mathtt{L}\}$, with markings defined by μ as above. The marked tiles, for $n = 3$, are illustrated in Figures 4, 5 and 6.

For any tile $g\mathsf{A}$, $g \in \mathrm{Isom}(\mathbb{E}^n)$, $\mathsf{A} \in \mathcal{T}$, define $\mu_{g\mathsf{A}}(g\mathsf{A}) :\to \{\text{white}, \text{black}\}$ as follows: for any $g\mathbf{x} \in g\mathsf{A}$, $\mu_{g\mathsf{A}}(g\mathbf{x}) := \mu_{\mathsf{A}}(x)$.

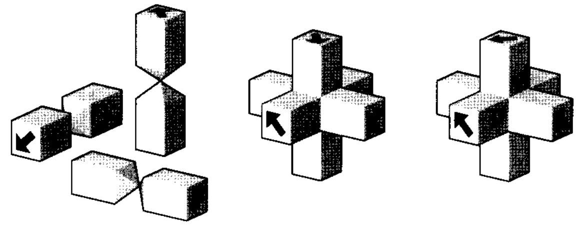

FIGURE 5. Several views of I, two typical X-tiles.

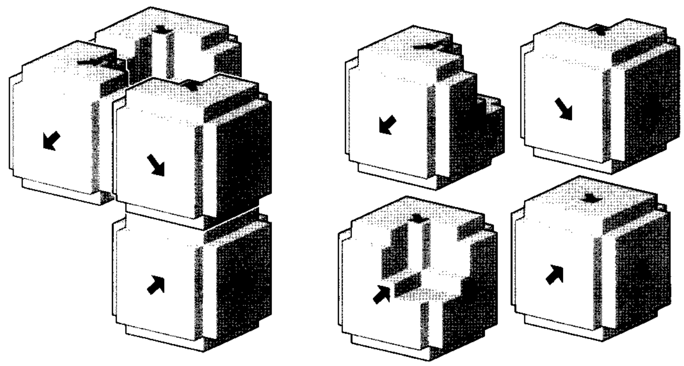

FIGURE 6. Several views of L.

We can now phrase our matching rule $\mathcal{M}$ in precise set-theoretic terms: a tiling τ of $\mathbb{E}^n$ by $\mathcal{T}$ **satisfies** $\mathcal{M}$ if and only if for all tiles $g\mathsf{A}, h\mathsf{B} \subset \tau$, $g, h \in \mathrm{Isom}(\mathbb{E}^n)$, $\mathsf{A}, \mathsf{B} \in \mathcal{T}$, we have for all $\mathbf{x} \in g\mathsf{A} \cap h\mathsf{B}$,

$$\mu_{g\mathsf{A}}(\mathbf{x}) = \mu_{h\mathsf{B}}(\mathbf{x}).$$

More simply, then, a tiling satisfies $\mathcal{M}$ if the colours match. Let $\Sigma(\mathcal{T}, \mathcal{M})$ be the set of all tilings by images of the tiles in $\mathcal{T}$ under $\mathrm{Isom}(\mathbb{E}^n)$, that satisfy $\mathcal{M}$. Note that for tilings and configurations satisfying $\mathcal{M}$, we can drop the subscripts and simply discuss the colouring $\mu : \tau \to \{\mathsf{white}, \mathsf{black}\}$.

We will assume tiles are always marked; thus when we say $g\mathsf{A} = h\mathsf{B}$, we mean that the tiles not only coincide but that $\mu_{g\mathsf{A}} = \mu_{h\mathsf{B}}$.

LEMMA 3.1. *In any tiling $\tau \in \Sigma(\mathcal{T}, \mathcal{M})$, every tile $g\mathrm{I}$, $g \in \mathrm{Isom}(\mathbb{E}^n)$ is in a unique configuration in τ with support $g\mathrm{X}$.*

That is, the I-tiles can only fit together to form X-shaped configurations. However, note that there are lots of ways these configurations can be marked, this is why we do not simply take a marked X to be one of our tiles instead of I. An X-tile will be any configuration of I-tiles with support $g\mathrm{X}$ for some $g \in \mathrm{Isom}(\mathbb{E}^n)$, and the lemma thus states that every tiling in $\Sigma(\mathcal{T}, \mathcal{M})$ can be uniquely viewed as a tiling by X and L-tiles. We omit the proof.

LEMMA 3.2. *A configuration $hg\mathrm{L} \cup h\mathrm{L}$, $h, g \in \mathrm{Isom}(\mathbb{E}^n)$, satisfies $\mathcal{M}$ if and only if $hg\mathrm{L}$ and $h\mathrm{L}$ do not coincide on some* black *point or $g\mathrm{L} = g'\mathrm{L}$ where $g' \in \mathrm{Isom}(\mathbb{E}^n)$ is one of:*

(i) $g'(\mathbf{x}) = (\sigma(-, + \cdots +))\mathbf{x} + \sigma(1, 0, \ldots, 0)$ for some $\sigma \in \mathcal{S}$; (that is g' is a reflection across one of the planes $x_i = 1$) or

(ii) $g'(\mathbf{x}) = (\sigma(+, - \cdots -))\mathbf{x} + \sigma(2, 0, \ldots, 0)$.

In other words, if two L-tiles meet at a black point, they can only meet in one of the two ways (up to $\mathrm{Isom}(\mathbb{E}^n)$) described in the lemma. Note that these two forms of g' are the forms (i) and (ii) of Lemma 2.2. The lemma can be easily verified and proof is omitted.

LEMMA 3.3. *A configuration* $hg\mathrm{L} \cup h\mathrm{L}$, $h, g \in \mathrm{Isom}(\mathbb{E}^n)$, *satisfies* $\mathcal{M}$ *if and only if* $h\frac{1}{4} \notin hg\mathrm{L} \cap h\mathrm{L}$ *or* $g\mathrm{L} = g'\mathrm{L}$ *where* $g' \in \mathrm{Isom}(\mathbb{E}^n)$ *is:*
(iii) $g'(\mathbf{x}) = \mathbf{r}\mathbf{x} + \mathbf{1}$ *for some* $\mathbf{r} \in \mathcal{R} - \{(- \cdots -)\}$.
Moreover, for any $g, h \in \mathrm{Isom}(\mathbb{E}^n)$, $h\frac{1}{4} \notin hg\mathrm{X} \cap h\mathrm{L}$

Note that the form of g' is the form (iii) of Lemma 2.2. In other words, if one L-tile meets the inside corner of another, it can only meet in one of $2^n - 1$ ways. Moreover, the inside corner of an L-tile cannot meet an X-tile. In particular, the inside corner of any L-tile can only be incident to the outside corner of some other L-tile. Again, the lemma can be easily verified and proof is omitted. Finally,

LEMMA 3.4. *A tiling* $hg\mathrm{L} \cup h\mathrm{L}$, $h, g \in \mathrm{Isom}(\mathbb{E}^n)$, *satisfies* $\mathcal{M}$ *if and only if* $hg\mathrm{L}$ *and* $h\mathrm{L}$ *do not coincide on* any *point or they coincide at some black point or an outside corner of one coincides with the inside corner of the other.*

This is tedious and trivial to show; if the reader wishes, it is not hard to add additional markings to guarantee this lemma holds.

4. APERIODICITY

We now state and prove the main theorem.

THEOREM 4.1. *Every tiling in* $\Sigma(\mathcal{T}, \mathcal{M})$ *is non-periodic, and* $\Sigma(\mathcal{T}, \mathcal{M}) \neq \emptyset$. *That is, the marked tiles* I *and* L *are a pair of aperiodic tiles in* $\mathbb{E}^n$, $n \geq 3$.

We will prove the theorem in a series of lemmas culminating in Proposition 4.6. We first define a map f from congurations of **L**-tiles to configurations of L-tiles as follows.

$$f(\mathsf{L}) \quad := \quad \mathrm{L} \tag{1}$$

$$\forall g \in \mathrm{Isom}(\mathbb{E}^n), \quad f(g\mathsf{L}) \quad := \quad g\mathrm{L} \tag{2}$$

$$\forall \mathcal{G} \subset \mathrm{Isom}(\mathbb{E}^n), \quad f\Big(\bigcup_{g \in \mathcal{G}} g\mathsf{L}\Big) \quad := \quad \bigcup_{g \in \mathcal{G}} g\mathrm{L}. \tag{3}$$

LEMMA 4.2. *For all* $k \in \mathbb{N}$, $f(S^k(\mathsf{L}))$ *satisfies* $\mathcal{M}$.

PROOF. This follows immediately from Lemmas 2.2, 3.2 and 3.3. □

LEMMA 4.3. *For all* $k \in \mathbb{N}$, *there is a configuration of* $\mathcal{T}$ *satisfying* $\mathcal{M}$, *containing* $f(S^k(\mathsf{L}))$, *and with support that is a topological ball contained in* $2^k(\mathsf{L} \cup \mathrm{X})$

That is, we can fill in the 'holes' in $f(S^k(\mathsf{L}))$ (using X-tiles). Note that the last condition forces us to add all the images of X to the 'inside' of the configuration.

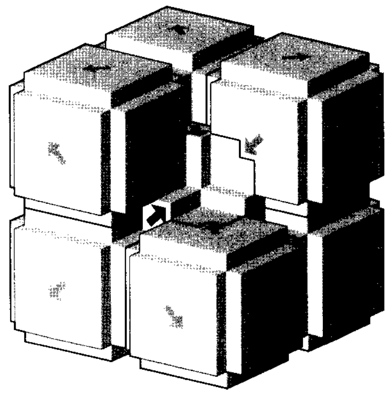

FIGURE 7. $f(S^1(\mathsf{L}))$.

PROOF. From the definitions of the tiles X and L it should be clear that *unmarked* X-tiles can be inserted into $f(S^k(\mathsf{L}))$ (in particular, note that the 'holes' in $f(S^k(\mathsf{L}))$ lie on a subset of the vertices and edges of the cubic lattice). We must only show that we can arrange for the markings of the X to match.

But this is not much of a problem! It is not hard to verify that every edge e of the cubic lattice, parallel to the vector $\sigma_e(1, 0, \dots, 0)$, in $f(S^k(\mathsf{L}))$ satisfies one of the following conditions:

(a) Either e lies on a straight chain of edges that begins and ends at markings on a pair of L-tiles of the form $h\mathrm{L} + \mathbf{x}$, $(\sigma_e(-, + \cdots +))h\mathrm{L} + \mathbf{x} + \sigma(k, 0, \dots, 0)$ where $h \in \mathrm{Isom}(\mathbb{E}^n)$, $\mathbf{x} \in \mathbb{E}^n$ and $k \in \mathbb{Z}$,

(b) or e lies on a chain of edges that begin at markings on some L-tile but continues to the boundary of $f(S^k(\mathsf{L}))$.

(c) e lies on a chain of edges that has both ends on the boundary of $f(S^k(\mathsf{L}))$ (and, for $n \geq 4$, may or may not cut through the interior of $f(S^k(\mathsf{L}))$). In the first two cases, we say e is **determined**.

Now, we are free to place I tiles along the chains of determined edges. The markings of these tiles are fixed by the marked centres of the L-tiles at the ends of these chains.

We then fill in the remaining edges with chains of I tiles with markings in whatever orientation we please (along a chain the markings must be consistent).

COROLLARY 4.4. *$\Sigma(\mathcal{T}, \mathcal{M})$ is non-empty.*

PROOF. The above lemma points out that every $S^k(\mathsf{L})$ corresponds to a configuration of tiles in $\mathcal{T}$ satisfying $\mathcal{M}$. It follows that for every $\tau \in \Sigma(\mathsf{L}, S)$, $f(\tau)$ is a tiling by $\mathcal{T}$ satisfying $\mathcal{M}$. And so because $\Sigma(\mathsf{L}, S)$ is non-empty, $\Sigma(\mathcal{T}, \mathcal{M})$ is non-empty. □

LEMMA 4.5. *For any tiling $\tau \in \Sigma(\mathcal{T}, \mathcal{M})$, there exists an image of* L *in τ.*

This can be verified immediately, because X cannot tile by itself.

PROPOSITION 4.6. *For any $\tau \in \Sigma(\mathcal{T}, \mathcal{M})$, for any tile $g\mathrm{L}$ ($g \in \mathrm{Isom}(\mathbb{E}^n)$) in τ, for any $k \in \mathbb{N}$ there is a $g' \in \mathrm{Isom}(\mathbb{E}^n)$ such that $g\mathrm{L} \subset g'f(S^k(\mathsf{L})) \subset \tau$. Moreover, $g'f(S^k(\mathsf{L}))$ is unique.*

PROOF. Let $\tau \in \Sigma(\mathcal{T}, \mathcal{M})$ and let $g\mathrm{L}$ be any tile in τ. We will induct on k. When $k = 0$, there is nothing to show! By the definition of f the statement holds.

So assume the statement is true for fixed k; that is, each L-tile lies in a unique image of $f(S^k(\mathsf{L}))$. Now it can be verified (consider Lemma 4.3) that if two $hf(S^k)(\mathsf{L})$, $hgf(S^k)(\mathsf{L})$ intersect on an $(n-1)$-dimensional set, they can only meet in the following way:

Let $g' \in \mathrm{Isom}(\mathbb{E}^n)$ be such that $2^k g'(\mathbf{x}) = g2^k(\mathbf{x})$. Then g' must be a type (i), type (ii) or type (iii) isometry (cf. Lemmas 2.2, 2.3 and 2.4).

Thus, by 'deflation',it is really sufficient to show that if the proposition holds for $k = 0$ then it holds for $k = 1$. But this is not too hard:

Let us define a **central** L-tile to be an L-tile that meets the inside corners of $2^n - 1$ other L-tiles, one at each of its outside corners. Now we will show that every L-tile is either a central L-tile, or meets the outside corner of a central L-tile, and that moreover, the inside corner of a central L-tile cannot meet the outside corner of some other central L-tile. Once we show this, we are done, because every central L-tile thus lies in the centre of some image of $f(S(\mathsf{L}))$, and every L-tile therefore lies in a unique image of $f(S(\mathsf{L}))$.

So consider a particular tile $g\mathrm{L}$ in τ.

First, if any outside corner $gr\mathbf{1}$, $\mathbf{r} \neq \pm(+\cdots+)$ is incident to the inside corner of an L-tile, we claim that $g\mathrm{L}$ must be a central L-tile. Note first that the matching rules force every outside vertex on the same $(n-1)$-plane containing $r\mathbf{1}$ to be incident to the inside corner of an L-tile (because the tile with inside corner at $g\mathbf{r}\mathbf{1}$ precludes the positioning of a tile $gh\mathrm{L}$ with h a type (i) or type (ii) isometry such that $gh\mathrm{L}$ meets $g\mathrm{L}$ on this $(n-1)$-plane.) Walking around outside vertices of $g\mathrm{L}$ we then have that *all* other outside vertices of $g\mathrm{L}$ are incident to the inside vertices of L-tiles, and in this case our original $g\mathrm{L}$ is central.†

So now we suppose that no outside corner $gr\mathbf{1}$, $\mathbf{r} \neq \pm(+\cdots+)$ is incident to the inside corner of an L-tile. We will show that the inside corner of $g\mathrm{L}$ is incident to the outside corner of a central tile. The inside corner of $g\mathrm{L}$ is incident to the outside corner of *some* tile $h\mathrm{L}$; if this outside corner is not $h(-1, \ldots, -1)$ then we are done by the above paragraph. So suppose this outside corner is $h(-1, \ldots, -1)$ and thus $h\mathrm{L} = h'\mathrm{L}$ where $h'(\mathbf{x}) = g((\mathbf{x}) + \mathbf{1})$. Now consider any outside corner $h'\mathbf{r}\mathbf{1}$, $\mathbf{r} \neq \pm(+\cdots+)$; suppose $h'h_2\mathrm{L}$ meets $h'\mathrm{L}$ on $(n-1)$-dimensional set in the $(n-1)$-plane containing both $h'(-1, -1, \ldots, -1)$ and $h'\mathbf{r}\mathbf{1}$. Then h_2 cannot be a type (i) or (ii) isometry and so the inside vertex of $h_2\mathrm{L}$ must meet an outside vertex of $h'\mathrm{L}$. In this way, we can find that $h\mathrm{L}$ is a central L-tile.

Finally, note that there is simply not enough room for the inside vertex of one central L-tile to meet the outside corner of another central L-tile. □

PROOF OF THEOREM 4.1. The theorem follows almost immediately from the above proposition. In particular, let $\tau \in \Sigma(\mathcal{T}, \mathcal{M})$. Then the proposition implies that there exists a unique tiling $\tau' \in \Sigma(\mathsf{L}, S)$ such that $f(\tau') \subset \tau$ (this follows from the definition of $\Sigma(\mathsf{L}, S)$ and f). As no isometry leaves τ' invariant and τ' is unique, for all $\mathbf{x} \in \mathbb{E}^n$, $\tau + \mathbf{x}$ cannot be equivalent to τ. (For if so, $f(\tau' + \mathbf{x})$ is also a configuration in τ, but because τ' is unique, $\tau' + \mathbf{x} = \tau'$, a contradiction.) □

5. VARIATIONS

We gave an aperiodic pair of tiles in $\mathbb{E}^n$, $n \geq 3$, one of which had a disconnected interior. In the interest of upholding tradition, we should see what we can accomplish if we require tiles to have *connected* interior. Our primary goal is to keep the number of kinds of tiles as low as possible.

†Curiously, this is one way this proof fails when $n = 2$. See [5].

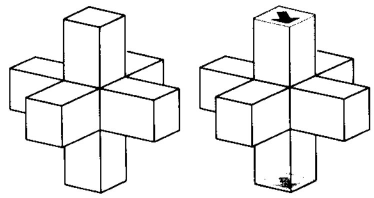

FIGURE 8. Two marked X tiles.

The problem stems from Lemma 4.3, in which we must fill in the determined chains of edges with I tiles. It can be noted that in fact, at every vertex of the cubic lattice in an $f(S^k(\mathsf{L}))$, at most one chain of edges is determined, and when $n = 3$, exactly one chain is determined at each vertex.

Thus, our first option is to define two kinds of marked X tiles:

Let X_1 be the X-tile left completely white. Let X_2 be the X-tile marked through the map $\mu_X : X \to \{\text{black}, \text{white}\}$ defined as:

for any $\mathbf{x} \in X$, if $\mathbf{x} \notin (\frac{1}{4}\mathsf{H}' + \mathbf{v})$, $\mathbf{v} = (\pm 1, 0, \dots, 0)$ let $\mu_I(\mathbf{x}) := \text{white}$. Otherwise let $\mu_I(\mathbf{x}) := \mu_I(4(\mathbf{x} - \mathbf{v}))$.

For $n > 3$ we can take as our tiles $\mathcal{T} := \{X_1, X_2, L\}$, and for $n = 3$ take $\mathcal{T} := \{X_2, L\}$. Then these will be aperiodic sets of tiles.

It would appear that another option is to mark every face of the X tile; that is, perhaps there is a way to use a single marked configuration of I tiles. However, even in $\mathbb{E}^3$ is is easy to verify this cannot suffice.

So our second option is to have the marked tile I_0 defined as the portion of I with first coordinate non-negative. A matching rule is then defined that these I_0-tiles must form marked I tiles. This author finds this sort of matching rule to be mathematically acceptable but aesthetically unappealing.

Finally, it seems plausible to the author that for odd dimension n, certain undetermined edges can be marked so that every vertex in $f(S^k(\mathsf{L}))$ meets exactly one marked chain of edges. If this is true then $\mathcal{T} := \{X_2, L\}$ is an aperiodic set of tiles in $\mathbb{E}^n$, n odd and greater than 2.

We would also like to repeat that our markings defined by μ can be replaced with simple bumps and nicks, in which case our only matching rule would be that the tiles fit together.

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