For instance, when

$$f(H) = \exp\left\{-\frac{1}{2}\operatorname{tr} H^2 - \frac{s}{N}\operatorname{tr} H^4\right\}$$

we should obviously substitute the following into (7.16):

$$f(\Lambda) = \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{s}{N}\sum_{i=1}^{N}\lambda_i^4\right\}.$$

Unfortunately, this method alone does not allow us to compute the integral (7.5).

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The Birkhoff–von Neumann theorem for polystochastic matrices*

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Introduction

The well-known Birkhoff–von Neumann theorem asserts that the extremal points of the space of (square) bistochastic matrices are permutation matrices.

The interest in this fact is explained to a significant extent by the fact that it allows the reduction of the so-called assignment problem to a linear programming problem, namely, to the transportation problem.

It was observed by Motzkin [3] that not all of the extremal points of the space of k-dimensional k-stochastic matrices are integral and, hence, the solution of the k-index transportation problem for $k > 2$ is, generally speaking, not integral.

A. M. Vershik pointed this fact out to the author and suggested the problem of classifying the extremal points of spaces of polystochastic matrices. We have not succeeded in constructing a complete classification of these extremal points, but in the present paper we find a version of the Birkhoff–von Neumann theorem for k-dimensional k-stochastic matrices, which allows us to describe the arithmetic structure of the extremal points.

Plan of the paper. In §1 we state the main result of the paper, Theorem 1.5, and in addition we give the notations that are needed later on in the paper. In §2 we introduce the concept of a homomorphism of matrices, used throughout the paper, and using which we prove (in §2) the necessity of the conditions of Theorem 1.5. In fact, the necessity of the conditions of Theorem 1.5 represents a direct generalization of the Birkhoff–von Neumann theorem. It would be

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possible to give a more direct and simpler proof of this part of the theorem, but we have chosen a presentation that is compatible with what we do later, using it simultaneously for the introduction of definitions and lemmas that are needed for the second part. The proof of sufficiency, which is trivial in the two-dimensional case, presents the fundamental difficulty in the k-dimensional case for $k > 2$.

In §3 the sufficiency of the conditions of Theorem 1.5 is reduced to a lemma in combinatorial algebra, Lemma 3.3. In §§4 and 5 we give some preliminary facts, using which we prove Lemma 3.3 in §6. In §7 we carry out the proofs of the facts from matrix algebra that are used in the paper. The proof of these facts is very simple, but it is rather complicated to give it in matrix language, so we translate the statements into an invariant language, after which they become tautologies. The invariant presentation has yet another advantage as it explains the algebraic meaning of the concept of a homomorphism of a matrix into a matrix, introduced in §1. In the appendix we discuss (without a detailed proof) the restrictiveness of conditions (1) and (2) of Theorem 1.5.

The author thanks A. M. Vershik, who initiated the study of multidimensional extremal points, as well as L. A. Oganesyan and O. G. Fayans for a detailed critical reading of the paper.

§1. Notation and statement of results

1.1. Following I. V. Romanovskiĭ [**4**], a vector with component set S will be denoted $x[S]$ and interpreted as a function on S. We denote by $x[s]$, $s \in S$, and $x[S']$, $S' \subset S$, the value of the function at the point s and its restriction to S', respectively. A matrix is interpreted as a function on the Cartesian product of the sets S of rows and T of columns, and is denoted $A[S, T]$. The symbol "$\times$" denotes both the scalar product of vectors and the action of matrices on a row. We denote by $\mathbf{1}[S]$ the vector consisting only of ones.

Vectors, matrices, etc. are said to be positive, rational, etc., if all of their entries are such. The empty vector is the vector with an empty set of components.

We denote by $m:n$ $(m \leq n)$ the set of numbers $m, m+1, \dots, n$. The vector $x[m:n]$ is said to be *decreasing* if $x[i] < x[j]$ for $i > j$, $i, j \in m:n$. The symbol $|P|$ denotes the number of elements in the set P.

1.2. Extremal k-matrix; its spectrum. A *k-dimensional matrix* will be a function on a set J, which is represented as the Cartesian product of k copies of the set I. Fixing the index i in the l-th copy distinguishes the set J, $(l, i) \subset J$, $l \in 1:k$, $i \in I$.

We say that a k-dimensional nonnegative matrix $M[J]$ is *k-stochastic* if

$$\sum_{j \in J(l,i)} M[j] = 1, \qquad l \in 1:k, \qquad i \in I. \tag{1}$$

In the n^k-dimensional space (with $n = |I|$) of all k-dimensional matrices the k-stochastic matrices form a bounded convex polyhedron (of dimension $n^k - nk + k - 1$). Its extremal points are called *extremal k-matrices*.

The *spectrum*[1] spec M of the matrix $M[J]$ is the (strictly!) decreasing positive vector σ, among whose components each of the numbers $M[j]$, $j \in J$, occurs, except for zero (in other words, spec ignores zero and also the multiplicity with which the numbers occur in the matrix).

In these terms the Birkhoff–von Neumann theorem asserts that the spectrum of any extremal 2-matrix is the vector with a single component equal to one.

1.3. Matrix of relations. A *relation* for a vector $\sigma[1:p]$ is a nonnegative integer vector $r[1:p]$ such that $\sigma[1:p] \times r[1:p] = 1$. The matrix whose rows are all the (pairwise distinct) relations for σ is called the *matrix of relations* for σ and is denoted by $R(\sigma)$. It is clear that the matrix of relations of a positive vector is a finite (possibly empty) matrix.

1.4. Cancellation; k-series; k-height. Consider a matrix $A[S, T]$, and in it the row $A[s, T]$, $s \in S$, with $\sum_{t \in T} A[s, t] < k$ and remove from $A[S, T]$ all the columns that pass through the nonzero elements of the row $A[s, T]$.

We call a *k-series* a finite sequence of matrices, where each successive one is obtained from the preceding by the procedure of cancellation.

We call the *k-height* $h_k(A)$ of the matrix A the smallest number λ for which there exists a k-series beginning with A, ending with the empty matrix, and containing (together with A) $\lambda + 1$ terms.

The condition $h_k(A) \leq \lambda_0$ is equivalent to the possibility of exhausting the matrix A by λ_0 cancellations, and the condition $h_k(A) = \infty$ means the appearance in the process of cancellation of a nonempty matrix for which each nonzero row has the sum of entries not less than k.

We remark also that for $k_1 \leq k_2$ the inequality $h_{k_1}(A) \geq h_{k_2}(A)$ holds, and the equality $h_k(A) = \infty$ for all $k = 1, 2, \dots$ is equivalent to A's having a zero column.

[1] The term was proposed by A. M. Vershik.

1.5. Statement of the main theorem. *Let σ be a positive decreasing vector and let $k \geq 2$ be a natural number. For there to exist an extremal k-matrix with spectrum σ it is necessary and sufficient that two conditions hold*:

(1) $h_k(R(\sigma)) < \infty$.
(2) *the columns of the matrix $R(\sigma)$ are linearly independent.*

We shall indicate three corollaries, whose derivation from 1.5 is obvious.

1.6. *If the vector $\sigma[1:p]$ realizes the spectrum of an extremal k-matrix, then it is rational, $\sigma[1] \geq (k-1)^{-1}$, and if we represent the components of $\sigma[i]$ as fractions of natural numbers, $a[i]/b[i]$, then the inequalities*

$$\sigma[i+1] \geq \left((k-1) \prod_{j \in 1:i} b[j]\right)^{-1}$$

hold. In particular, for $k = 2$ we obtain the assertion of the Birkhoff–von Neumann theorem.

1.7. *For any positive decreasing vector $\sigma[1:p]$ whose matrix of relations is not empty and whose columns are independent, and for any $k \geq \max_{i \in 1:p}(\sigma[i])^{-1}$, there is an extremal k-matrix with spectrum $\sigma[1:p]$.*

1.8. *For any positive rational decreasing vector $\sigma[1:p]$ with components less than one, there is a natural number b such that the vector $\sigma'[0:p+1]$ with $\sigma'[0] = (b-1)/b$, $\sigma'[p+1] = 1/b$, $\sigma'[i] = \sigma[i]$, $i \in 1:p$, is the spectrum of some extremal 3-matrix.*

§2. k-partitions; homomorphism of a matrix into a matrix; proof of necessity for Theorem 1.5

2.1. The notation sp(A). We associate with a matrix $A = A[S, T]$ a system of linear equations $A[S, T] \times x[T] = \mathbf{1}[S]$. If the columns of the matrix A are linearly independent and the system has a (unique!) solution, then we denote by sp(A) the vector with strictly decreasing components, among which all the components of the solution of $x[T]$ occur. If the system does not have a solution, then the vector sp(A) is empty by definition. We remark that, for a positive decreasing vector σ whose matrix of relations has linearly dependent columns, we have: sp$(R(\sigma)) = \sigma(R(\sigma))$, just as in 1.3.

2.2. k-partitions. A *k-partition* of a matrix $A[S, T]$ is a decomposition of its set of rows into k disjoint subsets $S_1, \ldots, S_k$. A matrix with a fixed k-partition is called a *k-partitioned matrix*, and the submatrices (k of them) into which the original matrix is partitioned are called the *blocks* of the partition.

We say that a k-partitioned matrix is *admissible* if each of its entries is either 0 or 1, and if there is not more than one 1 in the intersection of each block.

A column in an admissible k-partitioned matrix is said to be *complete* if it contains exactly k 1's; *incomplete* if the number of 1's is less than k; *elementary* if it contains exactly one 1. We say that an admissible k-partitioned matrix is *complete* if each column is complete.

We shall show how to pass from k-dimensional matrices to two-dimensional k-partitioned matrices.

2.3. *With each extremal k-matrix M we can associate a complete k-partitioned matrix A with independent columns and with* sp$(A) =$ spec M. *Conversely, to each matrix A with independent columns, which consists of* 0's *and* 1's, *possesses a complete k-partition and for which the vector* sp(A) *is nonempty and positive, we can associate a k-matrix M with* spec$(M) =$ sp(A).

Proof. The left-hand side of the system of linear equations (1) from 1.2 is defined by some (two-dimensional) matrix A', consisting of 0's and 1's, whose rows are partitioned in k sets corresponding to the k possible values of the index l. The vector whose components are elements of the k-matrix M gives a solution of the system (1). The matrix obtained from A' by removing the columns corresponding to the zero components of this vector is nondegenerate (this follows from the extremality of the k-matrix M; see [**1**]) and, together with the k-partition inherited from A', gives the desired k-partitioned matrix A.

The obvious inversion of this argument proves the second assertion in 2.3.

The following simple fact is decisive for the proof of the necessity in 1.5.

2.4. *If a matrix* (*consisting of* 0's *and* 1's) *with independent columns possesses a complete k-partition, then its k-height is finite.*

Proof. Since the matrix has a complete k-partition it follows that its nonzero rows are linearly dependent and, since the columns are linearly independent, the number of nonzero rows is greater than the number of columns. It follows from the completeness of the partition that there is a nonzero row containing fewer than k 1's, so that we can perform the removal (see 1.4). Iterating this argument (and the removal), we are led in a finite number of steps to the empty matrix, as required.

2.5. Homomorphisms. A *mapping* $f: A \to A'$ of a matrix $A = A[S, T]$ into a matrix $A' = A'[S', T']$ is a pair of mappings $F_S: S \to S'$ and $f_T: T \to T'$.

A *homomorphism* is a mapping $f\colon A \to A'$ such that for any $s \in S$ and $s' \in S'$, connected by the relation $f_S(s) = s'$, and any $t' \in T'$ we have

$$\sum_{t \in f_T^{-1}(t')} A[s, t] = A'[s', t']. \tag{2}$$

We note simple properties of homomorphisms that follow immediately from the definition.

(a) If $s' = f_S(s)$, then $\sum_{t \in T} A[s, t] = \sum_{t' \in T'} A'[s', t']$.

(b) The mapping $g \circ f\colon A \to A''$, which is the composition of homomorphisms $f\colon A \to A'$ and $g\colon A' \to A''$, is a homomorphism.

(c) Consider a mapping $f\colon A[S, T] \to A'[S', T']$, a submatrix $A'' = A'[S'', T'']$, $S'' \subset S'$, $T'' \subset T'$, of the matrix A', and a submatrix $A_0 = A[f_S^{-1}(S''), f_T^{-1}(T'')]$ of the matrix A. If the mapping f is a homomorphism, then its restriction $A_0 \to A''$ is also a homomorphism.

(d) Let $f\colon A \to A'$ be a homomorphism such that f_T is a surjective mapping (i.e., maps T onto all of T'). If the matrix A is nonnegative, then $h_k(A') \le h_k(A)$, and if the columns of the matrix A are linearly independent, then the columns of A' are also linearly independent.

(e) Let $f\colon A \to A'$ be a homomorphism. If the columns of the matrices A and A' are linearly independent and the vector $\mathrm{sp}(A')$ is nonempty, then $\mathrm{sp}(A) = \mathrm{sp}(A')$.

2.6. *Let $\sigma = \sigma[1:p]$ be a vector with positive strictly decreasing components, $A = A[S, T]$ a matrix with linearly independent columns, consisting of nonnegative integer elements, and $\mathrm{sp}(A) = \sigma$. Then there exists a homomorphism $f\colon A \to R(\sigma)$ for which the mapping f_T is surjective.*

Proof. We denote by $x = x[T]$ the solution of the system $A[S, T] \times x[T] = \mathbf{1}[S]$. First we construct the mapping f_T, defining it uniquely by the condition: $f_T(t) = i$, $t \in T$, $i \in 1:p$, if $x[t] = \sigma[i]$. We now construct f_S. We choose a row of $A[s, T]$ and define the row $r_s[1:p]$ by the formula: $r_S[i] = \sum_{t \in f_T^{-1}(i)} A[s, t]$. It is clear that such a row r_s is a relation for σ, i.e., is a row of the matrix $R(\sigma)$. Thus, we have constructed a mapping f_S which together with f_T gives a mapping $f\colon A \to R(\sigma)$. From the construction it is easy to see that f is a homomorphism.

2.7. Proof of the necessity of the conditions of Theorem 1.5. Let $\sigma = \mathrm{spec}\, M$. By 2.3 there exists a complete k-partitioned matrix A with linearly independent columns and with $\mathrm{sp}(A) = \sigma$. By Lemma 2.4 $h_k(A) < \infty$. By 2.6 there exists a homomorphism $A \to R(\sigma)$ with a surjective mapping f_T. By property (d) of 2.5 we have $h_k(R(\sigma)) \le h_k(A) < \infty$ and the columns of the matrix $R(\sigma)$ are linearly independent.

§3. Faithful homomorphism. Statement of the main lemma

3.1. We associate with a homomorphism $f\colon A[S, T] \to A'[S', T']$ two subspaces in the space of the vectors $x = x[T]$. For a pair $t_1, t_2 \in T$, $f_T(t_1) = f_T(t_2)$, we consider a vector with $x[t_1] = 1$, $x[t_2] = -1$ and $x[t] = 0$ for t different from t_1 and t_2. We denote by L the space generated by all such vectors. For a pair $s_1, s_2 \in S$ with $f_S(s_1) = f_S(s_2)$ we consider the vector $A[s_1, T] - A[s_2, T]$ and denote by K the space generated by all these vectors. From the definition of homomorphism it follows that there is an obvious inclusion $K \subset L$.

We say that the homomorphism f is *faithful* if, first, $K = L$ and, second, f_S is a surjective mapping.

3.2. We begin with an important property, for our purposes, of faithful homomorphisms.

If a homomorphism $f\colon A \to B$ is faithful and the columns of the matrix B are linearly independent, then the columns of the matrix A are also independent.

See §7 for information about the proof.

3.3. Statement of the main lemma. *Let R be a matrix consisting of nonnegative integers and having finite k-height. If $k \ge 2$, then there exist a complete k-partitioned (see* 2.2*) matrix A (consisting of* 0's *and* 1's*) and a faithful homomorphism $A \to R$.*

The proof of this lemma is carried out in §6 by induction on the height, and §§4 and 5 prepare the induction.

3.4. We derive the sufficiency of the conditions of the main theorem from 3.3. Using 3.3, we construct a complete k-partitioned matrix A and a faithful homomorphism $f\colon A \to R(\sigma)$. By 3.2 the matrix A is composed of linearly independent columns. Since f is a homomorphism, then by property (e) of 2.5 we have $\mathrm{sp}(A) = \mathrm{sp}(R(\sigma)) = \sigma$, and it remains to apply the second assertion of 2.3.

§4. Properties of faithful homomorphisms and their construction

4.1. Only (a) of the properties of faithful homomorphisms given below is somewhat nontrivial; see §7 for the proof.

(a) *The composition of faithful homomorphisms is faithful.*

(b) *In the situation of* 2.5(c) *if we assume that* $S'' = S'$, *and if the homomorphism* f *is faithful, then its restriction to* A_0 *is also faithful. In other words, a compatible with the homomorphism* $f: A \to A'$ *removal of columns in* A *and* A' *does not affect faithfulness.*

(c) *Let*

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}, \qquad A' = \begin{pmatrix} B' & C' \\ 0 & D' \end{pmatrix}$$

and suppose that a homomorphism $f: A \to A'$ *maps* B *into* B', C *into* C' *and* D *into* D'. *If the corresponding homomorphisms* $B \to B'$ *and* $D \to D'$ *are faithful, then* f *is also a faithful homomorphism.*

4.2. Condensation. Consider a homomorphism $f: A[S, T] \to A'[S', T']$ and its decomposition into a composition of homomorphisms $g: A[S, T] \to \bar{A}[S, \bar{T}]$ and $\bar{f}: \bar{A}[S, \bar{T}] \to A'[S', T']$. If the mapping g_S is the identity and g_T is surjective, then the homomorphism $\bar{f}$ is uniquely determined by the mapping g_T. In this case we call the matrix $\bar{A}$ a *condensation of the matrix* A, and the homomorphism $\bar{f}$ a *condensation of the homomorphism* f. We can imagine $\bar{A}$ as being obtained by a "gluing" of the columns of A that are mapped by g_T into a single element of $\bar{T}$, in such a way that the column of $\bar{A}$ obtained as a result of the "gluing" is the sum of the corresponding columns of A. It is clear that every condensation can be obtained as a result of several simpler condensations, which "glue" at most two columns of A.

We state an obvious property of condensation.

4.3. *A condensation of a faithful homomorphism is a faithful homomorphism.*

4.4. Splitting of a column. We decompose the first column of the matrix $A[1:m, 1:n]$ into a sum $A[1:m, 1] = A'[1:m, 1] + A''[1:m, 1]$. We set $A''' = A[1:m, 2:n]$ and construct the matrix

$$A_1 = \begin{bmatrix} A' & A' & 0 & A''' \\ 0 & A'' & A' & A''' \\ A' & 0 & A'' & A''' \\ A'' & A' & 0 & A''' \end{bmatrix}. \tag{3}$$

A homomorphism $A_1 \to A$ is defined in a natural way, and this homomorphism is easily seen to be faithful. Moreover, if all the elements of A, A' and A'' are assumed to be nonnegative, then the relation $h_k(A_1) \le 4h_k(A)$ holds.

4.5. *Let* A *be a matrix with nonnegative integer elements and finite* k*-height. Then there exist a matrix* B *consisting of* 0's *and* 1's *and a faithful homomorphism* $B \to A$.

Proof. Using a permutation of the columns we can apply the splitting to an arbitrary column, so that after a finite number of splittings we can construct a matrix B consisting of 0's and 1's and a homomorphism $B \to A$, which is faithful in view of 4.1(a).

§5. k-homomorphisms

5.1. We consider a k-partitioned matrix $A[S, T]$ (i.e., S is decomposed into the union of the sets $S_1, \dots, S_k$).

A (faithful) homomorphism $f: A[S, T] \to A'[S', T']$ is called a (*faithful*) k*-homomorphism* if, for any $s' \in S'$, the equality

$$|S_{l_1} \cap f_S^{-1}(s')| = |S_{l_2} \cap f_S^{-1}(s')|, \qquad l_1, l_2 \in 1:k. \tag{4}$$

holds.

We mention two properties of k-homomorphisms, the first of which follows immediately from the definition, and the second of which follows from the first.

(a) Consider a k-homomorphism $f: A[S, T] \to A'[S', T']$, an arbitrary $t' \in T'$, $l_1, l_2 \in 1:k$ and set $X = f_T^{-1}(t')$. In these notations we have

$$\sum_{t \in X, s \in S_{l_1}} A[s, t] = \sum_{t \in X, s \in S_{l_2}} A[s, t].$$

(b) In the situation of part (a) we assume that the partition is admissible (see 2.2) and we form sets $X_l \subset X$, $l \in 1:k$, in the following way: $t \in X_l$ if and only if there exists an $s \in S_l$ such that $A[s, t] = 1$. With these notations we have

$$|X_{l_1}| = |X_{l_2}|, \qquad l_1, l_2 \in 1:k.$$

5.2. Condensation lemma. *Let* A *be an admissible* k*-partitioned matrix of the form* $\begin{pmatrix} B & C & 0 \\ 0 & 0 & D \end{pmatrix}$, *suppose that the matrix* A' *has the form* $\begin{pmatrix} B' & C' \\ 0 & D' \end{pmatrix}$ *and let* $f: A \to A'$ *be a* k*-homomorphism that maps* B *into* B', C *into* C' *and* D *into* D'.

Suppose that the induced partition on B is complete, all the columns of the matrix C are incomplete, and for every column C in A′ the number $N_1(C)$ of elementary columns in the pull-back of C which (being columns in A) pass through D is related to the number $N_2(C)$ of the remaining incomplete columns in this pull-back (i.e. the columns passing through C and incomplete nonelementary columns passing through D) by $N_1(C) \geq kN_2(C)$. Then there exists a condensation of the homomorphism f such that the condensed matrix $\bar{A}$ has the form $\begin{pmatrix} B & \bar{C} \\ 0 & \bar{D} \end{pmatrix}$, where $\bar{D}$ is a condensation of D and the partition of $\bar{A}$ arising from the k-partition of A is complete.

Proof. We isolate an incomplete column in A. Using our assumption about the excess of elementary columns and applying (b) of 5.1, we conclude that we can "paste" some elementary columns passing through D to it, so that the isolated column remains complete. Iterating this procedure we will obtain the desired condensation.

5.3. *In the situation of* 5.2 *if the restrictions $B \to B'$ and $D \to D'$ of the homomorphism f are faithful, then the homomorphism $\bar{A} \to A'$ obtained as a result of condensation is also faithful.*

The proof follows immediately from 4.3 and part (c) of 4.1.

5.4. Basic construction. *For the one-rowed matrix* $\mathbf{1}[1, 1:q]$ *with $q \leq k$ and an arbitrary number N there exist: a matrix $E = E_N$ consisting of* 0's *and* 1's, *an admissible k-partition of the matrix E containing* 2k *incomplete nonelementary columns, a faithful k-homomorphism $f: E \to \mathbf{1}[1, 1:q]$ such that the inverse image of each column of the matrix* $\mathbf{1}[1, 1:q]$ *contains not less than N elementary columns.*

Proof. We start with the case $q = k - 1$. We construct E in the form $\begin{pmatrix} F & 0 \\ G & H \end{pmatrix}$. We set $m_1 = (k-2)km$, $m_2 = km$, $n = (k-2)(km+1)$ and $F = F[1:m_1, 1:n]$, $G = G[1:m_2, 1:n]$, $H = H[1:m_1, 1:m_2]$, where H is a diagonal matrix (with 1's on the principal diagonal and zeros at all the remaining places). The 1's in F are determined by the condition $F[i,j] = 1$ if and only if $i \leq j \leq i + k - 2$. The matrix G has the form

$$\begin{pmatrix} G_1 & 0 & \cdots & 0 & 0 \\ 0 & G_2 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & G_N & 0 \end{pmatrix}$$

where $G_\nu = G_\nu[1:k, 1:k(k-2)]$, $\nu \in 1:N$. The 1's in each matrix G_ν are determined by the condition $G_\nu[i,j] = 1$ if and only if $j \equiv i \pmod{k}$.

We partition the set $1:m_1 + m_2$ of rows of the matrix E into subsets S_l, $l \in 1:k$, relating $i \in 1:m_1 + m_2$ to the subset S_l, if one of two conditions holds:

(a) $i \leq m_1$ and $i \equiv 1 \pmod{k}$;

(b) $i > m_2$ and $i \equiv l - 1 \pmod{k}$.

We now construct a homomorphism f, giving f_T in the following way: if $j \leq n$, $f_T(j)$ is defined by the condition $f_T(j) \equiv j \pmod{(k-1)}$, and if $j > n$, then it is defined by the condition $f_T(j) \equiv j + 1 - k\,\mathrm{entier}((j+2)/k) \pmod{(k-1)}$, where entier$(*)$ denotes the greatest integer less than $*$.

The verification that this matrix, partition and homomorphism are the desired ones is automatic.

The case $q < k - 1$ reduces to the one just proved by removing the redundant columns in E and using property (b) of 4.1.

§6. Proof of the main lemma (from 3.3)

6.1. We begin with an obvious fact. Let A' be a matrix of the form $(B' \; C')$ and $f: B \to B'$ a homomorphism. Then there is a matrix A of the form $(B \; C)$ and a homomorphism $A \to A'$, whose restriction to B coincides with f. In case the matrix C' consists of 0's and 1's, C can also be chosen in this form, and in such a way that each of its columns would contain at most one 1.

6.2. By 4.5 and part (a) of 4.1, the main lemma reduces to the following assertion.

For any matrix A', consisting of 0's *and* 1's, *with finite k-height there are a complete k-partitioned matrix A and a faithful k-homomorphism $A \to A'$.*

We shall prove this by induction on the height. We limit ourselves to the induction step from $t - 1$ to t, since the basis of the assertion, corresponding to the case $h_k(A') = 1$, is proved analogously and even more simply.

By a permutation of the rows and columns, a matrix A' with $h_k(A') = t$ can be reduced to the form $\begin{pmatrix} B' & C' \\ 0 & D' \end{pmatrix}$, where $h_k(B') < t$, and $D' = \mathbf{1}[1, 1:q]$, $q < k$. Applying the induction hypothesis to B', we construct a complete k-partitioned matrix B and a faithful k-homomorphism $B \to B'$. In view of 6.1 we may assume that this homomorphism extends to a homomorphism $f_0\colon (B\ C) \to (B'\ C')$ (which, in general, is not faithful, but *is* a k-homomorphism). We now choose a sufficiently large natural number N and we construct $E = E_N$ and a homomorphism $E \to D'$ as in 5.4. The last homomorphism together with f_0 defines a homomorphism

$$\begin{pmatrix} B & C & 0 \\ 0 & 0 & E \end{pmatrix} \to \begin{pmatrix} B' & C' \\ 0 & D' \end{pmatrix}.$$

In view of 5.2 and 5.3 this homomorphism can be condensed to the desired one (the condensation of a k-homomorphism is, obviously, again a k-homomorphism). Thus, our assertion, and with it, Lemma 3.3, are proved.

§7. Diagrammatic formulation of faithfulness

7.1. For a set P we denote by L_P the space of vectors $x = x[P]$.

A mapping $f\colon P \to Q$ defines a linear mapping $f^*\colon L_Q \to L_P$ by the formula $(f^*x)[p] = x[f(p)]$, $x \in L_Q$, $p \in P$.

With a matrix $A[S, T]$ we associate a linear mapping $u_A\colon L_T \to L_S$, taking $x[T]$ to $A[S, T] \times x[T]$. Thus the mapping $f\colon A[S, T] \to A'[S', T']$ is associated with a diagram of linear mappings:

$$\begin{array}{ccc} L_{T'} & \xrightarrow{f_T^*} & L_T \\ {\scriptstyle u_{A'}}\downarrow & & \downarrow{\scriptstyle u_A} \\ L_{S'} & \xrightarrow{f_S^*} & L_S \end{array} . \qquad (*)$$

7.2. The proofs of the following assertions follow from an unraveling of the definitions.

(a) *For a mapping f to be a homomorphism it is necessary and sufficient that the diagram* $(*)$ *be commutative* (i.e., $u_A \circ f_T^* = f_S^* \circ u_{A'}$).

(b) *For a homomorphism f to be faithful it is necessary and sufficient that* $\operatorname{Ker} f_S^* = 0$ *and* $\operatorname{Im} f_T^* = (u_A)^{-1}(\operatorname{Im} f_S^*)$.

(Recall that Ker denotes the kernel and Im the image of a mapping.)

7.3. This interpretation of faithfulness allows us to reduce all our assertions about faithful homomorphisms to facts in linear algebra, whose proofs can be obtained by a trivial "diagram chase" type argument (see [**2**]).

In fact, we have not proved only 3.2 and part (a) of 4.1, so that it is only for them that we give the diagrammatic formulation.

(a) Rephrasing of assertion 3.2. *If the conditions of part* (b) *of* 7.2 *hold for a diagram of the form* $(*)$, *then the equality* $\operatorname{Ker} u_{A'} = 0$ *implies the equality* $\operatorname{Ker} u_A = 0$.

(b) Rephasing of assertion (a) of 4.1. *Suppose that in the commutative diagram*

$$\begin{array}{ccccc} X_1 & \to & X_2 & \to & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \to & Y_2 & \to & Y_3 \end{array}$$

(*consisting of linear spaces and linear mappings*) *the left and right subdiagrams are faithful* (i.e., *satisfy the requirements of* (b) *of* 7.2). *Then the same is also true for the ambient diagram*:

$$\begin{array}{ccc} X_1 & \to & X_3 \\ \downarrow & & \downarrow \\ Y_1 & \to & Y_3 \end{array}$$

§8. Appendix

8.1. Definition of density. We denote by Ω the set consisting of finite strictly decreasing sequences of positive rational numbers not exceeding one, and by $\Omega_n \subset \Omega$ the set of those subsequences whose terms are all representable by fractions (with natural numbers in the numerator and denominator) with denominator n.

The upper density of a set $A \subset \Omega$ is the limit $\limsup_{n\to\infty} 2^{-n}|A \cap \Omega_n|$, and the lower density is the limit $\liminf_{n\to\infty} 2^{-n}|A \cap \Omega_n|$. (Note that $|\Omega_n| = 2^n$, i.e., Ω_n consists of 2^n sequences.)

If the upper and lower densities of the set A coincide, then we say that A has a density.

8.2. The constant μ. For two subsets A and B of the set of natural numbers we shall write $A > B$ if each number $b \in B$ is representable as a sum of natural numbers (among which some can be the same), each of which lies in A.

In the usual way a subset of the set of natural numbers is identified with the points of the interval $[0, 1]$: a set A corresponds to the number in whose binary expression there are 1's in the places corresponding to numbers from A and zeros elsewhere.

In this way the pairs (A, B) are identified with the points of the unit square, and it is easy to see that the pairs (A, B) with $A > B$ correspond to a measurable set in the square. We denote the measure of this set by μ. In other words, μ is the probability of the event $A > B$.

This definition allows us to compute μ with arbitrary accuracy. A rough computation gives the estimate $0.58 < \mu < 0.64$.

8.3. We denote by $\Sigma \subset \Omega$ the set of sequences σ for which the matrix of relations $R(\sigma)$ has independent columns, and by $\Sigma_k \subset \Sigma$ the set of those σ for which $R(\sigma)$ has finite k-height (see 1.3 and 1.4). It is clear that $\Sigma_l \subset \Sigma_k$ for $l \leq k$ and $\bigcup_{k=1}^{\infty} \Sigma_k = \Sigma$.

8.4. *The sets* $\Sigma, \Sigma_1, \ldots, \Sigma_k, \ldots$ *have a density. The sets* Σ_1 *and* Σ_2 *have density zero. The sets* $\Sigma, \Sigma_3, \Sigma_4, \ldots$ *have the same density, equal to* μ. *In other words, the probability of the occurrence of a sequence* $\sigma \in \Omega$ *in* Σ *or in one of the* Σ_k *with* $k \geq 3$ *is equal to* μ.

Sketch of the proof. For Σ_1 and Σ_2 everything is obvious. We shall now relate the occurrence of a sequence $\sigma \in \Omega$ in Σ with the event $A > B$. For $\sigma = \sigma[1:p]$ we denote by σ_A the set of those $\sigma[i]$, $i \in 1:p$, that do not exceed $\frac{1}{2}$, and by σ_B the set of numbers $1 - \sigma[i]$, $i \in 1:p$, with $\sigma[i] > \frac{1}{2}$. Reducing the numbers from σ_A and σ_B to a common denominator and replacing them by the numerators of the corresponding fractions, we obtain two finite sets A and B of natural numbers. It is clear that for $\sigma \in \Sigma$ it is necessary that the condition $A > B$ hold. On the other hand, simple calculations show that with probability 1 the condition $A > B$ is sufficient for $\sigma \in \Sigma_3$.

8.5. Combining 8.4 with Theorem 1.5, we conclude: *for any* $k \geq 3$ *the probability that a rational positive decreasing vector* σ *with components that do not exceed 1 is representable by the spectrum of some extremal k-matrix is well defined. This probability does not depend on k and is equal to the number* μ, $0.58 < \mu < 0.64$, *cf* 8.2.

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Asymptotic Aspects of the Representation Theory of Symmetric Groups[1]

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Perhaps the simplest combinatorial entity is the group of the $n!$ permutations of n objects. This group has a different constitution for each individual number n. The question is whether there are nevertheless some asymptotic uniformities prevailing for large n or for some distinctive class of large n. Mathematics has still little to tell about such problems.

H. Weyl, "Philosophy of Mathematics and Natural Science"

In this article we collect some facts that have a direct relation to representations of symmetric groups and their applications that were not touched on by James in his book. These facts are concerned mainly with articles of recent years, and their selection reflects the interests of the author of this supplement. We concentrate on the following questions: 1. Young's lattice and combinatorial foundations. 2. The RSK (Robinson–Schensted–Knuth) correspondence and its applications. 3. The limiting form of Young diagrams and asymptotic questions. 4. Symmetric functions and the K-functor. Our account does not contain proofs and is necessarily brief; each of these themes deserves several sections in a book on representations of symmetric groups and their applications. We give a list of references on the themes that we touch on that is far from exhaustive (see also the list at the end of the editor's preface to the translation).

S.1. Young's lattice and combinatorial foundations

In reading the literature on the representation theory of symmetric groups, the first appearance of Young diagrams and tableaux may leave the reader with an impression of purely technical innovation. This sensation is only reinforced when we learn that the correspondence between Young diagrams with n cells and irreducible representations of $\mathfrak{S}_n$ has a very complicated form, and there is still no transparent account of the construction of the irreducible representation corresponding to a given diagram (see Section 4[2] and later).

[1] Originally published as an editor's supplement to the Russian translation ("Mir", Moscow, 1982) of the book "The representation theory of symmetric groups" by G. D. James, Lecture Notes in Math. 682, Springer, Berlin, 1978. Translated by E. Primrose.

[2] References to Sections and §, unless preceded by an S., pertain to James' book (see footnote 1).