Cutting a Polygon into Triangles of Equal Areas

This tale, like so many in mathematics, begins with a simple question, answers it, and ends with questions that have yet to be resolved.

Fred Richman in 1965 wondered whether it is possible to cut a square into an odd number of triangles of equal areas. The key word here is "odd," for a moment's reflection shows that a square can be cut into any even number of triangles of equal areas.

Before I go on to describe the research that grew out of that question over the last third of a century, I will stop to introduce a few terms for the sake of clarity.

A dissection of a polygon into triangles of equal areas I will call an *equidissection.* An equidissection into m triangles I call an *m-equidissection.* An m-equidissection with m odd will be called an *odd equidissection*, and with m even, an *even equidissection.* Richman was asking whether every equidissection of a square is even.

Richman's colleague, John Thomas, got interested in the problem and proved that there is no odd equidissection of a square into triangles if the coordinates of the vertices of the triangles are rational with odd denominators. When he submitted his work to *Mathematics Magazine*, "The referee thought the problem might be fairly easy (although he could not prove it) and possibly well-known (although he could find no reference to it)." The referee suggested that Thomas submit it as a *Monthly* problem and if no one solved it, the paper should be published. It appeared in 1968.

Paul Monsky in 1970, building on Thomas's proof, showed that the answer to Richman's question is, "No, there is no odd equidissection of a square." His argument uses two tools, Sperner's Lemma from combinatorial topology, and 2-adic valuations from algebra. I will describe both.

In 1928 Emanuel Sperner published a theorem which he used to investigate the dimensions of Euclidean spaces, and was soon applied by others to give a short proof of Brouwer's fixed point theorem. He stated it for simplices in all dimensions, but I will present it just for polygons in the xy-plane.

Consider a polygon cut into triangles. For simplicity, assume that two triangles that touch each other either intersect in a complete edge of both or in a vertex of both. All the vertices are labeled A, B, or C. Figure 1 is an example.

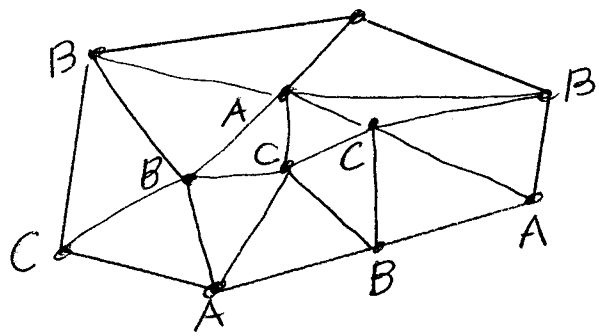

Fig. 1

An edge of a triangle whose ends are labeled A and B will be called *complete.* A triangle whose vertices are labeled A, B, and C will also be called *complete*. Sperner's reasoning shows that the number of complete edges on the boundary of the polygon has the same parity as the number of complete triangles. In Figure 1 the respective numbers are 3 and 9. In particular, *if there are an odd number of complete edges on the boundary there must be at least one complete triangle*. That implication is what Sperner used and so will we.

To prove Sperner's lemma, imagine going into each triangle in the labeled dissection of a polygon and placing a pebble in the triangle next to any complete edge. Then add up the total number of pebbles in two ways, by edges and by triangles. Next to each complete edge inside the polygon are two pebbles; next to each complete edge on the boundary is one pebble. So the parity of the total number of pebbles is the same as the parity of the number of complete edges on the boundary.

Next, count the pebbles triangle by triangle. Each triangle contains 0, 1, or 2 pebbles. Only the complete triangle contains exactly one pebble. Each of the other triangles contains an even number of pebbles. Hence the parity of the total number of pebbles is the same as the parity of the number of complete triangles. Comparing the two counts establishes the lemma.

The other tool is a *2-adic valuation, Ø*, which is a function defined on the real numbers extended by ∞. First, for a non-zero integer *n*, Ø(*n*) is the number of 2's in the prime factorization of *n*. If *a* and *b* are non-zero integers, Ø(*a/b*) is defined as Ø(*a*)- Ø(*b*). At 0, Ø is set equal to ∞. It turns out that Ø can be extended to all the reals in many ways and that all such extensions have the following properties:

Ø(xy) = Ø(x) + Ø(y)

Ø(x + y) $\geq$ minimum of Ø(x) and Ø(y)

Ø(x) = ∞ only when x is 0.

It follows from these properties that for x < y, Ø(x + y) = Ø(x).

As examples we have Ø(1) = 0, Ø(1/2) = -1, Ø($\sqrt{3}/2$) = -1, and Ø($\sqrt{2}$) = 1/2.

For each prime *p* there are *p*-adic valuations defined in a similar manner.

With the aid of a 2-adic valuations, we can divide the xy-plane into three sets, which I will call A, B, and C. The set A consists of the points (x,y) for which both Ø(x) and Ø(y) are positive. B consists of the points (x,y) for which Ø(x) $\leq$ 0 and Ø(x) $\leq$ Ø(y). C consists of the remaining points, namely, those (x,y) for which Ø(y) $\leq$ 0 and Ø(y) < Ø(x). We label each point A, B, or C, depending on which of the three sets contains it. For instance (0,0) is labeled A, (1,0) is B, (1,1) is B, and (0,1) is C. The point ($\sqrt{3}/2, \sqrt{3}/2$) is labeled B. It is easy to check that translating a point by a point labeled A (viewing them for a moment as vectors) does not change the label: if P is a point, then P and P - A have the same labels.

One of the key tools going back essentially to Thomas is that if the three vertices of a triangle have all three labels, A, B, and C, then the valuation of the area of the triangle is less than or equal to -1. To show this, first translate the complete triangle by the vertex labeled A. Let us say that he three vertices are now (0,0), (a,b), and (c,d) , with (a,b) labeled B and (c,d) labeled C. The absolute value of the area of the triangle is (ad - bc)/2. Since

Ø(ad - bc) = Ø(ad) $\leq$ 0, it follows that Ø((ad - bc)/2) $\leq$ -1.

Note that it follows that a line in the xy-plane cannot meet all three sets A, B, and C.

Now consider an m-equidissection of a polygon of area A. Assume that at least one of its m triangles is complete. Since the area of the triangle is A/m. It follows that

$\emptyset(A/\mathrm{m}) \leq -1$.

Thus $\emptyset(A) - \emptyset(\mathrm{m}) \leq -1$,

from which we conclude that

$\emptyset(\mathrm{m}) \geq \emptyset(A) + 1$.

So, if we knew that $\emptyset(A)$ is larger than -1, m must be even. In particular, if A is an integer, m is even.

In order to apply the information just obtained, we have to be sure that there is at least one complete triangle in the dissection. This is where Sperner's lemma enters the picture. After just one more observation, we will be ready to prove the Richman-Thomas-Monsky theorem.

Consider a finite set of points in a complete line segment. Each of these points is labeled either A or B. The points divide the segment into shorter sections. The number of these sections that are complete must be odd. One way to show this is to drop pebbles in each section next to an end labeled A, and then add them up in two ways, by points and by sections. When a line segment that is not complete is divided into sections, the number of complete sections is always even. For instance, there are no complete sections in when a segment with ends B and C or A and C is cut into sections. A segment with ends A and A or B and B has an even number of complete sections.

With all the machinery in place, we are ready to prove that a square has only even equidissections.

Now consider an equidissection of a square. It is no loss of generality to assume that the area of the square is 1 and that its vertices are (0,0), (1,0), (0,1), and (1,1). It is shown in Figure 2, with the labels of its four vertices.

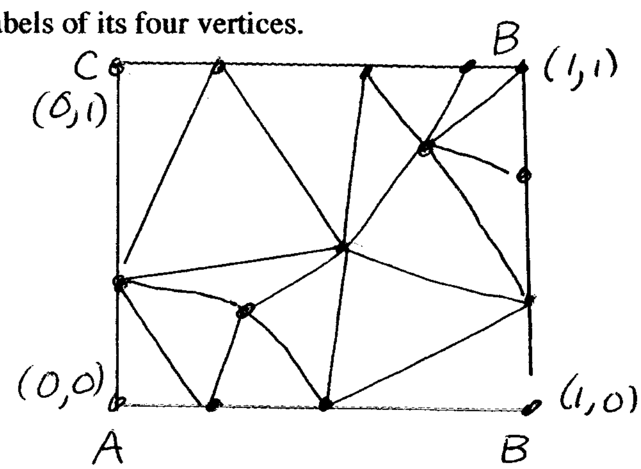

Fig. 2

All the vertices, not just the corners of the square, are labeled A, B, or C. No matter how those vertices are labeled there will be an odd number of complete sections in the bottom edge of the square, which is complete. The other three edges have no complete sections. Thus the total number of complete sections on the boundary of the square is odd. Hence there is at least one complete triangle in the dissection. It follows that the equidissection is even.

That is where the subject of equidissections remained until 1979 when David Mead obtained a generalization from a square to a cube in any dimension. He proved that when an n-dimensional cube is divided into simplices all of which have the same volume, the number of the simplices must be a multiple of n!. In addition to Sperner's Lemma in higher dimensions he used *p*-adic valuations for all primes *p*.

In 1985, when Elaine Kasimatis was presenting the result for a square in G. Donald Chakerian's geometry seminar, I wondered, "What about the regular pentagon?"

She found the answer and went on to prove that in any m-equidissection of a regular n-gon with at least five sides, m must be a multiple of n. In the proof she had to extend *p*-adic valuations to the complex numbers for the prime divisors of n. Her work appeared in 1989. In a sense it was another generalization of the theorem about equidissections of a square.

A year later she and I published the results of an investigation of equidissections of trapezoids and other quadrilaterals. Among these was yet another generalization of a square, namely quadrilaterals whose four vertices are (0,0), (1,0), (a,a), and (0,1), where a is any positive number, illustrated in Figure 3.

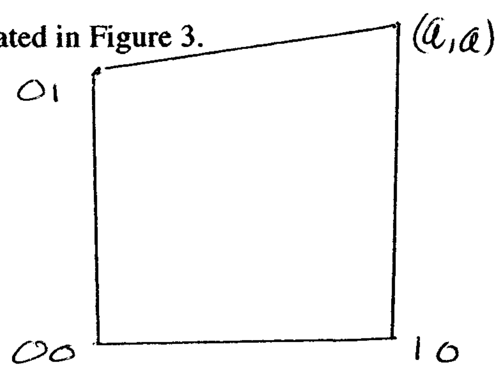

Fig. 3

The area of such a quadrilateral is *a*. If Ø(a) is greater than 0, then the boundary of the quadrilateral has two complete edges, and the hypothesis of Sperner's Lemma doesn't hold. Incidentally, had it held, we would have been able to conclude that in any m-equidissection of the quadrilateral m had to be a multiple of 4, since Ø(m) would be greater than 1. Since m can be as low as 2, we could have predicted that there are an even number of complete edges on the boundary.

In this case we apply the linear mapping that takes (x,y) to (x/a, y). The image of the original polygon has area 1 and vertices (0,0), (1/a,0), (1,a), and (0,1), as shown in Figure 4.

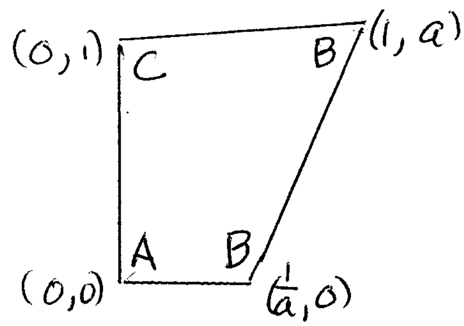

Fig. 4

Now there is only one complete edge on the boundary, and Sperner's Lemma applies. Hence m is even, as in the case of the square.

If $-1 < Ø(a) \leq 0$, the labeling of the initial quadrilateral has one complete edge, and there is no need to introduce a linear mapping. Again m must be even.

When Ø(a) = -1, there may be odd equidissections as the case a = 3/2 shows. Figure 5 illustrates this case, where the quadrilateral is cut into three right triangles, each of area 1/2.

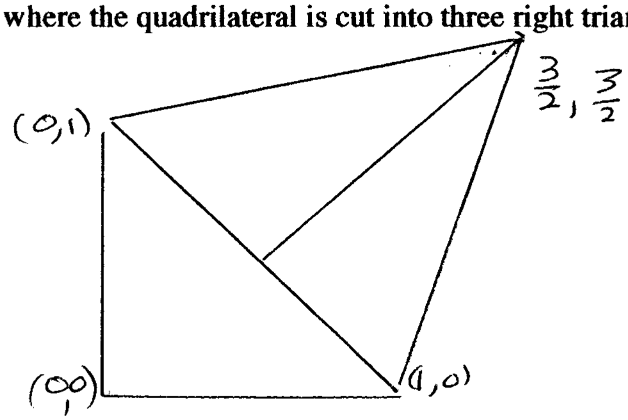

Fig. 5

More generally, if a = b/(2c), where b and c are odd integers, the corresponding quadrilateral has an odd equidissection, constructed, for instance, as follows. Choose an odd positive integer m. Introduce the point (1/m,0) and use it to divide the quadrilateral into three triangles, T, U, and V, as shown in Figure 6.

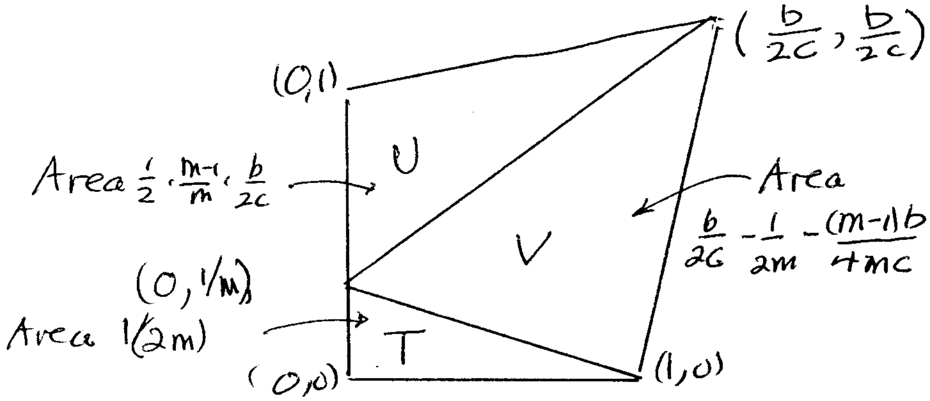

Fig. 6

The areas of T, U, and V, each expressed with the denominator 4mc, are in the proportions

$$\frac{2c}{4mc}:\frac{(m-1)b}{4mc}:\frac{mb-2c+b}{4mc},$$

hence in the proportions,

2c : (m-1)b : mb -2c + b.

Each of these three quantities is even. Therefore the proportion can be expressed as the proportions of three smaller integers,

$$c:\frac{(m-1)b}{2}:\frac{mb-2c+b}{2}.$$

Dissect triangle T into c triangles of equal areas, U into (m-1)b/2 such triangles, and V into (mb -2c +b)/2 such triangles. In this way we have obtained an equidissection of the quadrilateral into mb triangles.

However, if Ø(a) = -1 and a is irrational, I don't know what can be said. To be specific, even the case $a = \sqrt{3}/2$ is not settled. Does the quadrilateral with vertices $(0,0),(1,0),((\sqrt{3}/2,\sqrt{3}/2)$, and (0,1) have an odd equidissection?

In any case, this attempt to generalize the result for squares failed. The question remained: *What is there about a square that forces all its equidissections to be even?* In other words, *What is the most general class of polygons that have no odd equidissection?*

One simple generalization is that any parallelogram has no odd equidissection. This follows immediately from the result for a square since any parallelogram is the image of a square by a linear mapping. Since a linear mapping magnifies all areas by a constant, it takes an equidissection into an equidissection.

A parallelogram being centrally symmetric suggests that perhaps any centrally symmetric polygon has no odd equidissection. Kasimatis's theorem about regular n-gons, when n is even, gave me enough extra evidence that I investigated centrally symmetric polygons, trying to produce a counter-example. Instead I proved in 1989 that every centrally symmetric 6-gon or 8-gon has no odd equidissection. Monsky in 1990 proved the theorem in general.

Even so, I did not feel that that was the last word. There was another class of polygons that I suspected would generalize the square. To construct this type of polygon, I start with a unit square and then distort its boundary, changing opposite edges in the same way. The resulting polygon still tiles the plane by translates by all integer vectors. Figure 7 shows such a distorted square.

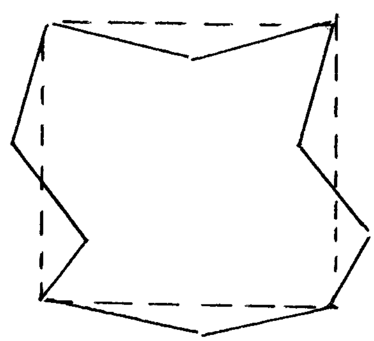

Fig 7

It seemed to me that complicated the boundary would lessen the chance that the resulting polygon would have an odd equidissection. I proved for a few simple families made this way that my suspicion was valid, such as polygons formed by adding one dent, as in Figure 8.

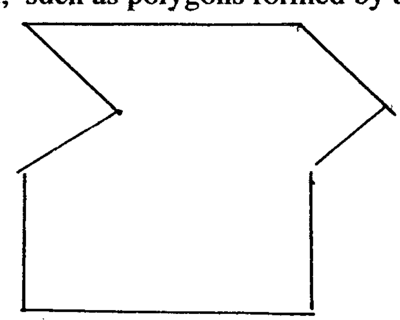

Fig 8

Some years later a surprising breakthrough occurred, which I described in a paper published in 1999. It concerns a unit square in the *xy*-plane whose corners have integer coordinates, such as the one in Figure 9.

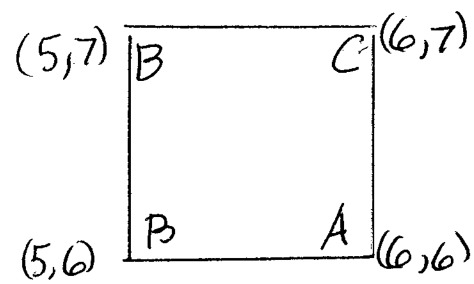

Fig. 9

Note that the square in Figure 9 has one complete edge. A moment's thought shows that exactly one vertex of any such square has one vertex with both coordinates even, hence labeled A. Its two neighboring vertices are then labeled B and C. That implies that the square has exactly one complete edge.

It follows immediately that any polygon in the *xy*-plane made up of an odd number of such unit squares has no odd equidissection. To see this, reason as in the proof of Sperner's Lemma, placing a pebble inside each square in the polygon next to its complete edge. Since there are an odd number of pebbles, there must be an odd number of complete edges on the boundary. Moreover, since the area in an integer, it follows that the number of triangles must be even.

It struck me as odd that by assuming that the polygon has an odd number of squares I was able to deduce that the number of triangles was even. I checked a few cases where the polygon had an even number of squares, enough to convince me that it was true in general, but left it to someone else to treat that case. Iwan Praton in 2002 disposed of the even case. His proof showed that if the number of squares is of the form 2^r b, where b is odd, then there is a translate of the image of the polygon by a linear mapping that takes (x,y) to $(x/2^u, y/2^v)$, where $u + v = r$, to which a stronger version of Sperner's lemma applies. The stronger version asserts the following. Orient the boundary of a polygon counter-clockwise, thus orienting each complete edge on the boundary. An AB edge is compatible with the orientation while a BA edge is not. A complete triangle also determines an orientation by taking the vertices in the order A, B, and C. The lemma asserts that the number of clockwise complete triangles minus the number of counter-clockwise complete triangles equals the number of BA edges minus the number of AB edges. This can be proved by an argument similar to the pebble approach.

Consequently any polygon composed of the unit squares described has no odd equidissection. There is another, more suggestive way to state this result. Consider any polygon in the xy-plane whose edges are parallel to the axes and have rational lengths. Such a polygon has no odd equidissection. To show this, first translate the polygon so that one of the vertices is at the origin.. Then magnify this image by a mapping that takes (x,y) to (mx,.my), where m is an integer divisible by all the denominators of the lengths of the edges. The image consists of congruent squares and has no odd equidissection. Hence the original polygon has no odd equidissection.

The next conjecture is inevitable. What if the assumption that the edges have rational lengths is removed? I conjectured that any polygon whose edges are parallel to the axes has no odd equidissection.

I then faced three classes of polygons that I either knew or suspected have not odd equidissections: centrally symmetric, distorted square, edges parallel to the axes. The first case was already settled, and there was ample evidence for the remaining two cases. Figure ~~9~~ 10 illustrates the three types.

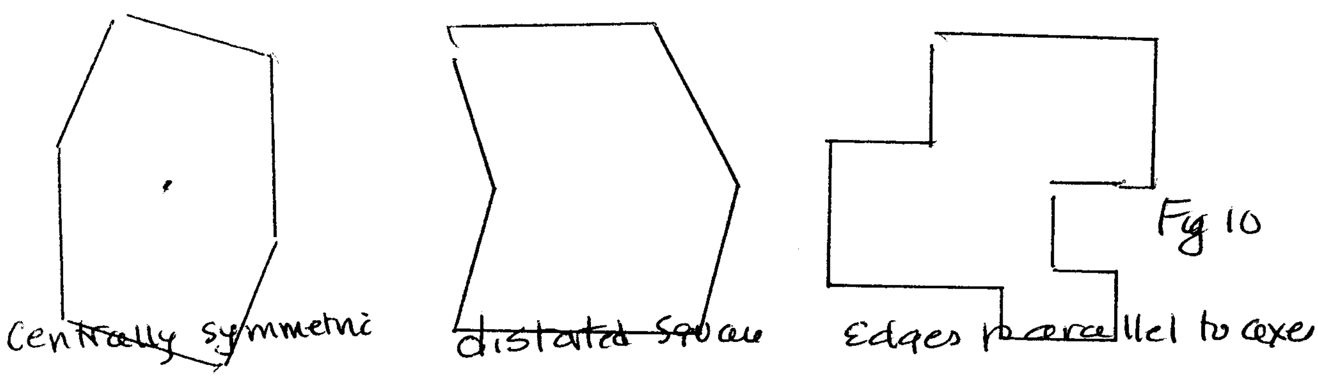

Fig 10

As I stared at polygons like those, I noticed a property that they all shared. To describe this property I orient the boundary, turning each edge into a vector whose direction is compatible with the orientation. Then I call two vectors on the boundary equivalent if they are parallel. Al three types have the property that *the sum of the vectors in each equivalence class is the zero vector*. I call such a polygon *special* and conjectured that each special polygon has no odd equidissection.

I showed that the conjecture is true when the special polygon has only a few edges. The smallest possible number of edges is four, and the polygon is then a parallelogram, for which the conjecture is true. There are no special polygons with five sides, as may easily be checked. There are three types of special polygons with six sides, constructed as follows.

The first step is to determine the number of edges in an equivalence class. There must be at least two in a class and at most three, for if there were four, two would be forced to be adjacent. The partitions of six meeting these conditions are 6 = 3 + 3 and 6 = 2 + 2 + 2.

The second step is to see how the equivalence classes could be arranged on the boundary. Take the 3 + 3 case first. Denoting parallel vectors by the same letter, the only possibility is to alternate the vectors of the two classes, as shown in Figure ~~10~~ 11.

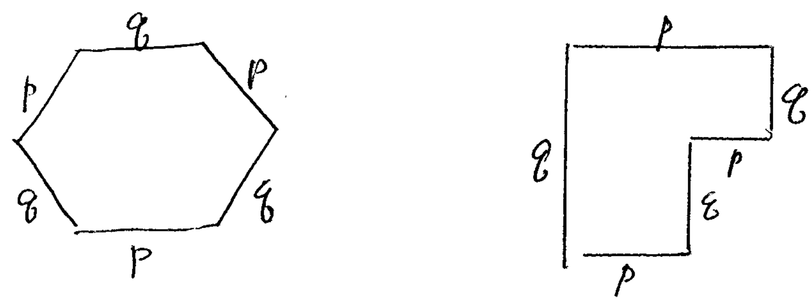

Fig 11

That schema can be realized by a special polygon, as shown in Figure 11. Without loss of generality, we can assume its edges are parallel to the axes.

The 2 + 2 + 2 case leads to two essentially different schemas, as shown in Figures 12 and 14.

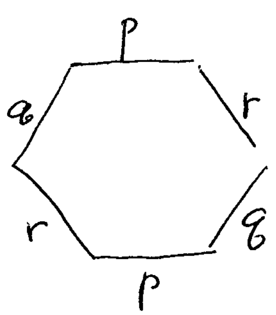

Fig 12

This schema can be realized by any centrally symmetric polygon with six sides, as shown in Figure 13.

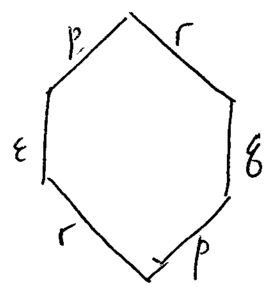

Fig 13

The other possible schema is shown in Figure 14.

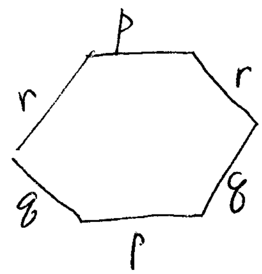

Fig 14

It, too, can be realized by a special polygon, shown in Figure 15.

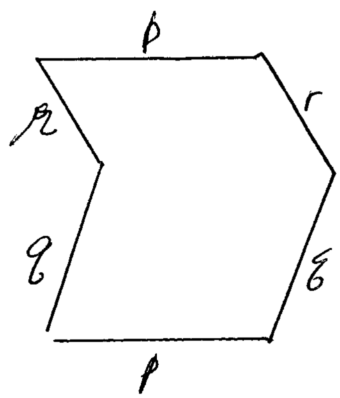

Fig 15

Each case can be treated with the aid of Sperner's Lemma, 2-adic valuations, and a variety of affine mappings, that is, mappings that take (x,y) to (ax + by + e, cx + dy + f), where a, b, c, d, e, and f are constants and ad - bc is not 0.

To determine the special polygons with seven sides, we first list the partitions of seven in which the summands are at least two and at most three. There is only one such partition, namely 7 = 3 + 2 + 2. It can be realized in two different ways by schemas and each schema has a geometric realization,, as shown in Figure 16.

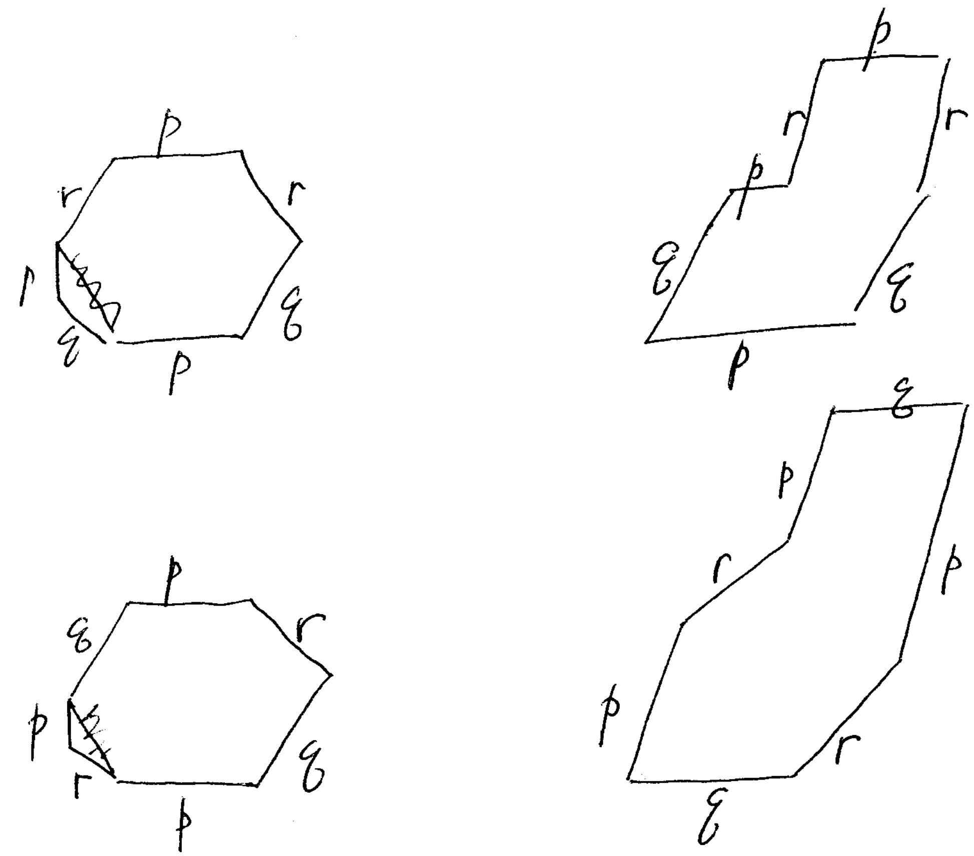

Fig 16

Again I managed to show that both of these types of special polygons have no odd equidissection. Because the proofs break into a couple dozen cases, I have hesitate to go on to the eight-sided special polygons. In any event, these 7-gons provide substantial evidence for the general conjecture, which I had wanted to call the "mother of all conjectures," but was restrained by the referee to refer to it simply as a "generalized conjecture."

As is customary in science, we are left with more questions than we had when we started. Perhaps we have found the fundamental property of the square that is the basis of the Richman-Thomas-Monsky theorem. Perhaps not. These are some of the open questions:

Does a special polygon have no odd equidissections?

How many partitions are there of a positive integer n if the summands are at least 2 at most n/2?

Is each such partition representable by a combinatorial schema?

If so, by how many?

Is each combinatorial schema representable by a special polygon?

Even if all these questions are answered, many questions about equidissections would remain. For instance, does a trapezoid whose parallel edges have lengths in the ratio of $\sqrt{2}$ to 1 have any equidissections? I think that the answer is no and make the following conjecture:

Consider a trapezoid whose parallel edges have lengths in the ratio of r to 1, where r is algebraic. I conjecture that if r has at least one negative conjugate, then the trapezoid has no equidissection.

That is where Richman's question that he raised a third oif a century ago has led. The path that he discovered seems to have no end.

References