Algebraic Graph Theory Without Orientation

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ABSTRACT

Let G be an undirected graph with vertices $\{v_1, v_2, \ldots, v_\nu\}$ and edges $\{e_1, e_2, \ldots, e_\varepsilon\}$. Let $\mathbf{M}$ be the $\nu \times \varepsilon$ matrix whose ijth entry is 1 if e_j is a link incident with v_i, 2 if e_j is a loop at v_i, and 0 otherwise. The matrix obtained by orienting the edges of a loopless graph G (i.e., changing one of the 1's to a -1 in each column of $\mathbf{M}$) has been studied extensively in the literature. The purpose of this paper is to explore the substructures of G and the vector spaces associated with the matrix $\mathbf{M}$ without imposing such an orientation. We describe explicitly bases for the kernel and range of the linear transformation from $\mathbf{R}^\varepsilon$ to $\mathbf{R}^\nu$ defined by $\mathbf{M}$. Our main results are determinantal formulas, using the unoriented Laplacian

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

matrix $\mathbf{MM}^t$, to count certain spanning substructures of G. These formulas may be viewed as generalizations of the matrix tree theorem. The point of view adopted in this paper also gives rise to a matroid structure on the edges of G analogous to the cycle matroid and its dual. In this setting, the analogue of a spanning forest can have components with one odd cycle, and the analogue of an edge cut has the property that its removal creates a new bipartite component.

1. INTRODUCTION

It is common knowledge that useful information about graphs can be obtained from certain matrices and vector spaces associated with the graphs. In particular, the adjacency matrix and the incidence matrix of a graph, as well as the collections of cycles and edge cuts of the graph, can be studied from the viewpoint—and with the powerful machinery—of linear algebra.

In most treatments of these topics (such as [1] or [2]), an incidence matrix for a graph is obtained by first choosing (arbitrarily) an orientation for each edge of the graph, so that the boundary of an edge can be defined as the *difference* of its endpoints. This trick seems to make the linear algebra work out well, and one shows that the results obtained about the graph are independent of the particular orientation chosen. Typical of such results is the matrix tree theorem, which allows one to compute the number of spanning trees of a graph as the determinant of a matrix obtained from its oriented incidence matrix.

Arbitrarily orienting the edges of an undirected graph seems artificial, however. Cannot the same—or analogous—results be obtained more naturally by looking simply at the *unoriented* incidence matrix? W. T. Tutte took this approach in what he described as "an expository paper on chain-groups" [10], but no one seems to have pursued it further. (See *Note added in proof*, page 307.) We were originally motivated to study the unoriented incidence matrix because of its use in integer linear programming [4], where the computation of its minors was an important issue. Since the determinant of the incidence matrix of an odd cycle is ± 2, powers of 2 and the presence of odd cycles in a graph took on crucial roles. As we will see here, adopting the unoriented point of view provides a rich and illuminating theory, in which spanning trees and edge cuts are supplanted by spanning substructures and disconnecting sets of slightly different natures.

The purpose of this paper is to explore some algebraic graph theory that arises from analyzing the unoriented incidence matrix $\mathbf{M}$ of a graph G. In terms of the structure of G, we compute the rank and nullity of $\mathbf{M}$, as well as exhibit explicit bases for its range and its kernel. We do the same for its transpose $\mathbf{M}^t$. In contrast to the oriented situation, these depend not only on the number of vertices, edges, and connected components of G, but also on how many components of G are bipartite (i.e., do not contain an odd cycle). As in the oriented approach, we then look at the unoriented Laplacian matrix $\mathbf{C} = \mathbf{M}\mathbf{M}^t$, obtaining essentially the adjacency matrix of G (augmented by vertex degree information along the diagonal). The determinant of $\mathbf{C}$ and the trace of its compounds give useful combinatorial information about the analogues of the spanning trees of G, in the spirit of the matrix tree theorem. (See [8] for a survey of interesting properties of the Laplacian matrix in the oriented setting.)

Not surprisingly, one way to look at what we have here is as a matroid associated with a graph, analogous to the usual cycle matroid. This point of view will shed more light on these structures, as well as put the power of matroid theory at our disposal. (For example, we immediately get a greedy algorithm to find a minimum-cost instance of our analogue of spanning tree.) The dual matroid, not unexpectedly, also has graph-theoretic significance.

The paper is organized as follows. In Section 2 we set the notation and define the generalizations of spanning tree and edge cut that we need. We prove in Section 3 that we have constructed a matroid. In Section 4 we develop—in our context—the classical algebraic theory relating the cycle space to the bond space, namely a theory relating the "even-circuit space" to the "star space" (or "quasibond space"). Section 5 contains our main results on the matrices $\mathbf{M}$ and $\mathbf{C}$; in particular, we get an unoriented generalized analogue of the matrix tree theorem.

2. DEFINITIONS

Graphs in this paper are undirected and may contain parallel edges and loops; we generally follow the terminology of [2]. Throughout, G is a graph with vertex set $V = \{v_1, v_2, \ldots, v_\nu\}$ and edge set $E = \{e_1, e_2, \ldots, e_\epsilon\}$. We will not distinguish between a subset of the edges of G and the subgraph of G induced by those edges. We also need to consider *substructures* of a graph in which we may have deleted some vertices but retained the edges incident with those vertices. In all cases, however, we will retain at least one endpoint of every edge, so that while one end may "dangle" freely, the other end will be pinned down to the vertex set. (R. Merris [7] colorfully calls an edge with one endpoint deleted a "marimba stick.") In particular, if we take a spanning tree of a connected graph G on ν vertices and delete one vertex (but not the edges incident to that vertex), then the resulting *rootless spanning tree* has $\nu - 1$ vertices and $\nu - 1$ edges. Note that a rootless spanning tree will be nonconnected if the deleted vertex had degree greater than 1.

As usual, we let $\nu = \nu(G)$, $\varepsilon = \varepsilon(G)$, and $\omega = \omega(G)$ denote the numbers of vertices, edges, and components of G, respectively. This notation applies just as well to substructures. We also let $\omega_0 = \omega_0(G)$ be the number of bipartite components (i.e., those that do not contain an odd cycle), and we let $\omega_1 = \omega_1(G)$ be the number of nonbipartite components (i.e., those that do contain at least one odd cycle). For example, a connected bipartite graph has $\omega = \omega_0 = 1$ and $\omega_1 = 0$. In every case, $\omega = \omega_0 + \omega_1$. Note that a rootless tree (including the special case of the empty substructure) is considered to be bipartite.

The *incidence matrix* of G is the $\nu \times \varepsilon$ matrix $\mathbf{M} = \mathbf{M}(G) = [m_{ij}]$ whose entries are given by $m_{ij} = 1$ if vertex v_i is incident with link e_j, $m_{ij} = 2$ if edge e_j is a loop at vertex v_i, and $m_{ij} = 0$ otherwise. Thus, every column of $\mathbf{M}$ consists of either exactly two 1's or exactly one 2 (with the remaining entries being 0's). This is in contrast to the traditional approach of first orienting the edges of G so that each column of $\mathbf{M}$ contains one 1 and one -1 (and forbidding loops altogether). Let $\mathbf{C} = \mathbf{M}\mathbf{M}'$. It is easy to see that the off-diagonal entries of $\mathbf{C} = [c_{ii'}]$ are the same as those of the adjacency matrix for G; that is, $c_{ii'}$ is the number of edges joining v_i with $v_{i'}$ if $i \neq i'$. It is equally easy to see that the diagonal entries of $\mathbf{C}$ are (almost) the degrees of the vertices; that is, c_{ii} is the number of edges incident to v_i, with, however, each loop contributing 4 to this count. Any submatrix of $\mathbf{M}$ in which every column has at least one nonzero entry corresponds to a substructure of the graph, as defined above. Furthermore, we note that the incidence matrix for a nonconnected graph (or substructure) can, with a rearrangement of rows and columns, be put into block form

$$\begin{bmatrix} \mathbf{M}_1 & & & 0 \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ 0 & & & \mathbf{M}_\omega \end{bmatrix}.$$

in which the (not necessarily square) blocks $\mathbf{M}_k$ are the incidence matrices of the components.

The definitions in this paragraph and the next are central in what follows: the objects they define play roles analogous to that of spanning forest in a graph. (Additional motivation is provided in the penultimate paragraph of this section.) For convenience, let us call a connected graph containing exactly one cycle, with that cycle having odd length, an *odd unicyclic* graph. (The graph may contain other edges and vertices as well, as long as they do not create another cycle or another component.) Thus, an odd unicyclic graph consists of an odd cycle (possibly a loop) together with (possibly trivial) trees growing out of each vertex in the cycle. We call a subgraph S of a connected graph G an *essential spanning subgraph* of G if either G is bipartite and S is a spanning tree of G, or else G is not bipartite, $V(S) = V(G)$, and every component of S is odd unicyclic. Note in particular that an essential spanning subgraph of a connected nonbipartite graph may be nonconnected, but each of its components H satisfies $\nu(H) = \varepsilon(H)$. An essential spanning subgraph S in a nonconnected graph is defined to be the union of one essential spanning subgraph from each component. It is easy to see that S must contain $\nu - \omega_0$ edges. Clearly, every graph G contains an essential spanning subgraph S; in fact, it contains one satisfying $\omega(S) = \omega(G)$. One of the goals of this paper is to count the essential spanning subgraphs of a graph; Corollaries 5.7 and 5.8 will essentially accomplish this goal.

The crucial property that we desire of an essential spanning subgraph S of a graph G is that it have an equal number of vertices and edges, so that $\mathbf{M}(S)$ will be a maximal square—and, as we will see, nonsingular—submatrix of $\mathbf{M}(G)$. Unfortunately, if G has a bipartite component, then we cannot achieve this goal, and every $\nu(G) \times \nu(G)$ submatrix of $\mathbf{M}(G)$ will be singular. (This will follow from Theorem 5.1.) We must, therefore, delete vertices in order to correct the imbalance. This leads us to the following definition. A *k-reduced spanning substructure* of a graph G on ν vertices is a substructure of G containing $\nu - k$ vertices, each component of which contains an equal number of edges and vertices and has no even cycles. It is easy to see that any k-reduced spanning substructure R of a graph G has rootless trees and odd unicyclic graphs as its components, and satisfies $\nu(R) = \varepsilon(R) = \nu(G) - k$. Every graph with at most k bipartite components has a k-reduced spanning substructure: we can simply take a spanning tree in k components (including all the bipartite ones) with one vertex (but not its incident edges) deleted, together with a spanning odd unicyclic subgraph in the remaining components. On the other hand, if a graph has more than k bipartite components, then it has no k-reduced spanning substructures. Theorem 5.6 will allow us to count the reduced spanning substructures of a graph.

Note that for graphs without bipartite components, a 0-reduced spanning substructure is the same thing as an essential spanning subgraph. More generally, we have:

THEOREM 2.1. *Let G be a graph with ω_0 bipartite components. Then the ω_0-reduced spanning substructures of G are in one-to-one correspondence with the essential spanning subgraphs of G with one vertex deleted from each bipartite component.*

Proof. This is clear from the definitions, once we realize that in order to obtain an ω_0-reduced spanning substructure of G, we must delete exactly one vertex from each bipartite component of G and hence can delete no vertices from the other components. ■

The philosophy behind these definitions can be viewed in another light. Spanning trees are not quite the "right" maximal substructures of a connected graph, because they have one more vertex than they have edges (and hence are singular – see Theorem 5.1). To correct this, we must either replace an offending vertex by an odd (i.e., nonsingular) cycle (the net effect being the addition of one extra edge), or else remove it. Since we want the result to be maximal, we resort to the latter action only if we cannot perform the former, i.e., if there are no odd cycles.

We need two more concepts to complete the analogy with classical concepts. First, the *star* at a vertex will be the set of edges incident to that vertex, including loops (counted double). Less familiarly, we define a *quasi edge cut* to be a set of edges whose removal increases the number of bipartite components of a graph, and a *quasibond* to be a minimal quasi edge cut. Necessarily, removing a quasibond will create exactly one new bipartite component, either by removing enough edges to kill off all the odd cycles in a previously nonbipartite component, or by splitting off a bipartite component from the rest of the graph. Not surprisingly, quasi edge cuts and quasibonds will play roles analogous to edge cuts and bonds in the traditional theory. The surprising fact is that the stars and the quasi edge cuts turn out to generate the same vector space. We note that the concepts of quasibond and bond are independent: an edge joining two disjoint copies of K_4, for example, is a bond but not a quasibond, whereas one edge of a K_3 is a quasibond but not a bond.

3. THE MATROID STRUCTURE

In this section we look briefly at what we are doing from the point of view of matroids on the edge set of a graph G. (This approach is somewhat implicit in [10], especially Theorems 8.6 and 8.7.) First we define the *even circuit matroid* of G. The bases for this matroid are the essential spanning subgraphs of G. The circuits in this matroid are the even cycles, as well as the graphs consisting of the vertex-disjoint unions of two odd cycles joined by a path. We call the latter *bow ties*. (We allow the path joining the two odd cycles of a bow tie to have length 0, in which case the cycles share one vertex.) It follows from the discussion above that the rank of the even circuit matroid is $\nu - \omega_0$. Dually, we define the *quasibond matroid*, whose bases are the complements of the essential spanning subgraphs, and whose circuits are the quasibonds. Its rank is necessarily $\varepsilon - \nu + \omega_0$. The following theorem justifies these definitions; see [11].

THEOREM 3.1. *The even circuit matroid of a graph G, and its dual, the quasibond matroid, are indeed matroids.*

Proof. First we need to verify that if S and T are essential spanning subgraphs of G, and x is an edge of $S - T$, then there is an edge y of $T - S$ such that removing x from S and adjoining y creates another essential spanning subgraph of G. If x is in a bipartite component, then this follows from the fact that spanning trees are the bases in the usual cycle matroid of a connected graph. If x is in a nonbipartite component, then regardless of whether x is part of an odd cycle of S or not, removing x from S creates a new tree W, together with possibly some remaining odd unicyclic components. If any edge y of T joins a vertex of W with a vertex not in W, then y is necessarily in $T - S$, and $(S - \{x\}) \cup \{y\}$ is an essential spanning subgraph of G. Otherwise, T contains an odd cycle C using only vertices in W. Imagine the vertices in W to be 2-colored, so that edges in W join vertices of opposite color. Then C necessarily has two adjacent vertices of the same color. Adjoining to W an edge y of C connecting two such vertices creates a unique odd cycle, so again $(S - \{x\}) \cup \{y\}$ is an essential spanning subgraph of G.

It is clear that the minimal dependent sets in the even circuit matroid are the even cycles and bow ties. The only other statements needing proof are that every essential spanning subgraph meets every quasi edge cut, and that if B is a set of edges that meets every essential spanning subgraph, then B is a quasi edge cut. For the first statement, if edges outside an essential spanning subgraph are removed from G, then no bipartite component of G

can become nonconnected, and every other component of what remains has an odd cycle; therefore, these edges do not contain a quasi edge cut. For the second statement, if B is not a quasi edge cut, then let S be an essential spanning subgraph of the graph obtained from G by removing the edges of B. Because the removal of B created no new bipartite components, it is clear that S is also an essential spanning subgraph of G. Thus, B fails to meet some essential spanning subgraph of G. ■

COROLLARY 3.2. *Let G be a graph in which the edges have been assigned nonnegative weights. Then there is an efficient algorithm for finding a minimum-weight essential spanning subgraph S of G.*

Proof. Because the essential spanning subgraphs are the bases in a matroid, the following Kruskal-like greedy algorithm [11] will do the job. Order the edges by weight, from smallest to largest. Start with S consisting of all the vertices of G and no edges, with a trivial 2-coloring of each component of S (each vertex colored red, say). For each edge, in order, add that edge to S if either (case 1) it joins two vertices previously in different components of S, as long as at least one of those components was 2-colored, or (case 2) it connects two identically colored vertices in the same 2-colored component of S (this includes the possibility that the edge is a loop). In case 1, if both of the components of S being joined were previously 2-colored, then 2-color the combined component of the new S (If the new edge joined vertices of opposite colors, then the new component is already 2-colored; otherwise, reverse all the colors in one of the previous components.) If one of the components of S joined by the new edge was not 2-colored, then mark the new component as not 2-colored. In case 2 mark the component of S in which the edge is added as not 2-colored. With appropriate "merge/find" data structures (enhanced to keep track of the coloring and update it efficiently), this algorithm has time complexity in $O((\nu+\varepsilon)\log\nu)$. ■

4. VECTOR SPACES ASSOCIATED WITH UNORIENTED GRAPHS

In this section we adapt the development in Section 12.1 of [2] to ou unoriented context. Analogues of some of these results can be found i [10], with different terminology and in a slightly different setting.

A real-valued function f on E is called a *circulation* if for each vertex the sum of $f(e)$ taken over all edges e incident to v is zero. It is understoo that a loop contributes twice to this sum for its single endpoint. If f and are any two circulations and r is any real number, then it is easy to verify that both $f+g$ and rf are also circulations. Thus, the set of all circulations in G is a vector space (a subspace of the set $\mathbf{R}^{\varepsilon}$ of all real-valued functions on E), which we denote by $\mathcal{C}_0$. (In the conventional, oriented setting, the contribution of $f(e)$ to the tail of e is multiplied by -1. We could replace the ground field of real numbers here by any field whose characteristic is not 2. The reason for this restriction will become apparent shortly. In [10], the coefficients are required to lie in the ring of integers rather than in a field, the functions on the edge set are called 1-chains, and circulations are called cycles.)

There are certain circulations of special interest. These are associated with closed walks in G having even length. For simplicity, we use the word *circuit* in place of *closed walk*. Let C be an even circuit, and let $e_{j_1}, e_{j_2}, \ldots, e_{j_r}$ be a listing of the edges of C in cyclic order. Note that a given edge can appear more than once in this list. We associate with C the function f_C defined by setting $f_C(e)$ equal to the number of appearances of e as e_{j_k} with k odd, minus the number of appearances of e as e_{j_k} with k even. In particular, $f_C(e) = 0$ if e is not in C. If C is an even cycle or a bow tie, then $f_C(e) = \pm 1$ for edges e in the cycle(s) and $f_C(e) = \pm 2$ for edges e (if any) in the path of the bow tie. Clearly, f_C is a circulation, since as we traverse the circuit, a contribution of $1 + (-1)$ occurs at each vertex. (Strictly speaking, f_C is defined only up to sign; whether we get f_C or $-f_C$ depends on where we start listing the edges of the circuit. This fact is irrelevant to our use of f_C, however.) We will see shortly (Theorem 4.4) that each circulation is a linear combination of the circulations associated with even circuits. For this reason we refer to $\mathcal{C}_0$ as the *even-circuit space* of G.

We next turn our attention to a related class of functions. Given a real-valued function p on the vertex set V of G, we define the *unoriented coboundary* δp of p on the edge set E by the rule that, if e is an edge with endpoints x and y, then $\delta p(e) = p(x) + p(y)$. [In particular, if e is a loop at x, then $\delta p(e) = 2p(x)$.] We call any function g on E such that $g = \delta p$ for some function p on V a *potential sum* in G. As with circulations, the set $\mathcal{B}_0$ of all potential sums in G is closed under addition and scalar multiplication, and hence is a vector space, a subspace of the vector space $\mathbf{R}^{\varepsilon}$ of all real-valued functions on E. (Tutte calls his version of $\mathcal{B}_0$ the coboundary-group of G, and he calls integer-valued functions on the vertex set 0-chains.)

As with circulations, there are potential sums of special interest. Analogous to the function f_C associated with each even circuit C, there is a function g_v associated with the star at each vertex v, as well as a function

g_B associated with each quasibond B. The former is given by $g_v(e) = 1$ for e incident with v (2 if e is a loop at v), and $g_v(e) = 0$ for e not incident with v. It is easy to see that $g_v = \delta p$, where p is the characteristic function of $\{v\}$ on V. For the latter, suppose that H is the new bipartite component of G created by the removal of the quasibond B, and assume that its vertices are properly 2-colored red and white. Let p be the function that has the value 1 on the red vertices of H, the value -1 on the white vertices of H, and the value 0 outside of H. Then $g_B = \delta p$. It is not hard to see that $g_B(e) = 0$ for $e \notin B$, but $g_B(e) = \pm 1$ or ± 2 for $e \in B$. As with f_C above, the definition of g_B is ambiguous as to sign (depending on which of the two 2-colorings of H we pick), but again this is of no consequence.

We will see below that each potential sum is a linear combination of potential sums associated with stars (Theorem 4.1), as well as a linear combination of potential sums associated with quasibonds (Theorem 4.5). For this reason, we refer to $\mathcal{B}_0$ as the *star space* of G; alternatively we could just as well call it the *quasibond space*.

In studying the two vector spaces $\mathcal{B}_0$ and $\mathcal{C}_0$ we will find it convenient to regard a function on E as a row or column vector (as appropriate) whose coordinates are labeled with the elements of E; in other words, the function f is identified with the vectors

$$[f(e_1) \quad f(e_2) \quad \cdots \quad f(e_\varepsilon)] \quad \text{and} \quad \begin{bmatrix} f(e_1) \\ f(e_2) \\ \vdots \\ f(e_\varepsilon) \end{bmatrix}.$$

Thus, we may regard the rows of the incidence matrix $\mathbf{M}$ of a graph G as the functions g_v defined above. We will also regard functions on V as vectors.

We can now state the unoriented analogue of the theorem that the cycle space and the star space are orthogonal complements. This theorem also shows that the functions associated with the stars span the star space.

THEOREM 4.1. *If* $\mathbf{M}$ *is the* $\nu \times \varepsilon$ *incidence matrix of a graph* G, *then* $\mathcal{B}_0$ *is the row space of* $\mathbf{M}$, *and* $\mathcal{C}_0$ *is its orthogonal complement in* $\mathbf{R}^\varepsilon$.

Proof. Let $g = \delta p$ be a potential sum in G. Then clearly $g(e) = \sum_{v \in V} p(v) g_v(e)$ for each edge e. Thus, g is a linear combination of the rows of $\mathbf{M}$. Conversely, since each row of $\mathbf{M}$ is a potential sum (namely, g_v), any linear combination of the rows of $\mathbf{M}$ is a potential sum. Hence, $\mathcal{B}_0$ is the row space of $\mathbf{M}$.

Now let f be a function on E. The condition for f to be a circulation can be rewritten as $\sum_{e \in E} g_v(e) f(e) = 0$ for all $v \in V$. This implies that f is a circulation if and only if it is orthogonal to each row of $\mathbf{M}$. Hence, $\mathcal{C}_0$ is the orthogonal complement of $\mathcal{B}_0$. ∎

This duality between $\mathcal{B}_0$ and $\mathcal{C}_0$ is further amplified in the next two lemmas. Recall that the support $\|f\|$ of a function f on E is the set of elements of E at which the value of f is nonzero.

LEMMA 4.2. *If f is a nonzero circulation, then $\|f\|$ contains an even circuit. Furthermore, this even circuit can be taken to be an even cycle or a bow tie.*

Proof. By focusing on just one component of $\|f\|$, we can assume that $\|f\|$ is connected. Since $\|f\|$ cannot contain a vertex of degree one, it must contain a cycle C. By adjoining one new edge of $\|f\|$ at a time, we can extend this cycle to a connected unicyclic subgraph H spanning $\|f\|$. If C has even length, then we are finished, so assume that C is odd. If $\|f\| = H$, then again, since it has no vertex of degree one, it is precisely an odd cycle. But this is impossible: clearly the cycle cannot have length 1, and if the length is greater than 1, then the sign of f must be the same on some pair of adjacent edges, making the sum of the values of f at their common vertex nonzero. Therefore, $\|f\|$ contains at least one edge e in addition to H. If e joins vertices in different components of H with the edges of C removed, then an even cycle is formed with the appropriate "half" of C. On the other hand, if e joins two vertices in the same component of H with the edges of C removed, then a cycle C' is formed in this component. If C' is even, then we are finished. If C' is odd, then we obtain an even circuit (a bow tie) by starting at the endpoint of e closest to C, traversing C', following the (possibly empty) path to C, traversing C, and returning to the starting point along the same path. The length of this circuit is even, because it consists of two odd cycles and a path traversed twice (once in each direction). The second statement follows from this construction. ∎

LEMMA 4.3. *If g is a nonzero potential sum, then $\|g\|$ contains a quasibond.*

Proof. Suppose $g = \delta p$. Since g is nonzero, there is a vertex v and an edge e incident to v such that $g(e) \neq 0$ and $p(v) \neq 0$. Consider the subgraph of G that remains when the edges of $\|g\|$ are removed. In some components of this subgraph, p may be identically 0. In every other component, p must necessarily take on only two values, one positive and the other its negative, and every edge must join a positive-valued vertex with a negative-valued vertex. Thus, each such component of the subgraph is bipartite. On the other hand, when the removed edge e is restored, either it will make a bipartite component nonbipartite (if both v and the other end of e are in the same component of the subgraph), or it will connect a bipartite component to another component. In either case, the removal of $\|g\|$ necessarily created at least one new bipartite component, and so $\|g\|$ is a quasi edge cut and therefore contains a quasibond. ■

Finally, we justify our name for $\mathcal{C}_0$, and our alternative name for $\mathcal{B}_0$, by showing that the functions associated with the even circuits span $\mathcal{C}_0$ and, dually, that the functions associated with the quasibonds span $\mathcal{B}_0$.

THEOREM 4.4. *Let f be a circulation of a graph G. Then f is a linear combination of the circulations f_C associated with the even circuits of G.*

Proof. If not, let f be a circulation that is not a linear combination of the f_C's, with support as small as possible. Then $f \neq 0$, and by Lemma 4.2 and the remarks made when defining f_C, $\|f\|$ contains an even circuit C such that $f_C(e) = \pm 1$ for some edge e of C. Let α be the coefficient of e in f. Then $f \pm \alpha f_C$ (with the sign chosen so as to make this function vanish on e) has support smaller than that of f. By the choice of f, this circulation is a linear combination of circulations associated with even circuits, and thus so is f, a contradiction. ■

THEOREM 4.5. *Let g be a potential sum of a graph G. Then g is a linear combination of the potential sums g_B associated with the quasibonds of G.*

Proof. If not, let g be a potential sum that is not a linear combination of the g_B's, with support as small as possible. Then $g \neq 0$, and by Lemma 4.3 and the remarks made when defining g_B, $\|g\|$ contains a quasibond B such that $g_B(e) = \pm 1$ or ± 2 for some edge e of C. Let α be the coefficient of e in g. Then $g + t\alpha f_C$ (with t chosen to be ± 1 or $\pm \frac{1}{2}$ so as to make this function vanish on e) has support smaller than that of g. By the choice of g, this potential sum is a linear combination of potential sums associated with quasibonds, and thus so is g, a contradiction. ■

Note that the proof of Theorem 4.5 requires that the ground field have characteristic different from 2. The proof of Theorem 4.4, however, does not.

5. PROPERTIES OF THE UNORIENTED INCIDENCE AND LAPLACIAN MATRICES

We begin with an analysis of the incidence matrix $\mathbf{M}$ of a graph G. When convenient to do so, we will think of $\mathbf{M}$ as a linear transformation from $\mathbf{R}^{\varepsilon}$, the vector space of all real-valued functions on $E(G)$, to $\mathbf{R}^{\nu}$, the vector space of all real-valued functions on $V(G)$. Thus, the row space and the kernel of $\mathbf{M}$, as well as the range (column space) of $\mathbf{M}^t$, are subspaces of $\mathbf{R}^{\varepsilon}$, while the kernel of $\mathbf{M}^t$ and the range of $\mathbf{M}$ are subspaces of $\mathbf{R}^{\nu}$.

We saw in Theorem 4.1 above that the row space of $\mathbf{M}$ is $\mathcal{B}_0$, whereas the kernel of $\mathbf{M}$ is $\mathcal{C}_0$. We will compute the dimensions of these two spaces and find explicit bases for them. We begin by calculating the determinants of submatrices of the incidence matrix [4, Theorem 2.2].

THEOREM 5.1. *Let* $\mathbf{N}$ *be the incidence matrix of a substructure* R, *containing an equal number of vertices and edges. If* R *does not satisfy the condition that every component has an equal number of edges and vertices, then* $\det \mathbf{N} = 0$. *If this condition is satisfied, then every component of* R *is a unicyclic graph or a rootless tree. If any of the cycles in the unicyclic components are even, then* $\det \mathbf{N} = 0$; *otherwise*, $\det \mathbf{N} = \pm 2^{\omega_1(R)}$.

Proof. The first claim follows from the expansion of $\det \mathbf{N}$ using Laplace development [6]. Thus, we assume that the stated condition holds, so that $\mathbf{N}$ can be put in block-diagonal form with square blocks, corresponding to the component of R. It is easy to see that each components of R either is a rootless tree or consists of a cycle with (possibly trivial) rooted trees growing out of the vertices on the cycle. We compute the determinant of the incidence matrix of each component by first repeatedly expanding along the rows corresponding to vertices of degree 1, until either nothing remains or all that remains is the incidence matrix for a cycle. In the former case, the determinant is ± 1. The determinant in the latter case is easily seen to be ± 2 if the cycle is odd and 0 if it is even. Since the determinant of $\mathbf{N}$ is equal to the product of the determinants of the submatrices corresponding to the components of R, the final sentence of the theorem follows. ∎

Applying this result, we obtain [3, p. 114]:

THEOREM 5.2. *The rank of the incidence matrix* $\mathbf{M}$ *of a graph* G *equals* $\nu - \omega_0$.

Proof. Because of the block structure of $\mathbf{M}$ induced by the components of G, it suffices to prove this result for connected graphs. That is, we need to show that the rank of $\mathbf{M}$ is $\nu - 1$ when G is bipartite and is ν otherwise. In the former case, let the vertices of G be 2-colored red and white, and consider the sum of the rows of $\mathbf{M}$ corresponding to the red vertices, minus the sum of the rows of $\mathbf{M}$ corresponding to the white vertices. Since each column of $\mathbf{M}$ has two 1's, one in a row corresponding to a red vertex and one in a row corresponding to a white vertex, this linear combination of the rows is the zero row vector. Hence, the rank of $\mathbf{M}$ is less than ν. On the other hand, the $(\nu - 1) \times (\nu - 1)$ square submatrix of $\mathbf{M}$ corresponding to a rootless spanning tree of G (obtained by taking a rooted spanning tree of G and removing its root) is nonsingular by Theorem 5.1. Therefore, the rank of $\mathbf{M}$ is at least $\nu - 1$, completing the proof in the bipartite case.

If G is a connected nonbipartite graph, consider the $\nu \times \nu$ submatrix of $\mathbf{M}$ corresponding to a connected essential spanning subgraph S of G, which is necessarily odd unicyclic. By Theorem 5.1, the determinant of this submatrix is ± 2. Therefore, $\mathbf{M}$ has rank ν. ■

COROLLARY 5.3. *The dimension of the range of* $\mathbf{M}$ *is* $\nu - \omega_0$, *and the dimension of the kernel of* $\mathbf{M}$ *is* $\varepsilon - \nu + \omega_0$.

COROLLARY 5.4. *The dimension of the star space* $\mathcal{B}_0$ *is* $\nu - \omega_0$, *and the dimension of the even-circuit space* $\mathcal{C}_0$ *is* $\varepsilon - \nu + \omega_0$.

Next we compute explicit bases for the kernel and range of $\mathbf{M}$. To this end, let S be a fixed essential spanning subgraph of G. For each edge e of G not in S (recall that there are $\varepsilon - \nu + \omega_0$ such edges), the graph obtained by adjoining e to S contains an even cycle or a bow tie containing e (Theorem 3.1); denote this even circuit by $C(e)$. Then $f_{C(e)}$ is an element of $\mathcal{C}_0$. Furthermore, since $e \in \|f_{C(e)}\|$ but $e \notin \|f_{C(e')}\|$ for any other e' in S, the $\varepsilon - \nu + \omega_0$ circulations $f_{C(e)}$, for e in S, are linearly independent. Since the cardinality of this set of circulations equals the nullity of $\mathbf{M}$, it must form a basis for the kernel of $\mathbf{M}$. (The analogous basis in the oriented setting is the usual fundamental cycle basis.)

As for the range, for each edge e in S, let $\chi_e \in \mathbf{R}^\varepsilon$ be the characteristic function for e, i.e., the column vector whose coordinates are all 0 except for a 1 in the row corresponding to e. We claim that the set $\{\mathbf{M}\chi_e \mid e \in S\}$ forms a basis for the range of $\mathbf{M}$. By Corollary 5.3, its cardinality is correct, so it suffices to show that its elements are linearly independent. Suppose to the contrary that there is some nontrivial relation $\sum \alpha_k \mathbf{M}\chi_{e_{i_k}} = \mathbf{0}$, with each $e_{i_k} \in S$ and each $\alpha_k \neq 0$. Then the edges of S involved in this sum form a disjoint union of cycles, because any vertex of degree 1 in the subgraph induced by these edges would have a nonzero coefficient in the sum. Pick one such cycle, which is necessarily odd. If it is a loop, then there will be a nonzero coefficient on its endpoint. Otherwise, the coefficients α_k corresponding to some pair of adjacent edges of the cycle have the same sign, giving a nonzero coefficient to their common vertex. In either case, we have a contradiction.

Summarizing, we have proved:

THEOREM 5.5. *Let S be any essential spanning subgraph of a graph G. Then the edges of $G - S$ induce a basis for the kernel of* $\mathbf{M}(G)$, *i.e., a basis for $\mathcal{C}_0$, consisting of certain even cycles and bow ties. The edges of S induce a basis for the range of* $\mathbf{M}(G)$, *consisting of certain pairs of adjacent vertices.*

Before moving on, let us find explicit bases for the star space $\mathcal{B}_0$. Theorems 4.1 and 4.5 guarantee that the stars and the quasibonds generate $\mathcal{B}_0$. We need to choose a linearly independent set of stars and a linearly independent set of quasibonds that do the same. Assume for the moment that G is connected, and let S be an essential spanning subgraph of G. If G is not bipartite, then the dimension of the star space is ν, so the set of all ν stars (i.e., the rows of $\mathbf{M}$) forms a basis. Alternatively, the removal from S of any edge $e \in S$ creates a new bipartite component of S; therefore, the removal from G of e and all edges not in S creates a new bipartite component of G. This set of edges contains a quasibond (necessarily including e). The set of these ν quasibonds is linearly independent and therefore forms a basis for $\mathcal{B}_0$. The situation is similar in the bipartite case. Here, the set of any $\nu - 1$ stars forms a basis for $\mathcal{B}_0$. (The set of all ν stars has the nontrivial relation induced by a 2-coloring of the vertices of G, in which the coefficient of each star is ± 1.) Again, each of the $\nu - 1$ edges in S induces a quasibond, and the set of these quasibonds forms a basis for $\mathcal{B}_0$. If G is not connected, then we just take the union of the basis vectors corresponding to each component of G.

Further information about $\mathbf{M}$, such as the possible values for its minors, can be found in [4]. Let us look briefly at $\mathbf{M}^t$. Its rank, of course, is the same as that of $\mathbf{M}$, namely $\nu - \omega_0$. Its range, i.e., its column space, is the row space of $\mathbf{M}$, namely the star space $\mathcal{B}_0$, having dimension $\nu - \omega_0$. Its kernel therefore has dimension $\nu - (\nu - \omega_0) = \omega_0$. To see what a basis for the kernel is, fix a red-white 2-coloring of the vertices in each bipartite component. For each such component, form the *2-coloring vector* **t** in $\mathbf{R}^\nu$ whose entries for vertices in that component are 1 or -1 according as the vertex is colored red or white, and whose entries for vertices outside that component are 0. It is not hard to see that the set of these vectors forms a basis for the kernel of $\mathbf{M}^t$.

Next we turn to the unoriented Laplacian matrix, $\mathbf{C} = \mathbf{MM}^t$. Recall from Section 2 that its off-diagonal entries are the same as the off-diagonal entries of the adjacency matrix of G, namely, $c_{ii'}$ equals the number of edges joining vertex v_i and vertex $v_{i'}$; and the diagonal entry c_{ii} is the number of edges incident to v_i, with, however, each loop contributing 4 to this count. We first note that the rank of $\mathbf{C}$ is the same as the rank of $\mathbf{M}$, namely $\nu - \omega_0$. This follows immediately from Theorem 2.3.4 in [9]. Similarly, the rank of $\mathbf{M}^t\mathbf{M}$ is also $\nu - \omega_0$. Note that $\mathbf{M}^t\mathbf{M}$ is closely related to the adjacency matrix of the line graph of G (see [7] for more on this in the classical case).

To gain a better understanding of $\mathbf{C}$, let us see what its kernel looks like. Assume for the moment that G is connected. If G is not bipartite, then the rank of $\mathbf{C}$ is ν, so the kernel is trivial. If G is bipartite, then the rank of $\mathbf{C}$ is $\nu - 1$, so its kernel is 1-dimensional; and the 2-coloring vector **t** defined above spans the kernel of $\mathbf{C}$, since $\mathbf{Ct} = \mathbf{MM}^t\mathbf{t} = 0$. For general G, the kernel of $\mathbf{C}$ has dimension ω_0, and the set of 2-coloring vectors (one for each bipartite component of G) forms a basis for it.

Before stating our main theorem, we must review briefly (see [5] or [9]) the concept of *compounds* of a matrix. If $\mathbf{A}$ is an $m \times n$ matrix, then the rth compound of $\mathbf{A}$, denoted $C_r(\mathbf{A})$, is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose ijth entry is the determinant of the matrix obtained from $\mathbf{A}$ by using the rows in the ith r-subset of the set of all rows of $\mathbf{A}$ and the columns in the jth r-subset of the set of all columns of $\mathbf{A}$. In particular, if $m = n$, then the nth compound is just det $\mathbf{A}$, and the $(n-1)$th compound is, except for the sign of some of the off-diagonal entries, the same as the adjoint of $\mathbf{A}$. It is convenient to think of the determinant of the empty (0×0) matrix as 1. The multiplicative property of compounds is known as the Cauchy–Binet theorem: if $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix, then $C_r(\mathbf{AB}) = C_r(\mathbf{A})C_r(\mathbf{B})$.

In its generality, our main theorem shows how the trace of the $(\nu - k)$th compound of $\mathbf{C}$ counts the k-reduced spanning substructures of G. Special cases will enable us to count the essential spanning subgraphs of G. We also remark that the quantities calculated in this theorem are the coefficients of the characteristic polynomial of $\mathbf{C}$, and therefore can provide information on its eigenvalues.

THEOREM 5.6. *Let G be a graph and k a nonnegative integer not exceeding $\nu(G)$. Then*

$$\operatorname{tr} C_{\nu-k}(\mathbf{C}) = \sum_R 4^{\omega_1(R)},$$

where the sum is taken over all k-reduced spanning substructures R of G.

Proof. By definition of compound, the left-hand side of this equation is the sum, over all choices of $\nu - k$ vertices of G, of the determinant of the matrix obtained from $\mathbf{C}$ by selecting the rows (and columns) corresponding to these vertices. By the Cauchy–Binet theorem, since $\mathbf{C} = \mathbf{M}\mathbf{M}^t$, each such determinant is the sum of the squares of the determinants of the square submatrices of $\mathbf{M}$ obtained by selecting $\nu - k$ edges of G. By Theorem 5.1, the only nonzero contributions to this sum come from substructures each component of which is a rootless tree or an odd unicyclic graph, and the contribution is clearly a factor 2^2 for each odd cycle in the substructure. ■

If we take $k = \omega_0(G)$ in this theorem, then by Theorem 2.1, the k-reduced spanning substructures are really just the essential spanning subgraphs, with a distinguished vertex chosen for deletion in each bipartite component. Therefore, the sum on the right-hand side of the equation in Theorem 5.6 is the same as the sum of $4^{\omega_1(S)}$, taken over all *rooted* essential spanning subgraphs S of G, where the rooting consists of selecting one vertex in each tree of S. (Intuitively, there is no need to "select a root" in the nonbipartite components, because the unique odd cycle in each such component serves as the root.) Thus, we have a counting formula for essential spanning subgraphs:

COROLLARY 5.7. *For any graph G,*

$$\operatorname{tr} C_{\nu-\omega_0}(\mathbf{C}) = \sum_S r(S) 4^{\omega_1(S)},$$

where the sum is taken over all essential spanning subgraphs S of G, and $r(S)$ is the product of the numbers of vertices in the bipartite (tree) components of S.

Another way to look at counting the essential spanning subgraphs of G is to treat the bipartite and nonbipartite components separately, since the number of essential spanning subgraphs is just the product of the numbers of essential spanning subgraphs for the components. The traditional matrix tree theorem for the case of bipartite graphs (which we obtain in this unoriented setting as Corollary 5.9 below) tells us about the number of essential spanning subgraphs in the bipartite components. Specializing Corollary 5.7 to the case of $\omega_0 = 0$ allows us to count the essential spanning subgraphs in the nonbipartite components:

COROLLARY 5.8. *If G has no bipartite components, then*

$$\det \mathbf{C} = \sum_S 4^{\omega(S)},$$

where the sum is taken over all essential spanning subgraphs of G.

Proof. Under the hypothesis, the essential spanning subgraphs of G are the same as the 0-reduced spanning substructures, so we take $k = 0$ in Theorem 5.6. The νth compound of $\mathbf{C}$ is then just its determinant. Furthermore, all the components of any essential spanning subgraph S are nonbipartite, so $\omega_1(S) = \omega(S)$. ∎

As an example of Corollary 5.8, let $G = K_6$. Then

$$\mathbf{C} = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 \end{bmatrix},$$

and $\det \mathbf{C} = 10{,}240$. [In fact, one can easily compute that $\det \mathbf{C} = 2(n-1)(n-2)^{n-1}$ when $G = K_n$.] It is not hard to count that G has 360 essential spanning subgraphs containing a pentagon, 2160 containing one triangle, and 10 containing two triangles. Hence, the desired sum is $(360 + 2160) \times 4 + 10 \times 4^2 = 10{,}240$.

COROLLARY 5.9. *If G is a connected bipartite graph, then each diagonal entry of* adj $\mathbf{C}$ *equals the number of spanning trees of G.*

Proof. We claim that adj $\mathbf{C}$ is constant up to sign. Indeed, since $\mathbf{C}$ adj $\mathbf{C} = (\det \mathbf{C})\mathbf{I}_\nu$ is the zero matrix, each column of adj $\mathbf{C}$ must be in the kernel of $\mathbf{C}$. Therefore, each column is a multiple of the 2-coloring vector $\mathbf{t}$, all of whose entries are ± 1. But since adj $\mathbf{C}$ is symmetric, we must always have the same multiple, up to sign. Furthermore, using the Cauchy–Binet theorem to compute the determinant of $\mathbf{C}$ with the ith row and ith column deleted, we see that each diagonal entry of adj $\mathbf{C}$ must be positive. Therefore, all the diagonal entries of adj **C** have the same value. Thus, the trace of adj **C**, which is the same as tr $C_{\nu-1}(\mathbf{C})$, is just ν times this common value. The result now follows from Corollary 5.7, since $r(S) = \nu$ and $\omega_1(S) = 0$ for the essential spanning subgraphs—i.e., spanning trees—of the graphs under consideration. ■

Note added in proof. Two additional references should be noted: S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, *SIAM J. Algebraic Discrete Methods* 3:319–329 (1982); and T. Zaslavsky, Signed graphs, *Discrete Appl. Math.* 4:47–74 (1982).

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Received 29 June 1993; final manuscript accepted 9 December 1993