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An Enumeration of Simplicial 4-Polytopes with 8 Vertices

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Abstract

An enumeration of all the different combinatorial types of 4-dimensional simplicial convex polytopes with 8 vertices is given. It corrects an earlier enumeration attempt by M. Brückner, and leads to a simple example of a diagram which is not a Schlegel diagram.

1. Introduction

Let P denote a *d-polytope*, that is, a d-dimensional convex polytope. Each $(d-1)$-dimensional face of P will be called a *facet* of P. The d-polytope P is called *simplicial* provided all its facets (and hence all its proper faces) are simplices. P is called *simple* if its dual P^* is simplicial; equivalently, P is simple if each vertex of P belongs to precisely d different facets. Two d-polytopes P and P' are of the same *combinatorial type* provided there exists a one-to-one inclusion-preserving correspondence between the set of all faces of P and the set of all faces of P'.

In 1909, Brückner published a paper [1] the main aim of which was the enumeration of all the different combinatorial types of simple 4-polytopes with 8 facets (or, by duality, of all simplicial 4-polytopes with 8 vertices). Brückner's method consists of considering the Schlegel diagrams[1] of representatives of all the combinatorial types of simple 4-poly-

[1] The terminology used here follows [5]; for the reader's convenience, most definitions and results used will be cited here.

topes with 7 facets and introducing an eighth 3-polytope into the Schlegel diagram by "cutting off" parts of the seven 3-polytopes present.

As far as we know, Brückner's enumeration has not been seriously questioned so far despite the fact that the objects obtained by Brückner are, at best, diagrams (and not necessarily Schlegel diagrams). However, it has been known for some time (see [4]) that not every diagram is a Schlegel diagram. Hence the validity of Brückner's enumeration was in doubt and we decided to check it. (As a matter of fact, there was also a suspicion that Brückner's work is incomplete, since a number of possible ways of "cutting off" the eighth 3-polytope are not discussed in [1].)

Our first step was to utilize the theory of Gale diagrams developed recently by M. A. Perles (see [5], Section 5.4), and to obtain representatives of many (hopefully all) combinatorial types of simplicial 4-polytopes with 8 vertices by letting the Michigan State University CDC 3600 computer pick (at random, or with some constraints) 8 points on the unit sphere in R^3 which were then interpreted as Gale transforms of the vertices. After about 2000 runs, the computer found (Gale diagrams of representatives of) 37 combinatorial types, out of the 39 types listed by Brückner,[2] and no type which was not in Brückner's list. The missing types, in Brückner's notation, were P_8^{20} and P_8^{39}.

A closer check of these two types revealed the following situation:

Regarding P_8^{20}, a number of errors were compounded by Brückner. The description of P_8^{20} in the table on page 27 of [1] would imply that the number of incidences of triangles with 3-faces of P_8^{20} is odd, which is impossible. The construction of P_8^{20} [1, p. 20] shows that in the table the two "1"'s should be replaced by "0" and "2," the description of P_8^{20} in terms of its facets then coinciding with that of P_8^{23}. A closer check shows that, as described by Brückner, P_8^{20} and P_8^{23} are indeed combinatorially equivalent (there exists a combinatorial automorphism of P_7^{1} carrying the edge CK onto edge the GL). But there exists another edge (e.g., MN) which is not equivalent to the edges considered by Brückner; however, the "cutting off" of this edge leads also to a type obtained previously (P_8^{19}). Hence, one of the types listed by Brückner is superfluous.

Regarding Brückner's P_8^{39}, the situation is much more interesting. The simplicial 3-complex which we shall denote by $\mathscr{M}$, and which is dual to the 3-complex associated with Bruckner's P_8^{39}, is realizable by a diagram in R^3 but is not combinatorially equivalent to the boundary complex of any 4-polytope. Hence Brückner's diagram P_8^{39} is not a Schlegel diagram and there exists no 4-polytope corresponding to it. Even more interesting is the property of $\mathscr{M}$ to be representable by a 3-diagram only if certain of its simplices are chosen as the "basis" of the diagram, but not if some others are chosen. Moreover, we were unable to represent the dual complex $\mathscr{M}^*$ by a 3-diagram; in other words, we could not corroborate even Brückner's assertion that the diagram P_8^{39} exists (although there is no doubt about its realizability if one allows curved "faces" for the 3-diagram). We believe that $\mathscr{M}^*$ is not representable by a 3-diagram, at least with certain of its facets as "basis" of the diagram.

Having thus established that, out of Brückner's list of 39, there exists 37 types of polytopes, we completed the rechecking by independently deriving all combinatorial types of simplicial 4-polytopes with 8 vertices. Our method was different from Brückner's; we used the "beyond-beneath" technique ([5], Section 5.2), which deals directly with 4-polytopes and automatically guarantees (in most cases) the existence of polytopes belonging to the types found.

[2] In Brückner's table [1, p. 27] there is a typographic error; the first "1" in the description of P_8^{31} and the first "3" in the description of P_8^{32} should be interchanged.

2. Statements of Results

The results of the present paper may be formulated as follows:

THEOREM 1. *The number $c_s(8,4)$ of different combinatorial types of simplicial 4-polytopes with 8 vertices (or of simple 4-polytopes with 8 facets) is 37.*

This corrects the value $c_s(8,4)$ claimed by Brückner [1]; a description of the combinatorial types is given in Tables 4 and 5. Our polytopes P_i^8 are dual to Bruckner's P_8^i for $1 \leq i \leq 22$, while our P_i^8 is dual to Brückner's P_8^{i+1} for $23 \leq i \leq 37$.

THEOREM 2. *There exists a simplicial 3-complex $\mathscr{M}$ with 8 vertices and 20 3-cells, homeomorphic to the 3-sphere and representable by a 3-diagram in Euclidean 3-space R^3, which is not combinatorially equivalent to the boundary complex of any 4-polytope.*

This simplifies the example mentioned in ([5], Section 11.5) and referred to in [4].

THEOREM 3. *There exists a simplicial 3-complex $\mathscr{M}$ with 8 vertices and 20 3-cells, homeomorphic to the 3 sphere, which is representable by a 3-diagram if certain of its 3-cells are chosen as basis for the 3-diagram, but is not representable in such a fashion with some other of its 3-cells serving as basis.*

This 3-complex may be used to construct a simplicial topological subdivision of a 3-simplex into 19 topological 3-simplices, which is not representable by a 3-diagram. More complicated examples to the same effect have been given by Cairns [2] and van Kampen [6].

The complete enumeration of simplicial 4-polytopes with 8 vertices yields on inspection

THEOREM 4. *The "lower bound conjecture" is true for simplicial 4-polytopes with 8 vertices.*

The only cases in which the truth of this conjecture was previously known are (i) simplicial polytopes of dimension at most 3; and (ii) simplicial d-polytopes with at most $d + 3$ vertices [5, Section 10.2]).

Another consequence of the existence of the polytopes P^8_{36} and P^8_{37}, elaborated in [5, Section 7.2], is a refutation of Motzkin's [7] conjecture (see also [3]) that cyclic 4-polytopes are the only neighborly 4-polytopes.

3. PROOF OF THEOREM 1

For the reader's convenience we begin by reformulating some of the definition and results of [5], Section 5.2].

Let Q be a d-polytope in the Euclidean d-space R^d, let V be a point of R^d not belonging to Q, and let H be a $(d-1)$-dimensional hyperplane such that $V \notin H$ and $H \cap \operatorname{int} Q = \emptyset$. We shall say that V is *beneath* H, or *beyond* H (with respect to Q) provided V belongs to the open half-space determined by H which contains int Q, or does not meet Q, respectively. If F is a facet, i.e., $(d-1)$-face, of Q, we shall say that V is beneath or beyond F provided V is beneath or beyond the hyperplane H determined by F.

Let Q be a d-polytope in R^d, and let $V \in R^d$ be a point not in Q and not belonging to any of the hyperplanes determined by the facets of Q. Let $P = \operatorname{conv}(Q \cup \{V\})$. The connection between the facial structures of Q and P is given by the following criterion [5, Theorem 5.2.1]:

The facets of P which do not contain V are precisely the facets of Q

beneath which V is. The facets of P which contain V are precisely the sets of the form $\operatorname{conv}(\{V\} \cup G)$, where G is a $(d-2)$-face of Q such that V is beneath one of the facets of Q containing G, and beyond the other such facet.

Therefore the combinatorial structure of P is completely determined by the combinatorial type of Q and the $(d-1)$-complex $\mathscr{C}$ consisting of those facets of Q beyond which V is, and their faces. For example, let Q be the polytope P_1^7 (Table 1) and let $\mathscr{C}$ consist of the facet A: 1256 (i.e., A is the convex hull of the vertices 1, 2, 5, and 6 of P_1^7) and its faces, and let the new vertex V be denoted by 8. Then the resulting polytope (denoted by P_1^8 in Table 4) has as facets all the facets of P_1^7 different from A, as well as 1258, 1268, 1568, and 2568 (which are the convex hulls of 8 with each of the triangles 125, 126, 156, 256).

The procedure for obtaining all the simplicial 4-polytopes with 8 vertices (called P^8's in the sequel) from those with 7 vertices (called P^7's) may therefore be described as follows: For Q, one of the five different combinatorial types (see Table 1 and Figures 1 and 2) of P^7's is chosen; on its boundary all the possible complexes $\mathscr{C}$ are chosen in turn, and the combinatorial type of the resulting P^8 determined. Since the most time-consuming part is the elimination of combinatorially

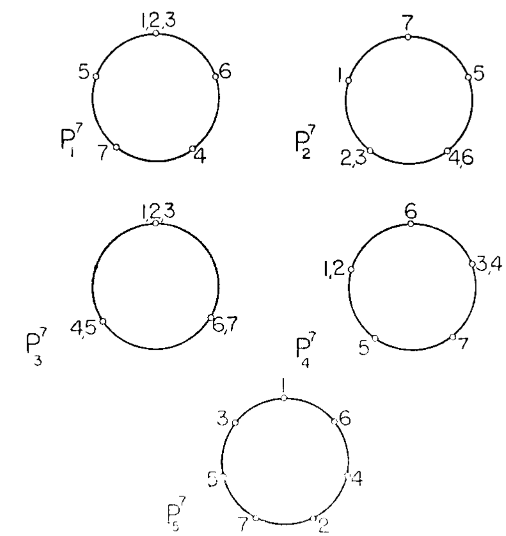


FIG. 1. Gale diagrams of P_i^7, $i = 1, 2, 3, 4, 5$.

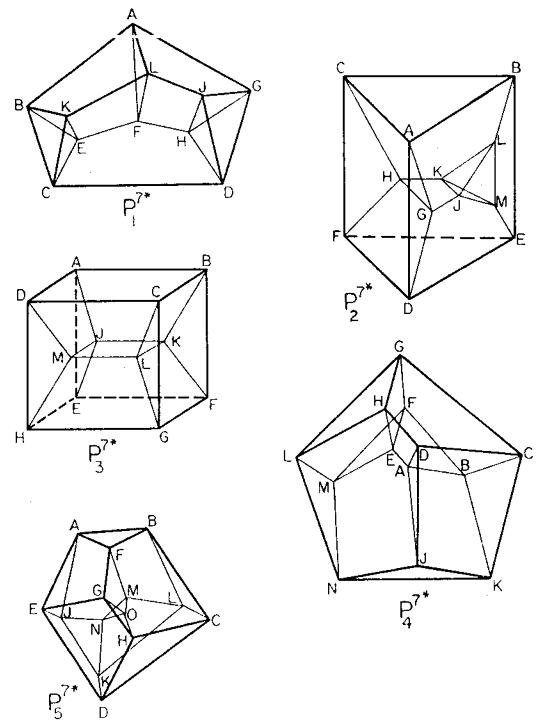


FIG. 2. Schlegel diagrams of P_i^{7*}.

equivalent P^8's obtained, it is convenient to adjust the choices of Q and $\mathscr{C}$ carefully.

First, we note that the valence of the new vertex 8 of P^8 (i.e., the number of edges incident to it) equals the number of vertices of $\mathscr{C}$. Therefore, if one first determines (by appropriate choice of Q and $\mathscr{C}$) all the P^8's having a vertex of valence at most k, in order to determine all the P^8's which have all vertices of valence at least $k + 1$ one has to consider only $\mathscr{C}$'s having at least $k + 1$ vertices, and Q may be restricted to those P^7's which either have no vertex of valence $\leq k$, or the vertices of valence k present will acquire an additional edge (to the vertex 8) because they belong to $\mathscr{C}$. An additional, easily exploitable, reduction in the number of cases to be considered arises from the observation that certain choices of $\mathscr{C}$ (denoted by $\mathscr{B}_2$, $\mathscr{C}_2$, $\mathscr{C}_3$ in Table 2 and in Figure 4) eliminate one of the edges of the P^7 involved.

Hence we have arranged (in Table 3, where the generation of the P^8's is given in detail) the P^8's according to the minimal valence of their vertices: first we generate all those which have a 4-valent vertex, then those having a vertex of valence 5, etc.

Second, if for a given P^7 two complexes $\mathscr{C}$ and $\mathscr{C}'$ are such that there exists a combinatorial automorphism of P^7 mapping $\mathscr{C}$ onto $\mathscr{C}'$, the resulting P^8's will clearly be of the same combinatorial type. The elimination of pairs of complexes $\mathscr{C}$, $\mathscr{C}'$ equivalent in this sense is easy using the Gale diagrams of the P^7's since [5, Section 6.3] two sets of vertices of P^7 are combinatorially equivalent if and only if the corresponding sets of points in the Gale diagram are equivalent (including multiplicities) under an orthogonal transformation of the Gale diagram.

In Table 3, only one representative is chosen for each class of $\mathscr{C}$'s equivalent under an automorphism of the P^7 considered.

Third, it is easy to determine all the complexes $\mathscr{C}$ we need to consider. Indeed, in any P^8 the vertex figure S of the vertex 8 (i.e., the simplicial 3-polytope obtained by intersecting P^8 with a hyperplane strictly separating 8 from the other vertices of P^8) has a simplicial decomposition which

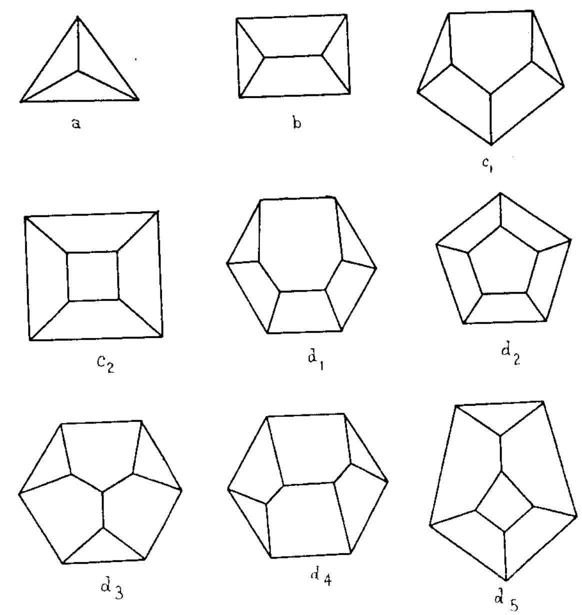


FIG. 3. Schlegel diagrams of simple 3-polytopes with at most 7 facets.

is combinatorially equivalent to the 3-complex $\mathscr{C}$. Since the number of vertices of S equals the valence of 8, only simplicial 3-polytopes with at most 7 vertices need to be considered[3] and, for each of them, those simplicial decompositions which do not introduce additional vertices. Moreover, for valence 7, only decompositions which contain no interior edges are interesting (because an interior edge would not be an edge of the resulting P^8, and therefore this P^8 would have a vertex of valence at most 6).

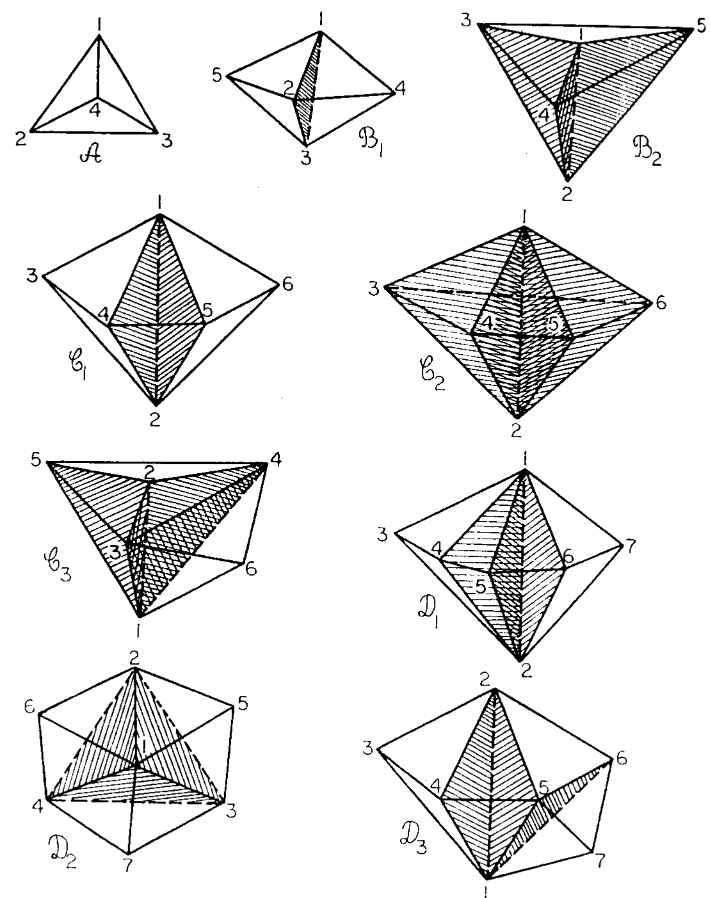


FIG. 4. The complexes $\mathscr{C}$.

[3] Schlegel diagrams of the duals of these 3-polytopes are shown in Figure 3.

Taking all the above into account, we can easily check that only 9 complexes $\mathscr{C}$ have to be considered; they are listed in Table 2, and the corresponding decompositions of S are indicated in Figure 4. In Table 3 we list all the combinations of P^7 and $\mathscr{C}$ needed, together with the resulting P^8. A detailed description of each of the P^8's is given in Table 4, while Table 5 contains a more compact listing of the P^8's.

One observation has to be borne in mind, however, in constructing the P^8's from the P^7's. Given P^8 and its vertex 8, the corresponding P^7 (i.e., the convex hull of the other seven vertices of P^8, which may, without loss of generality, be assumed in general position) and $\mathscr{C}$ are determined. However, with a given P^7 and a complex $\mathscr{C}$ on its boundary, there is, in principle, no guarantee for the existence of a point 8 that is beyond precisely those facets of P^7 which are in $\mathscr{C}$. For a given combinatorial type of P^7, and a fixed $\mathscr{C}$ on its boundary, such a point 8 may exist, or may fail to exist, depending on the particular polytope of type P^7 chosen. Hence, strictly speaking, what we have constructed so far are not 4-polytopes with 8 vertices but certain combinatorial schemes, or 3-complexes, which may, or may not, be the boundary complexes of 4-polytopes.

The greatest part of this question is easy to resolve in the particular circumstances which interest us here. As a matter of fact, it is very easy to see that, if the complex $\mathscr{C}$ is the star of some face G of Q in the boundary complex of Q, then, taking as the new vertex V any point not in Q but sufficiently near to it and belonging to a line passing through the relative interiors of G and of Q, V will precisely beyond the facets of Q which are in $\mathscr{C}$. Among the complexes $\mathscr{C}$ which interest us here, $\mathscr{A}$, $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{C}_2$ are of this type; hence the P^8's obtained by using those complexes clearly exist. It is also not hard to see (compare [5, Section 7.2]) that, if $\mathscr{C}$ consists of a number of facets of P^7 (and their faces), such that the facets have a common edge and form a chain in which neighboring members have a triangle in common, the existence or V is guaranteed, and hence the P^8's obtained are indeed 4-polytopes. Among the complexes $\mathscr{C}$ that interest us here, $\mathscr{C}_1$ and $\mathscr{D}_1$ are of this type.

Hence the only constructions which require a closer inspection are those involving $\mathscr{C}_3$, $\mathscr{D}_2$, or $\mathscr{D}_3$. However, as seen from Table 3, except for the very last case, all the P^8's obtained by the use of any of these complexes are combinatorially equivalent to some previously obtained by a construction of the former types, and therefore their existence is assured. This leaves us with only doubtful case, called "complex $\mathscr{M}$" in

Table 3, and, as we shall now see, $\mathscr{M}$ is indeed not combinatorially equivalent to the boundary complex of any 4-polytope. In other words, no representative of P_5^7 has its facets in such a position that there exists a point V beyond precisely the facets of $\mathscr{D}_3$.

Assume that P is a 4-polytope with boundary complex combinatorially equivalent to $\mathscr{M}$; let 1, 2, ..., 8 be the vertices of P labeled correspondingly to the labeling of the vertices of $\mathscr{M}$. Since P is a simplicial polytope, there is no loss of generality in assuming that its vertices are in general position. Let Q denote the convex hull of the seven vertices of P different from 6. Though we cannot claim at once a complete knowledge of the facial structure of Q, we know that it is one of the P^7's, and also that the faces of P which do not involve 6 are faces of Q. Hence Q is a neighborly 4-polytope (each pair of its vertices determines an edge), and thus Q is P_5^7. Moreover, each of the edges 14, 24, 34, 54, 74, 84 of P is incident, in P, to four 2-faces (triangles) of P not involving the vertex 6. Since all those edges and 2-faces of P are also edges and 2-faces of Q, it follows that in Q each edge incident to the vertex 4 will be contained in at least four 2-faces of Q. But this is impossible since in Q, which is P_5^7, for each vertex there are two edges incident to it which are contained in only three 2-faces of P_5^7. Hence there exists no 4-polytope P with boundary complex equivalent to $\mathscr{M}$.

This completes the proof of Theorem 1.

4. Proofs of Theorems 2 and 3

A d-*complex* $\mathscr{C}$ in R^n is a set of (convex) polytopes of maximal dimension d, with the properties:

(i) each face of a member of $\mathscr{C}$ is itself in $\mathscr{C}$;

(ii) the intersection of any two members of $\mathscr{C}$ is a common face of both.

A d-*diagram* $\mathscr{D}$ consists of a d-polytope D (the *basis* of $\mathscr{D}$) and a d-complex $\mathscr{C}_0$ such that D is the union of all the members of $\mathscr{C}_0$ and, for every $C \in \mathscr{C}_0$, the intersection of C with $bd\ D$ is a member of $\mathscr{C}_0$.

We shall say that a d-complex $\mathscr{C}$ is *representable* by a d-diagram $\mathscr{D}$, if the basis D of $\mathscr{D}$ is a member of $\mathscr{C}$, and if $\mathscr{C}_0$ is combinatorially equivalent to $\mathscr{C} \sim \{D\}$.

Clearly, a Schlegel diagram of a $(d+1)$-polytope P is a d-diagram which represents the boundary complex of P. In this case, each facet of P may serve as basis of the d-diagram.

The proof of Theorem 2 is easy, by indicating the coordinates (in R^3) of a 3-diagram representing the 3-complex $\mathscr{M}$, with basis 4567. A set of such coordinates, given in Table 6, was obtained by actually constructing a model, reading off the coordinates of its vertices, and checking them on a computer.

In order to prove Theorem 3, we take the same 3-complex $\mathscr{M}$, and we assume that it is representable by a 3-diagram with 2358 as basis. Next, we consider the points 4 and 6, and construct the simplices involving only 2, 3, 4, 5, 6, and 8 (i.e., 2345, 2458, 2368, and 3568). The "simply covered" faces of these simplices define an "inner surface" which has only the edges 23, 35, 58, and 28 in common with the boundary of 2358. The edge 46 is now determined. In order to locate the point 7, we note that the edge 46 is contained in the simplices 3456, 3467, and 4567. Since the "inner surface" has "saddle points" at 4 and 6, the point 7 must be located in such a position that the triangle 357 (which is not a face of $\mathscr{M}$) intersects the edge 46 in a point relatively interior to both. Note that the plane 357 separate also the vertices 2 and 8, with 2 being on the same side of 357 as 4, and 8 on the same side as 6. Hence the cone C' with vertex 5 spanned by the triangle 678 is contained in one of the closed half-spaces determined by the plane 357, while the other closed half-space determined by 357 contains the cone C'' with vertex 3 spanned by the triangle 247. Hence the intersection $C' \cap C''$ consists of the single point 7. Since the diagram contains the simplices 1567, 1568, 1578, the point 1 must be contained in C'. On the other hand, since the diagram contains 1234, 1237, 1347, the point 1 must belong to C''. Hence there is no position for 1, and the construction is not possible.

Table 1. Polytopes P_i^7 , $i = 1,2,3,4,5$.

Polytope	Number of Facets	List of facets	Facets of the dual polytope and their type
P_1^7	11	A: 1256 H: 1367 B: 1245 J: 2367 C: 1234 K: 2345 D: 1237 L: 2356 E: 1345 F: 1356 G: 1267	1: ABCDEFGH c_1 2: ABCDGJKL c_1 3: CDEFHJKL c_1 4: BCEK a 5: ABEFKL b 6: AFGHJL b 7: DGHJ a
P_2^7	12	A: 1245 H: 2356 B: 1246 J: 2347 C: 1256 K: 2367 D: 1345 L: 2467 E: 1346 M: 3467 F: 1356 G: 2345	1: ABCDEF b 2: ABCGHJKL c_1 3: DEFGHJKM c_1 4: ABDEGJLM c_1 5: ACDFGH b 6: BCEFHKLM c_1 7: JKLM a
P_3^7	12	A: 1246 H: 1347 B: 1256 J: 2346 C: 1257 K: 2356 D: 1247 L: 2357 E: 1346 M: 2347 F: 1356 G: 1357	1: ABCDFGH c_2 2: ABCDJKLM c_2 3: EFGHJKLM c_2 4: ADEHJM b 5: BCFGKL b 6: ABEFJK b 7: CDGHLM b
P_4^7	13	A: 2467 H: 1456 B: 2367 J: 1247 C: 1367 K: 1237 D: 1467 L: 1345 E: 2456 M: 2345 F: 2356 N: 1234 G: 1356	1: CDGHJKLN c_1 2: ABEFJKMN c_1 3: BCFGKLMN c_1 4: ADEHJLMN c_1 5: EFGHLM b 6: ABCDEFGH c_2 7: ABCDJK b
P_5^7	14	A: 1234 H: 1567 B: 1237 J: 2345 C: 1267 K: 2356 D: 1256 L: 2367 E: 1245 M: 3467 F: 1347 N: 3456 G: 1457 O: 4567	1: ABCDEFGH c_1 2: ABCDEJKL c_1 3: ABFJKLMN c_1 4: AEFGJMNO c_1 5: DEGHJKNO c_1 6: CDHKLMNO c_1 7: BCFGHLMO c_1

Table 2. Complexes $\mathcal{C}$.

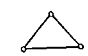
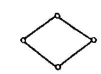
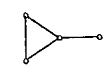
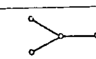
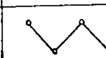
Number of vertices	Type of $\mathcal{C}$	Typical arrangement of facets in $\mathcal{C}$	Nerve of $\mathcal{C}$	Is $\mathcal{C}$ applicable to all polytopes ?
4	$\mathcal{A}$	1234		Yes
5	$\mathcal{B}_1$	1234 1235		Yes
	$\mathcal{B}_2$	1234 1235 1245 (Valence of 1, 2 reduced by one)		Yes
6	$\mathcal{C}_1$	1234 1245 1256		Yes
	$\mathcal{C}_2$	1234 1245 1256 1236 (Valence of 1, 2 reduced by one)		Yes
	$\mathcal{C}_3$	1234 1245 1235 1346 (Valence of 1, 2 reduced by one)		
7	$\mathcal{D}_1$	1234 1245 1256 1267		Yes
	$\mathcal{D}_2$	1234 1235 1246 1347		
	$\mathcal{D}_3$	1234 1245 1256 1567		

Table 3. Generation of polytopes P^8_4.

Type of complex $\mathcal{C}$	Polytope P^7_i	Facets included in $\mathcal{C}$	Resulting P^8_i (*)
$\mathcal{A}$	P^7_1	A: 1256	P^8_1
		B: 1245	P^8_2
		C: 1234	P^8_3
	P^7_2	B: 1246	P^8_4
		L: 2467	P^8_5
		C: 1256	P^8_6
	P^7_3	G: 1357	P^8_7
	P^7_4	B: 2367	P^8_8
		K: 1237	P^8_9
		N: 1234	P^8_{10}
	P^7_5	L: 2367	P^8_{11}
		A: 1234	P^8_{12}
$\mathcal{B}_1$	P^7_1	C: 1234 D: 1237	P^8_{13}
	P^7_2	K: 2367 L: 2467	P^8_{14}
		L: 2467 M: 3467	P^8_{15}
		H: 2356 K: 2367	P^8_{16}
	P^7_3	L: 2357 M: 2347	P^8_{17}
		G: 1357 L: 2357	P^8_{16} (32684751)

(*) If the resulting P^8_i has been obtained earlier, the permutation of the vertices which establishes the equivalence of the two polytopes is indicated in parentheses.

Table 3 (continued)

Type of complex $\mathcal{C}$	Polytope P^7_i	Facets included in $\mathcal{C}$	Resulting P^8_i
$\mathcal{B}_1$	P^7_4	B: 2367 C: 1367	P^8_{18}
		C: 1367 D: 1467	P^8_{19}
		C: 1367 K: 1237	P^8_{20}
		K: 1237 J: 1247	P^8_{21}
		C: 1367 G: 1356	P^8_{22}
		M: 2345 N: 1234	P^8_{19} (16342785)
	P^7_5	L: 2367 M: 3467	P^8_{23}
		H: 1567 O: 4567	P^8_{24}
		C: 1267 L: 2367	P^8_{25}
$\mathcal{B}_2$	P^7_3	A: 1246 J: 2346 E: 1346	P^8_{13} (12386574)
	P^7_4	N: 1234 J: 1247 K: 1237	P^8_{17} (45326178)
		L: 1345 H: 1456 G: 1356	P^8_{16} (54327618)
	P^7_5	A: 1234 F: 1347 B: 1237	P^8_{22} (56714238)
$\mathcal{C}_1$	P^7_3	K: 2356 M: 2347 L: 2357	P^8_{26}
	P^7_4	N: 1234 L: 1345 K: 1237	P^8_{27}
		M: 2345 A: 2467 F: 2456	P^8_{28}
		H: 1456 A: 2467 D: 1467	P^8_{29}
		C: 1367 H: 1456 D: 1467	P^8_{27} (25438167)

Table 3 (continued)

Type of complex $\mathcal{C}$	Polytope P_i^7	Facets included in $\mathcal{C}$	Resulting P_i^8
$\mathcal{C}_1$	P_5^7	J: 2345 K: 2356 L: 2367	P_{30}^8
		A: 1234 B: 1237 L: 2367	P_{31}^8
		A: 1234 B: 1237 C: 1267	P_{32}^8
		G: 1457 H: 1567 D: 1256	P_{33}^8
		E: 1245 F: 1347 G: 1457	P_{32}^8 (67341825)
		A: 1234 F: 1347 G: 1457	P_{31}^8 (67143825)
$\mathcal{C}_2$	P_3^7	J: 2346 K: 2356 L: 2357 M: 2347	P_{34}^8
	P_4^7	C: 1367 D: 1467 G: 1356 H: 1456	P_{26}^8 (48326571)
	P_5^7	A: 1234 E: 1245 F: 1347 G: 1457	P_{27}^8 (72453816)
$\mathcal{C}_3$	P_4^7	J: 1247 K: 1237 L: 1345 N: 1234	P_{26}^8 (76324158)
	P_5^7	A: 1234 B: 1237 E: 1245 F: 1347	P_{29}^8 (76542318)
		A: 1234 B: 1237 C: 1267 F: 1347	P_{27}^8 (61835427
		A: 1234 B: 1237 F: 1347 G: 1457	P_{28}^8 (14823567)

Table 3 (continued)

Type of Complex $\mathcal{C}$	Polytope P_i^7	Facets included in $\mathcal{C}$	Resulting P_i^8
$\mathcal{D}_1$	P_5^7	B: 1237 F: 1347 G: 1457 H: 1567	P_{35}^8
		C: 1267 B: 1237 F: 1347 G: 1457	P_{36}^8
		B: 1237 C: 1267 F: 1347 H: 1567	P_{37}^8
$\mathcal{D}_2$	P_5^7	E: 1245 D: 1256 G: 1457 J: 2345	P_{37}^8 (54328761)
$\mathcal{D}_3$	P_5^7	A: 1234 E: 1245 D: 1256 H: 1567	P_{36}^8 (82345671)
		A: 1234 B: 1237 E: 1245 L: 2367	P_{37}^8 (25781436)
		D: 1256 E: 1245 G: 1457 K: 2356	Non-existent (Complex $\mathcal{M}$)

Table 4. Polytopes P_i^8

Polytope	Number of facets	List of facets	Facets of the dual Polytope and their type
P_1^8	14	B: 1245 K: 2345 C: 1234 L: 235~~8~~6 D: 1237 M: 1258 E: 1345 N: 1268 F: 1356 O: 1568 G: 1267 P: 2568 H: 1367 J: 2367	1: BCDEFGHMNO d_3 2: BCDGJKLMNP d_3 3: CDEFHJKL c_1 4: BCEK a 5: BEFKLMOP c_1 6: FGHJLNOP c_1 7: DGHJ a 8: MNOP a
P_2^8	14	A: 1256 K: 2345 C: 1234 L: 2356 D: 1237 M: 1248 E: 1345 N: 1258 F: 1356 O: 1458 G: 1267 P: 2458 H: 1367 J: 2367	1: ACDEFGHMNO d_4 2: ACDGJKLMNP d_4 3: CDEFHJKL c_1 4: CEKMOP b 5: AEFKLNOP c_1 6: AFGHJL b 7: DGHJ a 8: MNOP a
P_3^8	14	A: 1256 K: 2345 B: 1245 L: 2356 D: 1237 M: 1238 E: 1345 N: 1248 F: 1356 O: 1348 G: 1267 P: 2348 H: 1367 J: 2367	1: ABDEFGHMNO d_1 2: ABDGJKLMNP d_1 3: DEFHJKLMOP d_1 4: BEKNOP b 5: ABEFKL b 6: AFGHJL b 7: DGHJ a 8: MNOP a
P_4^8	15	A: 1245 K: 2367 C: 1256 L: 2467 D: 1345 M: 3467 E: 1346 N: 1248 F: 1356 O: 1268 G: 2345 P: 1468 H: 2356 Q: 2468 J: 2347	1: ACDEFNOP c_1 2: ACGHJKLNOQ d_1 3: DEFGHJKM c_1 4: ADEGJLMNPQ d_3 5: ACDFGH b 6: CEFHKLMOPQ d_3 7: JKLM a 8: NOPQ a
P_5^8	15	A: 1245 J: 2347 B: 1246 K: 2367 C: 1256 M: 3467 D: 1345 N: 2468 E: 1346 O: 2478 F: 1356 P: 2678 G: 2345 Q: 4678 H: 2356	1: ABODEF b 2: ABCGHJKNOP d_1 3: DEFGHJKM c_1 4: ABDEGJMNOQ d_4 5: ACDFGH b 6: BCEFHKMNPQ d_4 7: JKMOPQ b 8: NOPQ a

Table 4 (continued)

Polytope	Number of facets	List of facets	Facets of the dual Polytope and their type
P_6^8	15	A: 1245 K: 2367 B: 1246 L: 2467 D: 1345 M: 3467 E: 1346 N: 1258 F: 1356 O: 1268 G: 2345 P: 1568 H: 2356 Q: 2568 J: 2347	1: ABDEFNOP c_1 2: ABGHJKLNOQ d_4 3: DEFGHJKM c_1 4: ABDEGJLM c_1 5: ADFGHNPQ c_1 6: BEFHKLMOPQ d_4 7: JKLM a 8: NOPQ a
P_7^8	15	A: 1246 K: 2356 B: 1256 L: 2357 C: 1257 M: 2347 D: 1247 N: 1358 E: 1346 O: ~~1278~~ 1378 F: 1356 P: 1578 H: 1347 Q: 3578 J: 2346	1: ABCDEFHNOP d_5 2: ABCDJKLM c_2 3: EFHJKLMNOQ d_5 4: ADEHJM b 5: BCFKLNPQ c_1 6: ABEFJK b 7: CDHLMOPQ c_1 8: NOPQ a
P_8^8	16	A: 2467 K: 1237 C: 1367 L: 1345 D: 1467 M: 2345 E: 2456 N: 1234 F: 2356 O: 2368 G: 1356 P: 2378 H: 1456 Q: 2678 J: 1247 R: 3678	1: CDGHJKLN c_1 2: AEFJKMNOPQ d_3 3: CFGKLMNOPR d_4 4: ADEHJLMN c_1 5: EFGHLM b 6: ACDEFGHOQR d_5 7: ACDJKPQR c_1 8: OPQR a
P_9^8	16	A: 2467 J: 1247 B: 2367 L: 1345 C: 1367 M: 2345 D: 1467 N: 1234 E: 2456 O: 1238 F: 2356 P: 1278 G: 1356 Q: 1378 H: 1456 R: 2378	1: CDGHJLNOPQ d_4 2: ABEFJMNOPR d_4 3: BCFGLMNOQR d_1 4: ADEHJLMN c_1 5: EFGHLM b 6: ABCDEFGH c_2 7: ABCDJPQR c_1 8: OPQR a
P_{10}^8	16	A: 2467 J: 1247 B: 2367 K: 1237 C: 1367 L: 1345 D: 1467 M: 2345 E: 2456 O: 1238 F: 2356 P: 1248 G: 1356 Q: 1348 H: 1456 R: 2348	1: CDGHJKLOPQ d_1 2: ABEFJKMOPR d_1 3: BCFGKLMOQR d_1 4: ADEHJLMPQR d_1 5: EFGHLM b 6: ABCDEFGH c_2 7: ABCDJK b 8: OPQR a

Table 4 (continued)

P^8_{11}	17	A: 1234 B: 1237 C: 1267 D: 1256 E: 1245 F: 1347 G: 1457 H: 1567 J: 2345 K: 2356 M: 3467 N: 3456 O: 4567 P: 2368 Q: 2378 R: 2678 S: 3678	1: ABCDEFGH 2: ABCDEJKPQR 3: ABFJKMNPQS 4: AEFGJMNO 5: DEGHJKNO 6: CDHKMNOPRS 7: BCFGHMOQRS 8: PQRS	c[illegible] c[illegible] d[illegible] c[illegible] c[illegible] d[illegible] d[illegible] a[illegible]
P^8_{12}	17	B: 1237 C: 1267 D: 1256 E: 1245 F: 1347 G: 1457 H: 1567 J: 2345 K: 2356 L: 2367 M: 3467 N: 3456 O: 4567 P: 1238 Q: 1248 R: 2348 S: 1348	1: BCDEFGHPQS 2: BCDEJKLPQR 3: BFJKLMNPRS 4: EFGJMNOQRS 5: DEGHJKNO 6: CDHKLMNO 7: BCFGHLMO 8: PQRS	d[illegible] c[illegible] d[illegible] d[illegible] c[illegible] c[illegible] c[illegible] a[illegible]
P^8_{13}	15	A: 1256 B: 1245 E: 1345 F: 1356 G: 1267 H: 1367 J: 2367 K: 2345 L: 2356 M: 1248 N: 1348 O: 2348 P: 1278 Q: 1378 R: 2378	1: ABEFGHMNPQ 2: ABGJKLMOPR 3: EFHJKLNOQR 4: BEKMNO 5: ABEFKL 6: AFGHJL 7: GHJPQR 8: MNOPQR	d_2 d_2 d_2 b b b b b
P^8_{14}	16	A: 1245 B: 1246 C: 1256 D: 1345 E: 1346 F: 1356 G: 2345 H: 2356 J: 2347 M: 3467 N: 2368 O: 2378 P: 3678 Q: 2468 R: 2478 S: 4678	1: ABCDEF 2: ABCGHJNOQR 3: DEFGHJMNOP 4: ABDEGJMQRS 5: ACDFGH 6: BCEFHMNPQS 7: JMOPRS 8: NOPQRS	b d_4 d_4 d_4 b d_4 b b
P^8_{15}	16	A: 1245 B: 1246 C: 1256 D: 1345 E: 1346 F: 1356 G: 2345 H: 2356 J: 2347 K: 2367 N: 2468 O: 2478 P: 2678 Q: 3468 R: 3478 S: 3678	1: ABCDEF 2: ABCGHJKNOP 3: DEFGHJKQRS 4: ABDEGJNOQR 5: ACDFGH 6: BCEFHKNPQS 7: JKOPRS 8: NOPQRS	b d_1 d_1 d_1 b d_1 b b

Table 4 (continued)

P^8_{16}	16	A: 1245 B: 1246 C: 1256 D: 1345 E: 1346 F: 1356 G: 2345 J: 2347 L: 2467 M: 3467 N: 2358 O: 2568 P: 3568 Q: 2378 R: 2678 S: 3678	1: ABCDEF 2: ABCGJLNOQR 3: DEFGJMNPQS 4: ABDEGJLM 5: ACDFGNOP 6: BCEFLMOPRS 7: JLMQRS 8: NOPQRS	b d_5 d_5 c_1 c_1 d_2 b b
P^8_{17}	16	A: 1246 B: 1256 C: 1257 D: 1247 E: 1346 F: 1356 G: 1357 H: 1347 J: 2346 K: 2356 N: 2358 O: 2578 P: 3578 Q: 2348 R: 2478 S: 3478	1: ABCDEFGH 2: ABCDJKNOQR 3: EFGHJKNPQS 4: ADEHJQRS 5: BCFGKNOP 6: ABEFJK 7: CDGHOPRS 8: NOPQRS	c_2 d_2 d_2 c_1 c_1 b c_2 b
P^8_{18}	17	A: 2467 D: 1467 E: 2456 F: 2356 G: 1356 H: 1456 J: 1247 K: 1237 L: 1345 M: 2345 N: 1234 O: 2368 P: 2378 Q: 2678 R: 1678 S: 1378 T: 1368	1: DGHJKLNRST 2: AEFJKMNOPQ 3: FGKLMNOPST 4: ADEHJLMN 5: EFGHLM 6: ADEFGHOQRS 7: ADJKPQRS 8: OPQRST	d_3 d_3 d_1 c_1 b d_2 c_1 b
P^8_{19}	17	A: 2467 B: 2367 E: 2456 F: 2356 G: 1356 H: 1456 J: 1247 K: 1237 L: 1345 M: 2345 N: 1234 O: 1368 P: 1378 Q: 3678 R: 1468 S: 1478 T: 4678	1: GHJKLNOPRS 2: ABEFJKMN 3: BFGKLMNOPQ 4: AEHJLMNRST 5: EFGHLM 6: ABEFGHOQRT 7: ABJKPQST 8: OPQRST	d_1 c_1 d_4 d_4 b d_2 c_2 b
P^8_{20}	17	A: 2467 B: 2367 D: 1467 E: 2456 F: 2356 G: 1356 H: 1456 J: 1247 L: 1345 M: 2345 N: 1234 O: 1678 P: 3678 Q: 1368 R: 1238 S: 1278 T: 2378	1: DGHJLNOQRS 2: ABEFJMNRST 3: BFGLMNPQRT 4: ADEHJLMN 5: EFGHLM 6: ABDEFGHOPQ 7: ABDJOPST 8: OPQRST	d_5 d_4 d_4 c_1 b d_5 c_1 b

Table 4 (continued)

P^8_{21}	17	A: 2467 L: 1345 T: 2378 B: 2367 M: 2345 C: 1367 N: 1234 D: 1467 O: 1248 E: 2456 P: 1478 F: 2356 Q: 2478 G: 1356 R: 1378 H: 1456 S: 1238	1: CDGHLNOPRS [illegible] 2: ABEFMNOQST [illegible] 3: BCFGLMNRST [illegible] 4: ADEHLMNOPQ [illegible] 5: EFGHLM [illegible] 6: ABCDEFGH [illegible] 7: ABCDPQRT [illegible] 8: OPQRST [illegible]
P^8_{22}	17	A: 2467 L: 1345 T: 3568 B: 2367 M: 2345 D: 1467 N: 1234 E: 2456 O: 1378 F: 2356 P: 1678 H: 1456 Q: 3678 J: 1247 R: 1358 K: 1237 S: 1568	1: DHJKLNOPRS d_5 2: ABEFJKMN c[illegible] 3: BFKLMNOQRT c[illegible] 4: ADEHJLMN c[illegible] 5: EFHLNRST c[illegible] 6: ABDEFHPQST d[illegible] 7: ABDJKOPQ c[illegible] 8: OPQRST b[illegible]
P^8_{23}	18	A: 1234 J: 2345 T: 3478 B: 1237 K: 2356 U: 4678 C: 1267 N: 3456 D: 1256 O: 4567 E: 1245 P: 2368 F: 1347 Q: 2378 G: 1457 R: 2678 H: 1567 S: 3468	1: ABCDEFGH c_1 2: ABCDEJKPQR d_4 3: ABFJKNPQST c_1 4: AEFGJNOSTU d_4 5: DEGHJKNO c_1 6: CDHKNOPRSU d_5 7: BCFGHOQRTU d_5 8: PQRSTU b_5
P^8_{24}	18	A: 1234 K: 2356 T: 4578 B: 1237 L: 2367 U: 4678 C: 1267 M: 3467 D: 1256 N: 3456 E: 1245 P: 1568 F: 1347 Q: 1578 G: 1457 R: 1678 J: 2345 S: 4568	1: ABCDEFGPQR d_4 2: ABCDEJKL c_1 3: ABFJKLMN c_1 4: AEFGJMNSTU d_4 5: DEGJKNPQST d_4 6: CDKLMNPRSU d_2 7: BCFGLMQRTU d_4 8: PQRSTU b_4
P^8_{25}	18	A: 1234 K: 2356 T: 2378 B: 1237 M: 3467 U: 3678 D: 1256 N: 3456 E: 1245 O: 4567 F: 1347 P: 1268 G: 1457 Q: 1278 H: 1567 R: 1678 J: 2345 S: 2368	1: ABDEFGHPQR d_1 2: ABDEJKPQST d_1 3: ABFJKMNSTU d_3 4: AEFGJMNO c_1 5: DEGHJKNO c_1 6: DHKMNOPRSU d_5 7: BFGHMOQRTU d_5 8: PQRSTU b_4

Table 4 (continued)

P^8_{26}	17	A: 1246 J: 2346 U: 3478 B: 1256 N: 2368 C: 1257 O: 2568 D: 1247 P: 3568 E: 1346 Q: 2578 F: 1356 R: 3578 G: 1357 S: 2348 H: 1347 T: 2478	1: ABCDEFGH c_2 2: ABCDJNOQST d_5 3: EFGHJNPRSU d_5 4: ADEHJSTU c_1 5: BCFGOPQR c_2 6: ABEFJNOP c_1 7: CDGHQRTU c_2 8: NOPQRSTU c_1
P^8_{27}	18	A: 2467 J: 1247 U: 1458 B: 2367 M: 2345 V: 3458 C: 1367 O: 1278 D: 1467 P: 1378 E: 2456 Q: 2378 F: 2356 R: 1248 G: 1356 S: 2348 H: 1456 T: 1358	1: CDGHJOPRTU d_5 2: ABEFJMOQRS d_4 3: BCFGMPQSTU d_5 4: ADEHJMRSUV d_4 5: EFGHMTUV c_1 6: ABCDEFGH c_2 7: ABCDJOPQ c_1 8: OPQRSTUV c_1
P^8_{28}	18	B: 2367 L: 1345 U: 2358 C: 1367 N: 1234 V: 3458 D: 1467 O: 2478 F: 2356 P: 2678 G: 1356 Q: 4678 H: 1456 R: 2568 J: 1247 S: 4568 K: 1237 T: 2348	1: CDFHJKLN c_1 2: BFJKNOPRTU d_4 3: BCFGKLNTUV d_4 4: DHJLNOQSTV d_2 5: FGHLRSUV c_1 6: BCDFGHPQRS d_2 7: BCDJKOPQ c_1 8: OPQRSTUV c_1
P^8_{29}	18	B: 2367 M: 2345 U: 1568 C: 1367 N: 1234 V: 4568 E: 2456 O: 2468 F: 2356 P: 2478 G: 1356 Q: 2678 J: 1247 R: 1478 K: 1237 S: 1678 L: 1345 T: 1458	1: CGJKLNRSTU d_5 2: BEFKMNJOPQ d_3 3: BCFGKLMN c_1 4: EJLMNOPRTV d_3 5: EFGLMTUV c_1 6: BCEFGOQSUV d_5 7: BCJKPQRS c_1 8: OPQRSTUV c_1
P^8_{30}	19	A: 1234 M: 3467 U: 2378 B: 1237 N: 3456 V: 2678 C: 1267 O: 4567 W: 3678 D: 1256 P: 2348 E: 1245 Q: 2458 F: 1347 R: 3458 G: 1457 S: 2568 H: 1567 T: 3568	1: ABCDEFGH c_1 2: ABCDEPQSUV d_2 3: ABFMNPRTUW d_4 4: AEFGMNOPQR d_1 5: DEGHNOQRST d_1 6: CDHMNOSTVW d_1 7: BCFGHMOUVW d_4 8: PQRSTUVW c_1

Table 4 (continued)

P^8_{31}	19	C: 1267 D: 1256 E: 1245 F: 1347 G: 1457 H: 1567 J: 2345 K: 2356	M: 3467 N: 3456 O: 4567 P: 1248 Q: 1348 R: 2348 S: 1278 T: 1378	U: 2368 V: 2678 W: 3678	1: CDEFGHPQST 2: CDEJKPRSUV 3: FJKMNQRTUW 4: EFGJMNOPQR 5: DEGHJKNO 6: CDHKMNOUVW 7: CFGHMOSTVW 8: PQRSTUVW
P^8_{32}	19	D: 1256 E: 1245 F: 1347 G: 1457 H: 1567 J: 2345 K: 2356 L: 2367	M: 3467 N: 3456 O: 4567 P: 1248 Q: 1348 R: 2348 S: 1378 T: 2378	U: 1268 V: 1678 W: 2678	1: DEFGHPQRSUV 2: DEJKLPRTUW 3: FJKLMNQRST 4: EFGJMNOPQR 5: DEGHJKNO 6: DHKLMNOUVW 7: FGHLMOSTVW 8: PQRSTUVW
P^8_{33}	19	A: 1234 B: 1237 C: 1267 E: 1245 F: 1347 J: 2345 K: 2356 L: 2367	M: 3467 N: 3456 O: 4567 P: 1258 Q: 1268 R: 2568 S: 1678 T: 5678	U: 1458 V: 1478 W: 4578	1: ABCEFPQSUV 2: ABCEJKLPQR 3: ABFJKLMN 4: AEFJMNOUVW 5: EJKNOPRTUW 6: CKLMNOQRST 7: BCFLMOSTVW 8: PQRSTUVW
P^8_{34}	16	A: 1246 B: 1256 C: 1257 D: 1247 E: 1346 F: 1356 G: 1357 H: 1347	N: 2468 O: 3468 P: 2568 Q: 3568 R: 2578 S: 3578 T: 2478 U: 3478		1: ABCDEFGH 2: ABCDNPRT 3: EFGHOQSU 4: ADEHNOTU 5: BCFGPQRS 6: ABEFNOPQ 7: CDGHRSTU 8: NOPQRSTU
P^8_{35}	20	A: 1234 C: 1267 D: 1256 E: 1245 J: 2345 K: 2356 L: 2367 M: 3467	N: 3456 O: 4567 P: 1238 Q: 1278 R: 2378 S: 1348 T: 3478 U: 1458	V: 4578 W: 1568 X: 1678 Y: 5678	1: ACDEPQSUWX 2: ACDEJKLPQR 3: AJKLMNPRST 4: AEJMNOSTUV 5: DEJKNOUVWY 6: CDKLMNOWXY 7: CLMOQRTVXY 8: PQRSTUVWXY

Table 4 (continued)

P^8_{36}	20	A: 1234 D: 1256 E: 1245 H: 1567 J: 2345 K: 2356 L: 2367 M: 3467	N: 3456 O: 4567 P: 1268 Q: 1678 R: 2678 S: 1238 T: 2378 U: 1348	V: 3478 W: 1458 X: 1578 Y: 4578	1: ADEHPQSUWX 2: ADEJKLPRST 3: AJKLMNSTUV 4: AEJMNOUVWY 5: DEHJKNOWXY 6: DHKLMNOPQR 7: HLMOQRTVXY 8: PQRSTUVWXY	d_4 d_4 d_1 d_1 d_4 d_4 d_1 d_1
P^8_{37}	20	A: 1234 D: 1256 E: 1245 G: 1457 J: 2345 K: 2356 L: 2367 M: 3467	N: 3456 O: 4567 P: 1568 Q: 1578 R: 5678 S: 1268 T: 2678 U: 1238	V: 2378 W: 1348 X: 1478 Y: 3478	1: ADEGPQSUWX 2: ADEJKLSTUV 3: AJKLMNUVXY 4: AEGJMNOWXY 5: DEGJKNOPQR 6: DKLMNOPRST 7: GLMOQRTVXY 8: PQRSTUVWXY	d_3 d_4 d_1 d_3 d_4 d_4 d_4 d_1
Complex $\mathcal{M}$	20	A: 1234 B: 1237 C: 1267 F: 1347 H: 1567 J: 2345 L: 2367 M: 3467	N: 3456 O: 4567 P: 2358 Q: 2368 R: 3568 S: 1268 T: 1568 U: 1248	V: 2458 W: 1478 X: 1578 Y: 4578	1: ABCFHSTUWX 2: ABCJLPQSUV 3: ABFJLMNPQR 4: AFJMNOUVWY 5: HJNOPRTVXY 6: CHLMNOQRST 7: BCFHLMOWXY 8: PQRSTUVWXY	d_4 d_4 d_4 d_5 d_4 d_5 d_4 d_4

Table 5. The polytopes P_i^8 (Summary).

i	Number of facets	Type and number of facets of the dual polytope: a	b	c_1	c_2	d_1	d_2	d_3	d_4	d_5
1	14	3	-	3	-	-	-	2	-	-
2		2	2	2	-	-	-	-	2	-
3		2	3	-	-	3	-	-	-	-
4	15	2	1	2	-	1	-	2	-	-
5		1	3	1	-	1	-	-	2	-
6		2	-	4	-	-	-	-	2	-
7		1	2	2	1	-	-	-	-	2
13		-	5	-	-	-	3	-	-	-
8	16	1	1	3	-	-	-	1	1	1
9		1	1	2	1	1	-	-	2	-
10		1	2	-	1	4	-	-	-	-
14		-	4	-	-	-	-	-	4	-
15		-	4	-	-	4	-	-	-	-
16		-	3	2	-	-	1	-	-	2
17		-	2	2	2	-	2	-	-	-
34		-	-	-	8	-	-	-	-	-
11	17	1	-	3	-	-	-	2	2	-
12		1	-	3	-	2	-	-	2	-
18		-	2	2	-	1	1	2	-	-
19		-	2	1	1	1	1	-	2	-
20		-	2	2	-	-	-	-	2	2
21		-	2	-	2	4	-	-	-	-
22		-	1	4	-	-	1	-	-	2
26		-	-	3	3	-	-	-	-	2
23	18	-	1	2	-	1	-	-	2	2
24		-	1	2	-	-	1	-	4	-
25		-	1	2	-	2	-	1	1	1
27		-	-	3	1	-	-	-	2	2
28		-	-	4	-	-	2	-	2	-
29		-	-	4	-	-	-	2	-	2
30	19	-	-	2	-	3	1	-	2	-
31		-	-	2	-	1	-	1	3	1
32		-	-	2	-	-	-	-	4	2
33		-	-	2	-	-	-	4	2	-
35	20	-	-	-	-	8	-	-	-	-
36		-	-	-	-	4	-	-	4	-
37		-	-	-	-	2	-	2	4	-
Complex $\mathcal{M}$	20	-	-	-	-	-	-	-	6	2

Table 6. Vertices of a 3-diagram representing the complex $\mathcal{M}$ in R^3

Vertex	Coordinates x	y	z
1	117	47	98
2	82	55	114
3	4	79	7
4	0	215	0
5	0	0	333
6	0	0	0
7	279	0	0
8	94	44	142

References

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Addendum (*November* 5, 1966). G. Wegner (Göttingen) kindly communicated to the authors the following construction of a 3-diagram representing the 3-complex $\mathcal{M}$; this construction yields a proof of Theorem 2 without recourse to tedious computations.

Choosing the notation so as to obtain $\mathcal{M}$ as above, Wegner's construction starts by taking three segments, 15, 27, 38 (see Fig. 5), which are mutually skew (such as those obtained from suitable disjoint edges

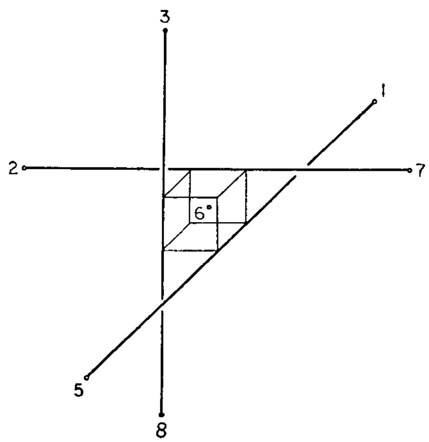

FIG. 5.

of a cube). From those segments, tetrahedra 1237, 1578, 2358 are constructed (see Fig. 6.). Each of the three tetrahedra has two faces in common with the (skew) octahedron Q which is the convex hull of

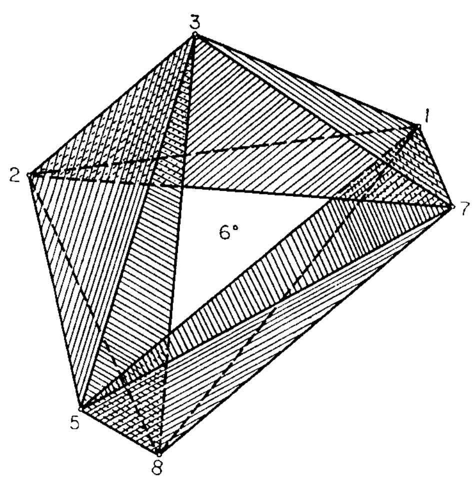

FIG. 6.

the three segments, while the other two faces of each tetrahedron are visible from each point of the interior of the small cube. Hence if 6 is such a point, a simplicial decomposition of Q is determined by the tetrahedra 1237, 1578, 2358; 1276, 1576, 1586, 2376, 2386, 3586; 1286, 3576. Taking a projective transform Q' of Q such that all the faces of Q' except the triangle (corresponding to) 128 are visible from a point 4 outside Q', deleting the tetrahedron 3576, and introducing the tetrahedra 3564, 3764, 5764; 1234, 1374, 1784, 2354, 2584, 5784, we obtain a simplicial decomposition of the tetrahedron 1284. An easy comparison shows that it is indeed a 3-diagram representing $\mathcal{M}$.

Note added in proof (May 24, 1967). The result of Theorem 4 was recently strengthened by David W. Barnette (private communication). Barnette proved the validity of the "lower bound conjecture" for simplicial 4-polytopes with at most 10 vertices, as well as in the following additional cases: 5-polytopes and 6-polytopes with at most 10 vertices, 7-polytopes with at most 11 vertices, and 8-polytopes with at most 12 vertices.

Regarding the cell-complex $\mathcal{M}^*$ dual to $\mathcal{M}$ the following results were obtained, which show that Brückner's tacit assumption about the existence of a 3-diagram realizing $\mathcal{M}^*$ was unjustified. G. Wegner has shown that the 2-skeleton of $\mathcal{M}^*$ is not realizable by a geometric cell complex in any Euclidean space. Using this result, D. W. Barnette has established that the 1-skeleton of $\mathcal{M}^*$ is not a 4-polyhedral graph.