Ambient Surfaces and T-Fillings of T-Curves

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Abstract

T-curves are piecewise linear curves which have been used with success since the beginning of the 1990's to construct new real algebraic curves with prescribed topology mainly on the real projective plane (see [11,7,3]). In fact T-curves can be used on any real projective toric surface. We generalize here the construction of the latter by departing from non-convex polygons and we get *ambient surfaces* that we characterize topologically and on which T-curves are also well defined. Associated to each T-curve we construct a surface, the *T-filling*, which is topologically the analog of the quotient of the complexification of a real algebraic curve by the complex conjugation. We use then the T-filling (1) to prove a theorem for T-curves on arbitrary ambient surfaces which is the analog of a theorem of Harnack for algebraic curves, and (2) to define the orientability and orientation of a T-curve, the same way it is defined in the real algebraic setting.

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Introduction

In the 1980's Oleg Viro invented a geometric method (called *patchworking*) to construct smooth algebraic hypersurfaces of real projective toric varieties [13] by gluing, like in a patchwork quilt, different toric varieties with given smooth embedded hypersurfaces. In particular, his method, when used with lines as patches, gives rise to a family of piecewise linear curves which have been commonly called *T-curves*. T-curves and related constructions have been used with success since the beginning of the 1990's to answer various questions around the first part of Hilbert 16^{th} Problem [6] (more precisely about the topology of real algebraic curves and surfaces, see for instance [11,7,3,8], and most recently about the number of limit cycles in polynomial vector fields (see [12]).

T-curves originally lie on (polygonal models of) real projective toric surfaces. They are constructed from the data of **(1)** an integral convex polygon, **(2)** a rectilinear triangulation of the polygon where the vertices of the triangles are the integer points of the polygon, and which satisfies some convexity property (see def. 25), and **(3)** a distribution of signs ("+" and "−") on the integer points of the polygon. The convex polygon must be viewed at the same time as a polygon defining the ambient toric surface and as the Newton polygon of some polynomial $\sum a_{i,j}x^iy^j$ (the Newton polygon is defined as the convex hull of the integer points (i,j) such that $a_{i,j} \neq 0$). The signs on the integer points (i,j) must be viewed as the signs of the $a_{i,j}$'s. The polynomial defines an algebraic curve on the toric surface, which is *congruent* to the T-curve, i.e. there is a homeomorphism of the toric surface which transforms the T-curve into the algebraic curve. Notice that not all algebraic curves are congruent to T-curves with same Newton polygon.

In the first part of his 16^{th} problem Hilbert asked what are all the possible non-singular real algebraic curves of given degree on the real projective plane $\mathbb{R}P^2$ up to congruence. This problem generalizes to curves with various constraints on various surfaces (for instance curves with a given Newton polygon on a given real projective toric surface). We study here, in this spirit, T-curves on some combinatorial surfaces, the ambient surfaces (see below), which generalize some classical models of non-singular real projective toric surfaces.

In the construction of a T-curve the restrictions to convex polygons and convex triangulations are necessary for the parallel construction of the polynomial defining the algebraic curve congruent to the T-curve. But these two restrictions are not necessary for the construction of the T-curve itself. Jesus De Loera [2] has studied some properties of T-curves on $\mathbb{R}P^2$ without the assumption of convexity on the triangulation. The purpose of this approach is not any more the construction of new algebraic curves, but the proof of properties of T-curves, in particular the ones inspired directly from algebraic curves on $\mathbb{R}P^2$ (for instance De Loera proved an analog of Hilbert's lemma).

The same approach would be interesting for T-curves on arbitrary real projective toric surfaces, but since we dropped the link with algebraic curves by dropping the convexity requirement on the triangulation, it is also natural to drop the link with real projective toric surfaces by dropping the requirement that the Newton polygon be convex. This is what we do in the first part of this article and we get this way the *ambient surfaces* on which T-curves are still well defined.

In this more general setting we construct in part II a surface from a T-curve, the *T-filling*, which is the analog of the quotient of the complexification of an algebraic curve by the complex conjugation. The latter surface is well known and has been classically used to prove many properties of real algebraic curves. Here we use the T-filling to prove in a similar way the analog of a theorem of Harnack [5] for algebraic curves on $\mathbb{R}P^2$: The maximum number of connected components of a T-curve on an arbitrary ambient surface is equal to the number of integer points in the interior of its Newton polygon (see subsect. 3.3.1). For each Newton polygon a remarkable T-curve, called the *Harnack T-curve* (see 3.3.2), has the maximum number of connected components (such a curve is called a *maximal curve*). Harnack T-curves are well known on $\mathbb{R}P^2$ where they are congruent to curves constructed by Harnack [5]. We also use the T-filling to define *orientability* and *orientations* of T-curves (see subsect. 3.3.3) in the same way it has been defined by Rokhlin [9] for algebraic curves on $\mathbb{R}P^2$.

Viro's patchworking method for curves, because of its real algebraic setting, deals with convex polygons. Allowing polygons to be non-convex leads to a patchworking method for T-curves with more freedom. In particular we show in [4] that every maximal T-curve is the patchworking of Harnack T-curves (most of which have non-convex Newton polygons). This is a first step in characterizing which ones among all maximal real algebraic curves on a given projective toric surfaces can be constructed as T-curves (indeed this *Harnack patchworking* holds in particular for maximal T-curves with convex Newton polygons and with convex triangulations).

The T-filling is a new object. The Harnack theorem for T-curves on $\mathbb{R}P^2$ and the notion of orientability and orientation for T-curves on $\mathbb{R}P^2$ were known by I. Itenberg and O. Viro but, as far as I know, never published by them (Jesús De Loera discusses orientability and orientation of T-curves on $\mathbb{R}P^2$ after Itenberg and Viro in [2]).

1 The Ambient Surfaces

1.1 Some Definitions in Lattice Geometry

Definition 1 *Here all segments are* integral rectilinear segments, *that is their two endpoints are integer points. A* (closed) polygonal line *is a finite union of segments* $s_1 \cup \cdots \cup s_r \subset \mathbb{R}^2$, *such that for* $i = 2, \ldots, r-1$, *(for* i *modulo* r*), the segment* s_i *shares one of its endpoints with* s_{i-1} *and its other endpoint with* s_{i+1}. *We always assume that the union of any two consecutive segments* $s_i \cup s_{i+1}$ *is not a segment. Here we call a* polygon *a subset of* $\mathbb{R}^2$ *bounded by a closed polygonal line and homeomorphic to a disk. An* edge of the polygon *is a segment of its boundary polygonal line. A* vertex of a polygon *is an endpoint of an edge.*

Notice that the vertices of a polygon are all integer points.

Definition 2 *A* triangulation of a polygon *is a cell decomposition of the polygon into simplexes. A simplex of dimension* 2 *is a* triangle of the triangulation, *a simplex of dimension* 1 *is an* edge of the triangulation, *and a simplex of dimension* 0 *is a* vertex of the triangulation.

Vertices and edges of a triangulation shouldn't be mixed up with vertices and edges of a polygon. We are going now to define a *parity* which assigns to various objects an element of $(\mathbb{Z}_2)^2 = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. We denote the four values of the parities by $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. A parity is said to be *even* if its value is $(0,0)$. A parity is *odd* if it is not even.

Definition 3 *The* parity of an integer point (p_1, p_2) *is* $(p_1 \bmod 2, p_2 \bmod 2)$. *The* parity of a segment *is the sum of the two values that the parity takes on all the integer points of the segment.*

Notice that the parity of the integer points which lie on a given segment takes exactly two values. Also, notice that the parity of a segment is never even.

Definition 4 *The* parity of a vertex of a polygon *is the sum of the parities of the two segments underlying the edges of the polygon meeting in the vertex. The* parity of an edge of a polygon *is the sum of the parities of the two vertices of the polygon which end the edge.*

Notice that the parity of a vertex (of an edge) of a polygon can possibly take all four values. Also the parity of a vertex (of an edge) of a polygon may be different than the parity of its underlying integer point (of its underlying segment).

Lemma 5 *If a polygon contains a vertex of odd parity, then it contains at least two vertices of odd parity.*

PROOF. Let us follow the boundary of the polygon, departing from the vertex of odd parity and coming back to it. Since the vertex has odd parity, the "departing-edge" has different parity than the "coming-back-edge". So on the way we must pass through a vertex where two edges of different parity meet, i.e. through a vertex of odd parity. □

Definition 6 *The vertices of odd parity of a polygon divide its boundary into connected components, the closure of which we call the* broken edges of the polygon.

Notice that the parity takes the same value on all the segments of a given broken edge.

Definition 7 *The* parity of a broken edge of a polygon *is the sum of the parities of all the edges of the polygon contained in the broken edge.*

Notice that if a polygon has vertices of odd parity, then the parity of a broken edge of the polygon is equal to the sum of the parities of the two vertices of the polygon which end the broken edge.

1.2 The Ambient Surface and its Local Topology

1.2.1 The Ambient Surface Associated to a Polygon

Let Π be a polygon in $\mathbb{R}^2_{\geqslant 0} := \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$, and for every $a, b \in \{0,1\}$, let $\sigma_{a,b}$ be the symmetry $(x,y) \mapsto ((-1)^a x, (-1)^b y)$. We glue the disjoint union of the four copies $(\sigma_{a,b} \cdot \Pi)$ by their boundary in the following way: we identify each point (x,y) of the boundary with the point $\sigma_{(a,b)} \cdot (x,y)$ where (a,b) is such that $\langle (a,b),(c,d) \rangle := ac + bd = 0$, when (x,y) lies on an edge of parity (c,d). We obtain in this way a surface without boundary.

Definition 8 *We call* ambient surface *(associated to a polygon Π), the surface constructed above, and we denote it by $S(\Pi)$.*

The four maps $\sigma_{a,b}$ glue to a map $\mu : S(\Pi) \to \Pi$ which is a fourfold ramified covering. The ramifications take place along the broken edges in the following way: A point in the interior of a broken edge lifts to two points, and an endpoint of a broken edge lifts to one point.

$$\begin{tikzcd} \bigsqcup_{a,b}(\sigma_{a,b}\cdot\Pi) \arrow[r, "q"] \arrow[dr, "\pi"'] & S(\Pi) \arrow[d, "\mu"] \\ & \Pi \end{tikzcd}$$

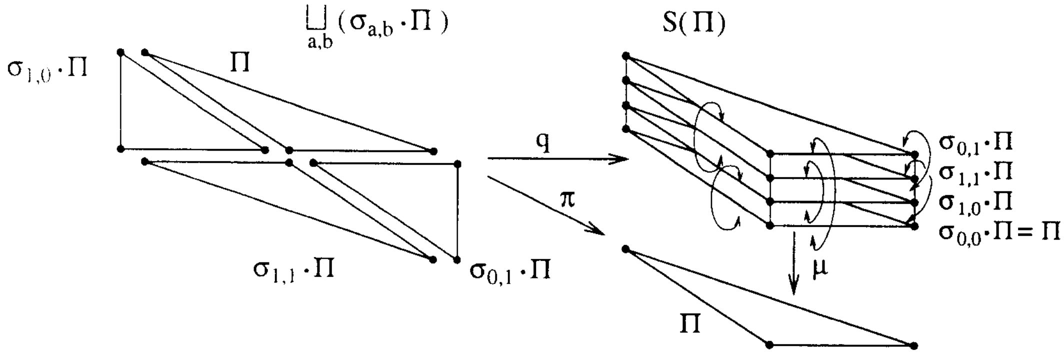

Fig. 1. The fourfold ramified covering structure of $S(\Pi)$.

Remark 9 *If the polygon Π is convex and the broken edges are just the edges of Π, then the ambient surface $S(\Pi)$ is a topological model of the real projective toric surface $X(\Pi)$ whose image by the moment map is Π. If Π is convex but some broken edge is the union of at least two edges of Π, then $X(\Pi)$ contains some conic points and $S(\Pi)$ is a desingularization of $X(\Pi)$. The following example will be used later. Let d be a positive integer and $\mathbf{T}_d$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(d,0)$ and $(0,d)$ then it is well known that $X(\mathbf{T}_d) = \mathbb{R}P^2$. Let's check that $S(\mathbf{T}_d)$ is homeomorphic to $\mathbb{R}P^2$: Let D_d be the* diamond *in $\mathbb{R}^2$ equal to the union of the four copies $(\sigma_{a,b}\cdot T_d)$, $a,b\in\{0,1\}$. Then $S(T_d)$ is obtained from D_d by identifying every two opposite points on its boundary (see fig. 2). As another example let $\tilde{\mathbf{T}}_d$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2d,0)$ and $(0,d)$, then $X(\tilde{\mathbf{T}}_d)$ is a pinched torus. Indeed the chart at $(0,d)$ gives an affine cone, and the charts at $(0,0)$ and $(2d,0)$ give affine planes. It is easy to check that $S(\Pi)$ is a sphere. The topology of an arbitrary ambient surface is described in proposition 21.*

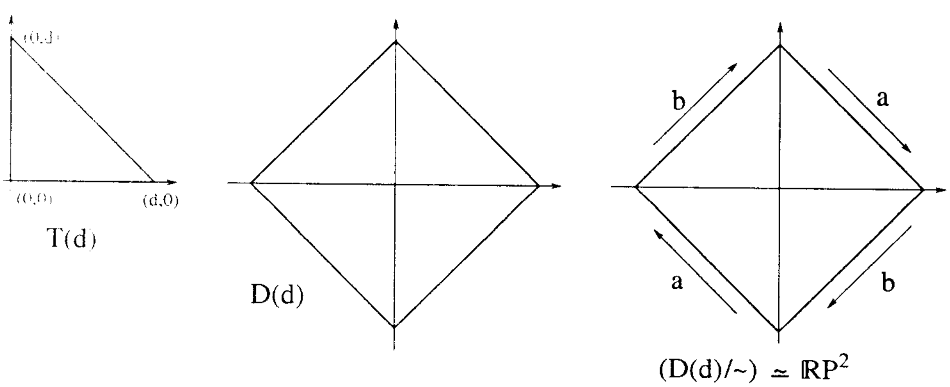

Fig. 2. Recovering the real projective plane $\mathbb{R}P^2$ from the triangle T_d.

Definition 10 *The image of a copy $(\sigma_{a,b}\cdot\Pi)$ by the quotient map $q:\bigsqcup_{a,b}(\sigma_{a,b}\cdot\Pi)\to S(\Pi)$ is a* quadrant *of $S(\Pi)$. Since q is one-to-one from a given copy*

$(\sigma_{a,b}\cdot\Pi)$ *to its image, we will identify (abusively) a quadrant with its pre-image* $(\sigma_{a,b}\cdot\Pi)$.

1.2.2 The Local Structure around the Lift of a Vertex

Let u be a vertex of Π, and ℓ and ℓ' be the two edges of Π meeting at u.

Assume first that u is of even parity, so ℓ and ℓ' are of same parity. The construction of $S(\Pi)$ (see def. 8) implies then that two copies of Π will be glued to one another by identifying the two corresponding copies of $\ell \cup \ell'$, and the two other copies of Π will be glued to one another by identifying the two other copies of $\ell \cup \ell'$ (see fig. 3).

Assume now that u is of odd parity. The construction of $S(\Pi)$ (see def. 8) implies that the four copies of Π are glued to one another in the following way: The union of ℓ with a reflection of ℓ through a coordinate axis is identified to the union of the two other copies of ℓ and the union of ℓ' with a reflection of ℓ' through the other coordinate axis is identified to the union of the two other copies of ℓ' (see fig. 3).

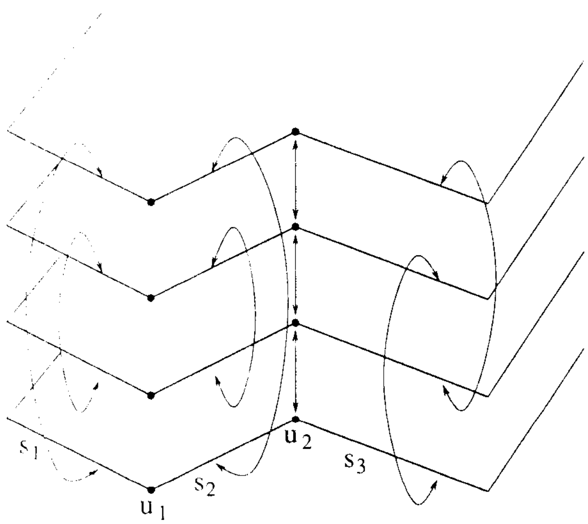

Fig. 3. How the segments in the lift of a broken edge are identified two by two according to their parity.

If all the segments of $\partial\Pi$ are of the same parity (that is if Π has only one broken edge), then two copies of Π will be glued to one another by all their edges, and the two other copies of Π will be also glued to one another by all their edges. We get this way two connected components, each homeomorphic to a sphere (see an example on fig. 4).

Assume now that the parity of the segments of $\partial\Pi$ takes at least two values. Notice that since the two endpoints u and u' of a broken edge ℓ are vertices of odd parity, they lift each to only one point in $S(\Pi)$. All the other points of

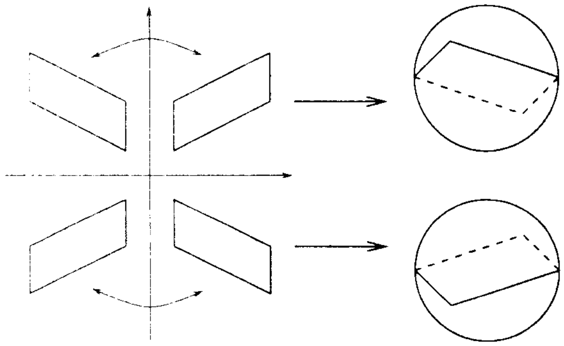

Fig. 4. Gluing the four copies of Π to one another gives a union of two spheres.

ℓ lift to two points. Therefore the lift $\mu^{-1}(\ell)$ of ℓ in $S(\Pi)$ is a circle which is a ramified twofold covering of ℓ, where the ramifications take place in the lifts $\mu^{-1}(u)$ and $\mu^{-1}(u')$.

1.2.3 *Canonical Charts for the Ambient Surface*

Let $\ell_1, \ldots, \ell_r$ be the broken edges of Π, indexed such that ℓ_i meets ℓ_{i+1}, i mod r, and let u_i be the vertex of Π where they meet. Let

$$U_i = S(\Pi) \setminus \Big(\bigcup_{\substack{j \neq i \\ j \neq i+1}} \mu^{-1}(\ell_j) \Big)$$

So U_i is an open neighborhood of $\mu^{-1}(u_i)$ in $S(\Pi)$ and is homeomorphic to $\mathbb{R}^2$. We will assume that the broken edges are oriented locally near their endpoints, away from their endpoints. So $\mu^{-1}(\ell_i)$ and $\mu^{-1}(\ell_{i+1})$ are oriented in U_i.

Definition 11 *The open neighborhood U_i equipped with a homeomorphism $U_i \to \mathbb{R}^2$ mapping $\mu^{-1}(\ell_i)$ and $\mu^{-1}(\ell_{i+1})$, with their orientations, onto the coordinate axis Ox and Oy is a* (oriented) chart of $S(\Pi)$. *The system of charts $U_1, \ldots, U_r$ will be called a* canonical system of charts *for* $S(\Pi)$.

Notice that the chart U_i tells how to glue the four quadrants of $S(\Pi)$ onto $\mu^{-1}(\ell_i)$ and $\mu^{-1}(\ell_{i+1})$.

Definition 12 *We call* (open) quadrants of the chart U_i *and denote $U_i^{0,0}$, $U_i^{0,1}$, $U_i^{1,1}$, and $U_i^{1,0}$ the connected components of $U_i \setminus (\mu^{-1}(\ell_i) \cup \mu^{-1}(\ell_{i+1}))$ such that $U_i^{a,b}$ is mapped to the set $\{(x,y) \in \mathbb{R}^2, (-1)^a x > 0, (-1)^b y > 0\}$ by the homeomorphism $U_i \to \mathbb{R}^2$.*

Notice that the closure in $S(\Pi)$ of an open quadrant $U_i^{a,b}$ is a quadrant $(\sigma_{c,d} \cdot \Pi)$ of $S(\Pi)$. So the chart is determined by the correspondence $(c,d) \mapsto (a,b)$.

Definition 13 *We call* the parity matrix of the chart U_i, *the matrix $M_i =$*

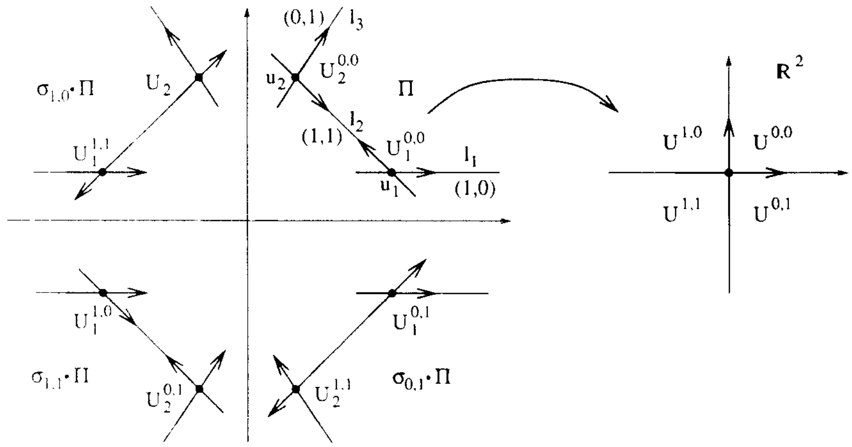

Fig. 5. A local orientation of all the broken edges determines the label of the quadrants by the $U_i^{a,b}$. The parity of the broken edges ℓ_i is in parentheses.

$\begin{pmatrix}\alpha_i & \alpha_{i+1}\\ \beta_i & \beta_{i+1}\end{pmatrix}$ *where* (α_j, β_j) *is the parity of the segments of the broken edge* ℓ_j.

Lemma 14 *If the closed quadrant* $\bar{U}_i^{a,b}$ *of the chart* U_i *is equal to the quadrant* $(\sigma_{c,d}\cdot\Pi)$ *of* $S(\Pi)$, *then* $(a,b) = ((c,d)\cdot M_i)$.

PROOF. With our choice of local orientations for the broken edges, $\bar{U}_i^{0,0}$ is always equal to $\Pi = (\sigma_{0,0}\cdot\Pi)$, so the permutation $(c,d)\mapsto(a,b)$ can be considered as an element of $GL(2,\mathbb{Z}_2)$. Since $\bar{U}_i^{0,1}$ and $\bar{U}_i^{1,0}$ correspond to the quadrants $(\sigma_{c,d}\cdot\Pi)$ glued to quadrant Π along $\mu^{-1}(\ell)_i$ and $\mu^{-1}(\ell)_{i+1}$ respectively, we get from the construction of $S(\Pi)$ (see def. 8) that

$$(c,d) = \begin{cases} (\beta_i,\alpha_i) & \text{if}\,(a,b) = (0,1)\\ (\beta_{i+1},\alpha_{i+1}) & \text{if}\,(a,b)=(1,0)\end{cases}$$

which is equivalent to $(c,d) = (a,b)\cdot M_i^{-1}$. $\square$

Notice that the transformations $M_i \mapsto M_j$ correspond to the gluing of chart U_i with chart U_j.

Definition 15 *We will call* $M_{i,j}: M_i\mapsto M_j$ *the* gluing transformation *of the parity matrices* M_i *and* M_j.

1.2.4 Gluing Charts of the Ambient Surface

Assume now that Π has more than two broken edges. Let $U_1,\ldots,U_r$ be a canonical system of charts of $S(\Pi)$.

Then the gluing of two consecutive charts U_i and U_{i+1} is easily stated:

Lemma 16 *The gluing transformation $M_{i,i-1}$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & \eta_i \end{pmatrix}$ and the right-product: $M_i = M_{i-1} \cdot M_{i,i-1}$, where $\eta_i = 0$ if the broken edge ℓ_i has even parity, and $\eta_i = 1$ if ℓ_i has odd parity.*

PROOF. We assume without loss of generality that $i = 2$, so we will compute with:

$$M_1 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \qquad \text{and} \qquad M_2 = \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}$$

Notice that the parities (α_1, β_1) and (α_3, β_3) of the segments of the broken edges ℓ_1 and ℓ_3 are equal if and only if ℓ_2 has even parity, so the lemma holds when $\eta_2 = 0$. Assume that $(\alpha_3, \beta_3) \neq (\alpha_1, \beta_1)$, so $\eta_2 = 1$. Since the parities belong to $(\mathbb{Z}_2)^2$ and are never even, and since $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, then $(\alpha_3, \beta_3) = (\alpha_1, \beta_1) + (\alpha_2, \beta_2)$. □

Notice that if a polygon Π has only two broken edges, then the canonical system of charts for $S(\Pi)$ has only two charts. Since the parity of the segments of $\partial\Pi$ takes only two values the gluing of the two charts gives a surface $S(\Pi)$ which is a sphere (see fig. 6).

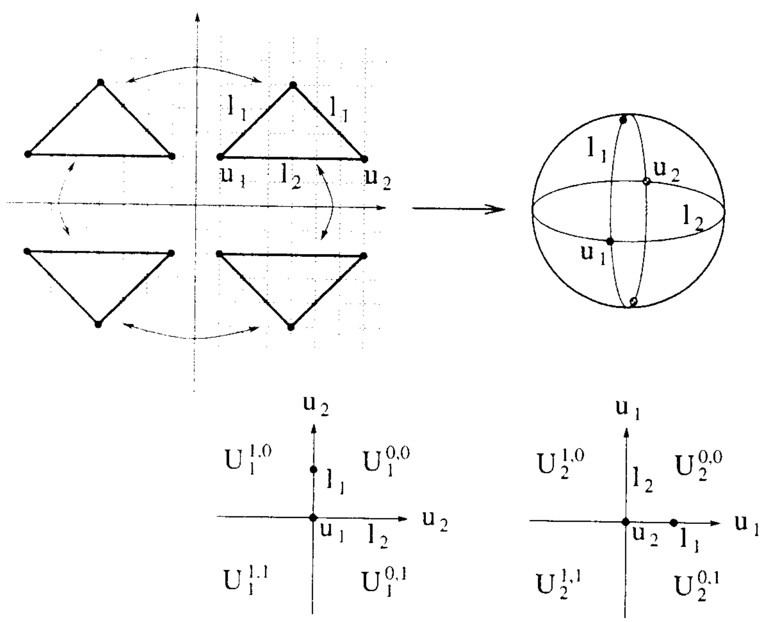

Fig. 6. Gluing the four copies of Π to one another gives a sphere.

Notice that the gluing transformation of any parity matrices is equal to the product of the transformations of consecutive parity matrices:

$$M_{i,j} = \prod_{k=0}^{k=j-i-1} M_{i+k,i+k+1}$$

Definition 17 *The canonical system of charts U_i equipped with the gluing transformations $M_{i,j}$ is the* canonical atlas structure *of $S(\Pi)$.*

Corollary 18 *If ℓ_i has even parity, then a tubular neighborhood of $\mu^{-1}(\ell_i)$ is an annulus.*
If ℓ_i has odd parity, then a tubular neighborhood of $\mu^{-1}(\ell_i)$ is a Moebius band.

PROOF. With our choice of local orientations for the broken edges of Π we have always

$$U_i^{0,0} = U_{i-1}^{0,0} \quad \text{and} \quad U_i^{0,1} = U_{i-1}^{1,0}$$

and either

$$U_i^{1,0} = U_{i-1}^{0,1} \quad \text{and} \quad U_i^{1,1} = U_{i-1}^{1,1}$$

In which case the tubular neighborhood of ℓ_i is an annulus (see fig. 7), or

$$U_i^{1,0} = U_{i-1}^{1,1} \quad \text{and} \quad U_i^{1,1} = U_{i-1}^{0,1}$$

In which case the tubular neighborhood of ℓ_i is a Moebius band (see fig. 7).

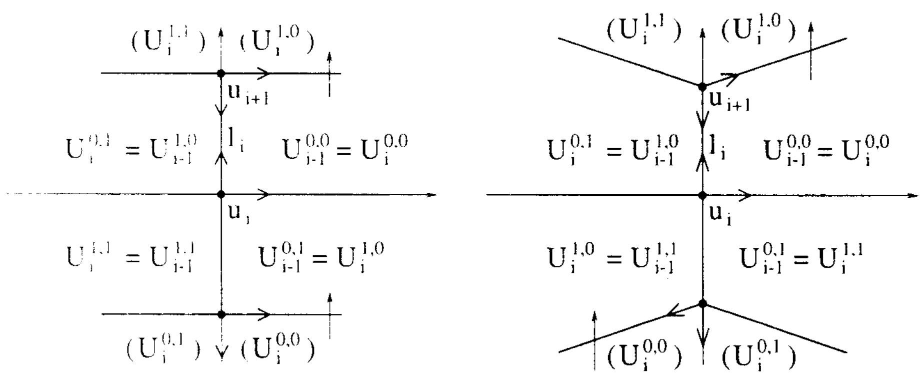

Fig. 7. The tubular neighborhood of the lift $\mu^{-1}(\ell)$ of a broken edge ℓ is either an annulus if ℓ is even, either a Moebius band if ℓ is odd.

The closed quadrants $\bar{U}^{0,0}$ are always represented by Π. Let $(\sigma_{c,d} \cdot \Pi)$ be the quadrant representing $\bar{U}_{i-1}^{0,1}$, so $(0,1) = (c,d) \cdot M_{i-1}$. Now $(\sigma_{c,d} \cdot \Pi)$ represents also a closed quadrant $\bar{U}_i^{a,b}$, so $(a,b) = (c,d) \cdot M_i$. Since $M_i = M_{i-1} \cdot M_{i,i-1}$ we get that $(a,b) = (0,1) \cdot M_{i,i-1}$. From lemma 16 we get directly that $(a,b) = (1,0)$ if ℓ_i is even (hence we get an annulus), and $(a,b) = (1,1)$ if ℓ_i is odd (hence we get a Moebius band). $\square$

Notice that, given a canonical system of charts, the gluing transformations of consecutive charts are determined by the 0–1 sequence of the numbers η_i of lemma 16. So two polygons with the same sequence of η_i's up to circular permutation give rise to homeomorphic ambient surfaces.

1.3 The Global Topology of an Ambient Surface

1.3.1 A Basis for the 1-Homology of the Ambient Surface

Let $S(\Pi)$ be an ambient surface, and let $U_1, \ldots, U_r$ be a canonical system of charts on $S(\Pi)$. We have seen that when $r = 1$, then $S(\Pi)$ is a homeomorphic to the disjoint union of two spheres (4), and when $r = 2$, then $S(\Pi)$ is homeomorphic to a single sphere (fig. 6). Assume now that $r \geq 3$. Recall that with the notations of the charts, the ℓ_i's are the broken edges of Π with endpoints u_{i-1} and u_i.

Lemma 19 *The lifts $\mu^{-1}(\ell_3), \ldots, \mu^{-1}(\ell_r)$ of the broken edges $\ell_3, \ldots, \ell_r$ form a basis for the 1-homology space $H_1(S(\Pi))$ (with coefficients in $\mathbb{Z}$ if $S(\Pi)$ is orientable, and in $\mathbb{Z}_2$ if $S(\Pi)$ is not orientable).*

PROOF. To prove this lemma it suffices to check two points:

- The set $\{\mu^{-1}(\ell_3), \ldots, \mu^{-1}(\ell_r)\}$ is free in H_1 since $S(\Pi) \setminus (\cup_3^r \mu^{-1}(\ell_i))$ is connected.
- The set $\{\mu^{-1}(\ell_3), \ldots, \mu^{-1}(\ell_r)\}$ is complete since $S(\Pi) \setminus (\cup_3^r \mu^{-1}(\ell_i))$ is simply connected.

□

1.3.2 The Topology of Ambient Surfaces

Assume here that Π has at least two broken edges (recall that if it has only one, $S(\Pi)$ is homeomorphic to the disjoint union of two spheres).

Lemma 20
- *If Π has a broken edge of odd parity, then it has two broken edges of odd parity.*
- *If Π has two consecutive broken edges of odd parity, then it has three broken edges of odd parity.*

PROOF. Let $U_1, \ldots, U_r$ be a canonical system of charts on the surface $S(\Pi)$. From lemma 16, we get that, to each broken edge of odd parity, correspond a gluing matrix $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and to each broken edge of even parity correspond a gluing matrix $A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The product $\prod$ of all gluing matrices of consecutive charts must be the identity. The product with A_0 just permutes the columns. So if $\prod$ has a matrix A_1 as factor then it must have at least two matrices A_1 as factors. Since $A_1^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, if Π has two matrices A_1 as consecutive factors, then it must have at least three matrices A_1 as factors. $\square$

Proposition 21 *Let $r > 1$ be the number of broken edges of Π.*

- *The surface $S(\Pi)$ is orientable if and only if the parity of the segments of $\partial\Pi$ takes only two values.*
- *If $S(\Pi)$ is orientable, then r is even, and $S(\Pi)$ is the connected sum of $r/2 - 1$ tori.*
- *If $S(\Pi)$ is not orientable, then it is the connected sum of $r - 2$ projective planes.*

PROOF. From corollary 18 we know that the tubular neighborhoods of the lifts of the broken edges of Π are all annuli if and only if all the broken edges are of even parity. From lemma 19, we know that the lifts of all the broken edges but two consecutive ones is a basis of the 1-dimensional homology space of $S(\Pi)$. From lemma 20 we know that if all but two consecutive broken edges are known to be of even parity, then all the broken edges are of even parity.

Therefore $S(\Pi)$ is orientable if and only if all its broken edges are of even parity. From definition 7 it is clear that the broken edges are all of even parity if and only if the parity of the segments of $\partial\Pi$ takes only two values.

From lemma 19 we know that the dimension of the 1-dimensional homology space of $S(\Pi)$ is $r - 2$. Since $r > 1$, the parity of the segments of $\partial\Pi$ takes more than one value, and then $S(\Pi)$ is connected. So if $S(\Pi)$ is orientable, it is the connected sum of $(r-2)/2$ tori, and if it is non-orientable, it is the connected sum of $r - 2$ projective planes. $\square$

Remark 22 *Notice that in the construction of an ambient surface, the requirement that the polygon be homeomorphic to a disk can be dropped. We could also construct ambient surfaces from generalized polygons, the boundary of which would be a collection of integral closed polygonal lines. However we tried to avoid here a too wide generalization. This is motivated by the necessity of this requirement for the ambient surfaces we use in [4] to construct maximal T-curves. The construction of ambient surfaces can also be generalized to higher dimensional* ambient varieties *by using piecewise integral linear subsets of $\mathbb{R}^n$. We may require certain constraints on these subsets. For instance we could ask that only n faces of dimensions $n - 1$ meet at each vertex (so we keep a nice local topological structure), or that each n-face be homeomorphic to a n-ball, etc.*

2 The T-Curves

2.1 T-Curves on an Ambient Surface

Definition 23 *A* primitive triangulation *of a polygon* Π *is a triangulation of the polygon such that the set of vertices of the triangulation is equal to the set of integer points* $\Pi \cap \mathbb{Z}^2$.

Choose a polygon $\Pi \subset \mathbb{R}^2_{\geqslant 0}$, and let $S(\Pi)$ be our ambient surface. We now define a T-curve on $S(\Pi)$ in five steps:

- Choose a primitive triangulation $\mathcal{T}$ of Π. The reflections through the two coordinate axis generate a triangulation of the disjoint union $\bigsqcup_{a,b}(\sigma_{a,b} \cdot \Pi)$, $a, b \in \{0, 1\}$, which induces a triangulation of $S(\Pi)$. With the notations of section 1.2.1 the triangulation of $\bigsqcup_{a,b}(\sigma_{a,b} \cdot \Pi)$ is $\pi^{-1}(\mathcal{T})$, and the triangulation of $S(\Pi)$ is $\mu^{-1}(\mathcal{T})$.
- Choose a sign function $\Pi \cap \mathbb{Z}^2 \to \{+1, -1\}$, $(x, y) \mapsto \delta(x, y)$. We extend this distribution on $\bigsqcup_{a,b}(\sigma_{a,b} \cdot \Pi) \cap \mathbb{Z}^2$ in the following way:

$$\delta((-1)^a x, (-1)^b y) = (-1)^{ax+by}\delta(x, y)$$

Or equivalently:

$$\delta(\sigma_{a,b} \cdot (x, y)) = (-1)^{\langle (a,b),(c,d) \rangle}\delta(x, y) \tag{1}$$

  Where $a, b \in \{0, 1\}$, (c, d) is the parity of the point (x, y), and $\langle (a, b), (c, d) \rangle = ac + bd$.
- We assign to each edge of the triangulation $\pi^{-1}(\mathcal{T})$ the sign equal to the product of the signs of its endpoints. This induces a sign function on the edges $\mu^{-1}(\mathcal{T})$. Indeed for an arbitrary edge e of $\pi^{-1}(\mathcal{T})$ which lies on the boundary of $\pi^{-1}(\Pi)$, we check that the sign of the edge $\sigma_{c,d}$ to which e becomes identified in $\mu^{-1}(\mathcal{T})$ is the same than the sign of e. From the construction of $S(\Pi)$ (see def. 8) we get that the parity (a, b) of e and (c, d) verify $\langle (a, b), (c, d) \rangle = 0$, and from formula 1 above we get that

$$\text{sign}(\sigma_{c,d} \cdot e) = (-1)^{\langle (a,b),(c,d) \rangle}\,\text{sign}(e) \tag{2}$$

  which is then equal to sign(e).
- The sign function on the edges of $\mu^{-1}(\mathcal{T})$ has the property that each triangle has either 0 or 2 edges of negative sign. Indeed the product of the three signs of the edges of a triangle is equal to $+1$ since it is the product of the squares of the three signs of the vertices of the triangle (see fig. 8).
- Consider a cell decomposition of $S(\Pi)$ dual to the triangulation $\mu^{-1}(\mathcal{T})$, and let G be the graph underlying its 1-skeleton. So each edge of G is dual to an edge of $\mu^{-1}(\mathcal{T})$. We assign to the former edge the sign of the latter edge. From the previous step we deduce that each vertex of G is the endpoint of exactly two edges of negative sign and one edge of positive sign. Therefore

the union of the edges of negative sign form a closed curve K on G (and hence on $S(\Pi)$).

Notice that the data $(\Pi, \mathcal{T}, \delta)$ and $(\Pi, \mathcal{T}, -\delta)$ define the same curves.

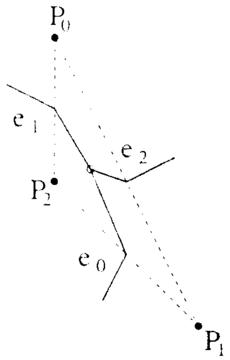

Fig. 8. $\text{sign}(e_i) = \delta(P_j)\delta(P_k)$. so $\prod \text{sign}(e_i) = \prod \delta(P_i)^2 = 1$. Therefore either 0 or 2 edges have negative sign.

Definition 24 *The* T-curve *defined by the data $(\Pi, \mathcal{T}, \pm\delta)$ is the curve constructed as above. It will be denoted by $K(\Pi, \mathcal{T}, \delta)$, (or $K(\Pi, \mathcal{T}, -\delta)$).*

2.1.1 T-Curves and Algebraic Curves

Definition 25 *A triangulation of a convex polygon Π is said to be* convex, or regular, *if there exist a convex function $\nu : \Pi \to \mathbb{R}$ which is linear on each triangle, and non linear on any union of two triangle.*

Remark 26 *If Π is a convex polygon in $\mathbb{R}^2$, then we have seen in remark 9 that the ambient surface $S(\Pi)$ is a topological model of (i.e. is homeomorphic to) the real projective toric surface $X(\Pi)$ defined by Π. In fact the construction of T-curves was initially motivated by the construction of new real algebraic curves on real projective toric surfaces as shows* Viro's theorem*: Let $K = K(\Pi, \mathcal{T}, \delta)$ be a T-curve. If $\mathcal{T}$ is a convex triangulation, then K is congruent to a real algebraic curve with Newton polygon Π. For instance T-curves constructed from a convex triangulation of the triangle $\mathbf{T}_d$ (see remark 9) are congruent to real algebraic curves of degree d.*

Definition 27 *In the remark above, observe that the convex polygon Π is the Newton polygon of an algebraic curve. By analogy, we will call the polygon Π in the data defining a T-curve $K = K(\Pi, \mathcal{T}, \delta)$ the* Newton polygon *of K.*

2.1.2 A digression to higher dimensions

As we can generalize ambient surfaces to higher dimensional ambient varieties (see remark 22), we can also generalize the construction of T-curves to

T-hypersurfaces. The data defining a T-hypersurface is similar: A Newton polytope Π, a triangulation $\mathcal{T}$ of Π and a distribution of signs on the integer points of Π. The triangulation $\mathcal{T}$ is a cell decomposition of Π into simplexes where the vertices of the simplexes are all the integer points of the Newton polytope. Notice that simplexes of dimension more than two may have different volumes. We can add more constraints and require that all simplexes of highest dimension of the triangulation have same volume.

Notice that Viro's theorem (see remark 26) for T-curves requires that the convex function (see def. 25) over the Newton polygon have integer values on integer points. This is not a serious restriction. Indeed we can allow rational values on the integer points since there are finitely many integer points and we can multiply the function by the least common multiple of the denominators. A perturbation of the vertices of a three dimensional polytope does not destroy the incidence structure of the faces if it is small enough. Therefore we can also allow the function to take real values on the integer points since an arbitrary small perturbation of it will transform the real values into rational values.

The definition of convex triangulation of a convex polytope in dimension more than two is similar than in dimension two (see def. 25): There should exist a convex real valued function on the Newton polytope which is affine on each simplex and non affine on any union of two simplexes. Notice that in high dimensions it is not true that the incidence structure of the faces of a polytope is preserved under a small enough perturbation of the vertices. Therefore it is safer to require in the definition of a convex triangulation in higher dimension that the convex function takes rational values on the vertices.

The connection of T-hypersurfaces with real algebraic hypersurfaces on projective toric varieties relies again on the convexity of the triangulation of the Newton polytope. In fact Viro's theorem is stated in all dimensions: If the triangulation of the Newton polytope of the T-hypersurface is convex, then there exist a real algebraic hypersurface which is congruent to the T-hypersurface.

2.2 *Isomorphic T-curves*

2.2.1 *Translation of the Newton polygon*

Let Π be a polygon in $\mathbb{R}^2_{\geqslant 0}$, and (s,t) be an integral vector such that the translated polygon $\Pi' = \Pi + (s,t)$ lies also in $\mathbb{R}^2_{\geqslant 0}$, and let $K(\Pi, \mathcal{T}, \delta)$ be a T-curve. Let $\mathcal{T}'$ be the triangulation of Π' translated from $\mathcal{T}$, and let δ' be the sign distribution on $\Pi' \cap \mathbb{Z}^2$ defined by $\delta'(x,y) = \delta(x-s, y-t)$.

Proposition 28 *The T-curves $K = K(\Pi, \mathcal{T}, \delta)$ and $K' = K(\Pi', \mathcal{T}', \delta')$ are equal.*

PROOF. It is clear that the construction of the ambient surface $S(\Pi)$ depends on Π up to translation by an integral vector, so $S(\Pi') = S(\Pi)$. Let (c,d) be the parity of the point (s,t), and let $a,b \in \{0,1\}$. If $(x,y) \in \Pi$, then

$$\delta'((-1)^a(x+s),(-1)^b(y+t)) = (-1)^{\langle(a,b),(c,d)\rangle}\delta((-1)^a x,(-1)^b y)$$

Recall that the sign of an edge of a triangulation is the product of the signs of its two endpoints. Therefore the edges of the triangulation of $S(\Pi')$ have the same sign than the corresponding edges of the triangulation of $S(\Pi)$, so $K = K'$. $\square$

2.2.2 Linear Transformations of the Newton Polygon

If K is a T-curve with Newton polygon Π, and $c,d \in \{0,1\}$, let $K^{c,d}$ be the restriction of K to the quadrant $(\sigma_{c,d} \cdot \Pi)$.

Let $K(\Pi,\mathcal{T},\delta)$ be a T-curve and let $A \in GL(2,\mathbb{Z})$. We will note A_2 the reduction of A in $GL(2,\mathbb{Z}_2)$. Let $\mathcal{T}'$ be the triangulation of $(A \cdot \Pi)$ resulting from applying A to $\mathcal{T}$, let $\delta'(x,y) = \delta(A^{-1}\cdot(x,y))$ and let $A \cdot K$ be the T-curve defined by $(A \cdot \Pi, \mathcal{T}', \delta')$.

Proposition 29 • *$A \cdot K$ and K are congruent.*
- *The homeomorphism $S(\Pi) \to S(A \cdot \Pi)$ which transforms K to $A \cdot K$, transforms each restriction $K^{c,d}$ into a restriction $(A \cdot K)^{s,t}$.*
- *$(c,d) = (s,t) \cdot A_2$.*

PROOF. The action of A_2 on parities underlies the action of A, so an edge e of the triangulation $\mathcal{T}$ is of even parity if and only if the edge $A \cdot e$ of $\mathcal{T}'$ is of even parity. So it is clear from prop. 21 that $S(A \cdot \Pi)$ and $S(\Pi)$ are homeomorphic.

Let U and U' be canonical charts of $S(\Pi)$ and of $S(A \cdot \Pi)$ around an arbitrary vertex (of odd parity) u and around $A \cdot u$ respectively. For any $a,b \in \{0,1\}$, let $(\sigma_{c,d} \cdot \Pi)$ be the quadrant of $S(\Pi)$ representing $\bar{U}^{a,b}$, and $(\sigma_{s,t} \cdot (A \cdot \Pi))$ be the quadrant of $S(A \cdot \Pi)$ representing $(\bar{U}')^{a,b}$. Let M be the parity matrix of U. Then $(A_2 \cdot M)$ is the parity matrix of U'. From lemma 14 we get that

$$(a,b) = (c,d) \cdot M \quad \text{and} \quad (a,b) = (s,t) \cdot (A_2 \cdot M)$$

Therefore $(c,d) = (s,t) \cdot A_2$.

To prove the proposition, we must show that the T-curve K is congruent to $A \cdot K$ by a homeomorphism $U \to U'$ such that for any $a,b \in \{0,1\}$, $U^{a,b} \mapsto U'^{a,b}$. It suffices then to show that the sign of an edge of the triangulation of

$(\sigma_{c,d} \cdot \Pi)$ is equal to the sign of the corresponding edge of the triangulation of $(\sigma_{s,t} \cdot (A \cdot \Pi))$:

Let e be a segment of $\partial\Pi$ and (a,b) its parity. So $((a,b) \cdot {}^tA_2)$ is the parity of $(A \cdot e)$. From the definition of the sign distribution δ' of $(A \cdot K)$, we get that the sign of $(A \cdot e)$ is equal to the sign of e. Then from the construction in section 1.2.1 we get that

$$\begin{aligned}
\mathrm{sign}(\sigma_{s,t} \cdot (A \cdot e)) &= (-1)^{\langle (s,t),(a,b)\cdot {}^tA_2\rangle}\,\mathrm{sign}(A \cdot e)\\
&= (-1)^{\langle (s,t)\cdot A_2,(a,b)\rangle}\,\mathrm{sign}(e)\\
&= (-1)^{\langle (c,d),(a,b)\rangle}\,\mathrm{sign}(e)\\
&= \mathrm{sign}(\sigma_{c,d} \cdot e)
\end{aligned}$$

□

2.3 T-Curves on $\mathbb{R}P^2$

Definition 30 *A* primitive segment *is a segment such that the only integer points it contains are its two endpoints. The* integral length *(or simply* length*) of a polygonal line is the number of primitive segments contained in it. Two integer points are* neighbors *if they can be connected by a primitive segment.*

Definition 31 *A piece of a curve homeomorphic to the segment $[0,1] \subset \mathbb{R}$ is called an* arc *of the curve.*

Definition 32 *Let K be a collection of disjoint embedded circles in a surface. A connected component of K is called an* oval *if it bounds a subset of the surface homeomorphic to a disk. The interior of this subset is called the* inside *of the oval. The interior of the complementary of the inside is the* outside *of the oval. An oval with no other ovals inside will be called an* empty oval, *and an oval inside no other ovals will be called an* outermost oval. *A connected component of K which is not an oval is called a* nontrivial component *of K.*

Recall from remark 9 that $\mathbf{T}_d$ is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(d,0)$ and $(0,d)$, and that $S(\mathbf{T}_d)$ is a model of the projective plane.

Proposition 33 *Let $K(\mathbf{T}_d, \mathcal{T}, \delta)$ be a T-curve on $\mathbb{R}P^2$. Then*

- *K has only ovals as connected components if its degree d is even.*
- *K has one and only one nontrivial component if its degree d is odd.*

PROOF. It is clear that on $\mathbb{R}P^2$ two nontrivial embedded circles which intersect transversally, intersect an odd number of time, and an oval which is intersected by any embedded circle transversally is intersected an even number of times. Therefore, since K is a disjoint union of embedded circles, it has at most one nontrivial connected component.

Consider now a simple path on the diamond $D(d)$ through the edges of the triangulation $\mathcal{T}$, going from one point of the boundary of the diamond to its opposite point. This path lifts to a loop on RP^2 which is a nontrivial embedded circle and if it cuts K, it cuts it transversally. So the loop intersects K an odd number of times if and only if K has a nontrivial connected component.

Let us follow the path from one endpoint to the other. If the loop intersects the T-curve r times, then the integer points on the path change signs r times. So r is odd if and only if the two endpoints of the path have opposite signs. The formula 1 page 14 for the extension of the sign distribution from the triangle $T(d)$ to the diamond $D(d)$ shows that two opposite points on the boundary of $D(d)$ have opposite signs if an only if the degree d of the T-curve is odd. □

Notice that this lemma holds if we replace "T-curve" by " nonsingular algebraic curve". In that case it can be proved in a similar way, which is a direct consequence of Bezout theorem applied to the curve and a generic line.

Notice now that if an oval O of a T-curve lies inside a quadrant of the ambient surface then all the integer points lying inside O and outside any oval which lies inside O have same sign. Therefore the following definition is coherent.

Definition 34 *Let O be an oval of a T-curve, which lies inside a quadrant of the ambient surface. The* sign of the oval O *is the sign of the integer points lying inside O and outside any oval which lies inside O.*

3 The T-Filling of a T-Curve.

Recall from definition 24 that a T-curve $K(\Pi, \mathcal{T}, \delta)$ lies on the graph G which underlies the 1-skeleton of a dual cell decomposition of the triangulation $\mathcal{T}_S$ of the ambient surface $S = S(\Pi)$. We note here $\mathcal{T}_S$ for the triangulation of S (that is $\mathcal{T}_S = \mu^{-1}(\mathcal{T})$ where μ is the projection $S \to \Pi$). The dual cell decomposition we use here is the one which can be refined into the barycentric cell decomposition of $\mathcal{T}_S$. Therefore an edge of G is the union of two segments linking the barycenters of two adjacent triangles t and t' and passing through the middle of the edge $t \cap t'$ of $\mathcal{T}_S$.

Definition 35 *The* incidence graph of the triangulation $\mathcal{T}_S$ *is the graph refined from G by subdividing the edges of G at the points of $G \cap \mathcal{T}_S$, (so the* *edges are now segments). We note this graph $G(S, \mathcal{T}_S)$ or simply*[1] *$G(S)$. The image by the projection μ of $G(S)$ will be called the* incidence graph of the triangulation *$\mathcal{T}$ and will be noted $G(\Pi, \mathcal{T})$ or simply $G(\Pi)$. Hence $G(S)$ is equal to the lift $\mu^{-1}(G(\Pi))$.*

In accordance with section 2.1, every edge of $G(S)$ inherit of the sign of the supporting edge of G (that is the sign of the edge of $\mathcal{T}$ it intersects). Notice that each edge of $G(\Pi)$ lifts to four edges of $G(S)$.

Lemma 36 *For each edge of $G(\Pi)$, there is exactly two edges, among its four lifts in $G(S)$, which have negative signs.*

PROOF. Let e be an edge of $\mathcal{T}$, and let (a, b) be its parity. Recall the formula 2 page 14 which gives the sign of a copy $(\sigma_{c,d} \cdot e)$ of e:

$$\text{sign}(\sigma_{c,d} \cdot e) = (-1)^{\langle (a,b),(c,d) \rangle} \, \text{sign}(e)$$

Notice that $\langle (a, b), (c, d) \rangle = 0$ if and only if $(c, d) = (0, 0)$ or $(c, d) = (b, a)$. Since the parity of a segment is never even, $(b, a) \neq (0, 0)$. Therefore there is exactly two edges, among the four which lift from e, which have same sign than e (so there is also two edges which have opposite sign than e). This proves the assertion, since every edge of $G(\Pi)$ inherits of the sign of the edge of $\mathcal{T}$ that it intersects. □

3.1 The Construction

Let $K = K(\Pi, \mathcal{T}, \delta)$ be a T-curve and $S = S(\Pi)$ its ambient surface. Each component $\tilde{\gamma}$ of K is a cycle (in the language of graph theory) in the incidence graph $G(S)$, i.e. $\tilde{\gamma}$ is a cyclic sequence of edges of $G(\Pi)$, each edge sharing one end with the previous edge and the other end with the next edge. These cycles don't intersect. The projection $\gamma = \mu(\tilde{\gamma})$ on Π will be considered as the cycle made up by the projections of the edges of $\tilde{\gamma}$. These cycles may intersect.

Definition 37 *a* thick-Y *is a tubular neighborhood of the three edges of $G(\Pi)$ lying in a triangle of the triangulation $\mathcal{T}$.*

More precisely let e_1, e_2 and e_3 be the three edges of $G(\Pi)$ lying in a triangle t of $\mathcal{T}$, and indexed counterclockwise, and let A and B_i be the endpoints of e_i. The thick-Y is obtained by gluing the three ribbons $r_i = e_i \times [-1, +1]$ by the following way: identify $A \times [-1, 0]$ in r_i with $A \times [0, +1]$ in r_{i+1} by $(A, x) \mapsto (A, -x)$ (see fig. 10).

[1] The notation is a little abusive since G is also a graph on S but acceptable since G and $G(S)$ have same geometric realization (the unions of their edges are equal).

From lemma 36 we get that every edge of $G(\Pi)$ is used twice by the cycles γ (see fig. 9). Then every union of two edges of $G(\Pi)$ which lie in a triangle is an arc of a cycle γ. In other words every cycle γ is the union of such arcs. We now describe how to glue adjacent thick-Y's together according to which arcs are consecutive on the cycles.

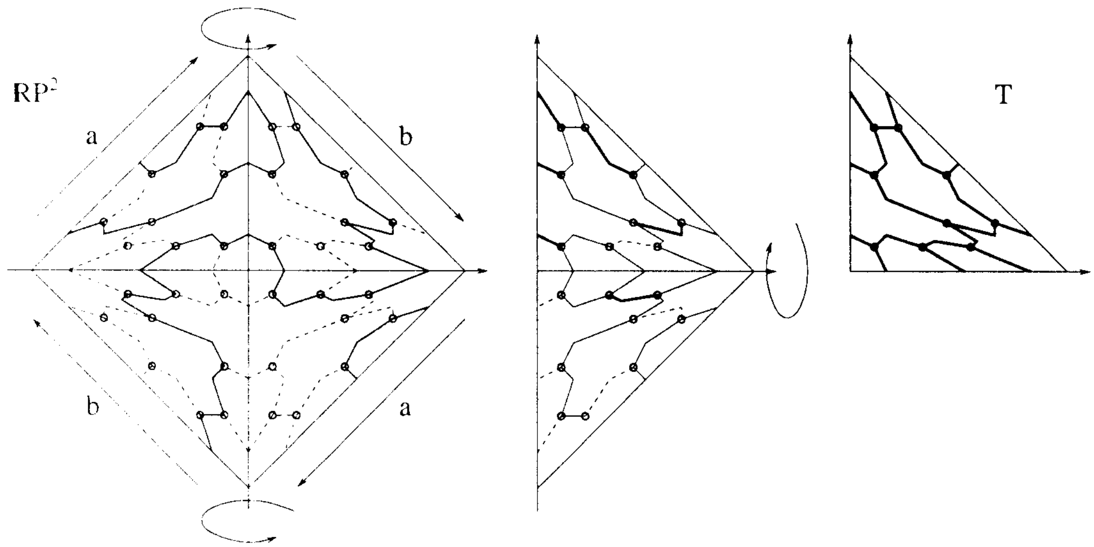

Fig. 9. The T-curve (plain lines) on the incidence graph $G(S)$ (dotted lines). The folded T-curve on $G(\mathbf{T}_d)$ uses every edge twice (thick plain lines).

Let t' be a triangle of $\mathcal{T}$ adjacent to t, and let e'_1, e'_2 and e'_3 be the three edges of $G(\Pi)$ lying in t', indexed counterclockwise, with endpoints A' and B'_i, and such that $B'_1 = B_1$. Let $s_1 = B_1 \times [-1, +1]$ be an end-segment of the thick-Y in t, and let $s'_1 = B'_1 \times [-1, +1]$ be the corresponding end-segment of the thick-Y in t'. From the disjoint union of the two thick-Y's in $t \cup t'$, we identify s_1 with s'_1:

(1) either by $(B_1, x) \mapsto (B'_1, -x)$ if $(e_2 \cup e_1)$ and $e'_1 \cup e'_3)$ are two consecutive arcs of some cycle γ. So $(e_3 \cup e_1)$ and $(e'_1 \cup e'_2)$ are two consecutive arcs of some cycle γ'.
(2) either by $(B_1, x) \mapsto (B'_1, x)$ if $(e_2 \cup e_1)$ and $(e'_1 \cup e'_2)$ are two consecutive arcs of some cycle γ. So $(e_3 \cup e_1)$ and $(e'_1 \cup e'_3)$ are two consecutive arcs of some cycle γ'.

Notice that in each case γ' may be the same cycle than γ.

Definition 38 *We will say that two thick-Y's are* glued with a twist *in the case (2) above, and* glued without a twist *in the case (1) above (see fig. 10).*

Since $G(\Pi)$ is connected, we obtain, by gluing in this way the thick-Y's of all the triangles of $\mathcal{T}$, a surface with boundary (see fig. 11).

Definition 39 *The surface with boundary constructed above from a T-curve*

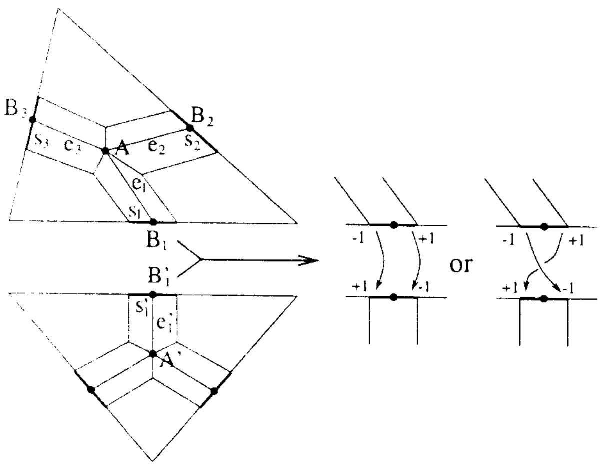

Fig. 10. A thick-Y is glued from three ribbons, and two thick-Y's are glued with or without a twist.

K will be called a T-filling, *and will be denoted $F(K)$.*

Notice from the construction of the T-filling $F(K)$, that the connected components of $\partial F(K)$ are in one-to-one correspondence with the connected components of K.

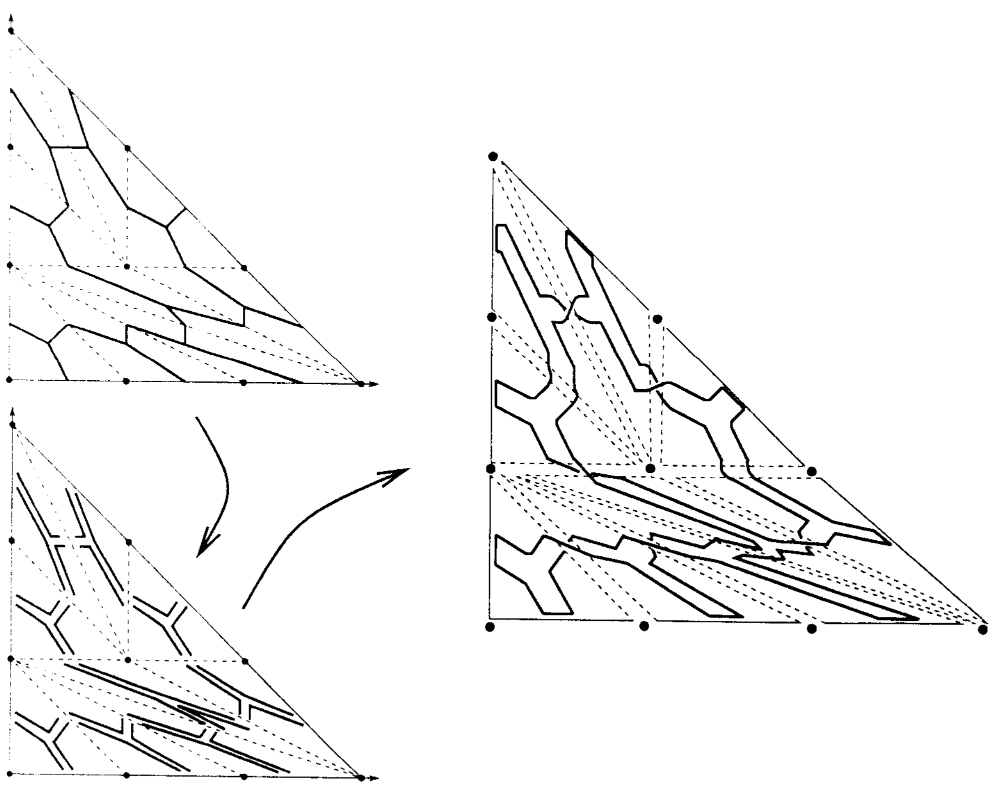

Fig. 11. To construct $F(K)$: Thicken the Y's in each triangle and glue any two adjacent thick-Y's with or without a twist (depending on K).

The T-filling of a T-curve is in fact a topological analog of the quotient of the complexification of an algebraic curve by the complex conjugation. Since the complexification of an algebraic curve is better known for curves on the projective plane than for curves on arbitrary projective toric surfaces, we restrict here the comparison to that case, bearing in mind that it holds in the more general case.

A real plane projective curve $C = C(f)$ of degree d is the subset $\{(x_0 : x_1 : x_2) \in \mathbb{R}P^2, f(x_0, x_1, x_2) = 0\}$ for some homogeneous polynomial f of degree d with real coefficients. The complexification $\mathbb{C}C$ of C is the surface $\{(z_0 : z_1 : z_2) \in \mathbb{C}P^2, f(z_0, z_1, z_2) = 0\}$ in $\mathbb{C}P^2$. The curve is non-(complex)-singular if the derivative is a nonzero function in all points of the curve (of its complexification). Notice that the complex conjugation leaves $\mathbb{C}C$ invariant, and that C is its set of fixed points. Therefore the quotient of $\mathbb{C}C$ by the complex conjugation is a surface with C as boundary.

Let $K = K(\mathbf{T}_d, \mathcal{T}, \delta)$ be a T-curve on $\mathbb{R}P^2 = S(\mathbf{T}_d)$ (see remark 9). We have seen in remark 26 that , when the triangulation $\mathcal{T}$ of T_d is convex, K correspond to a real plane projective non singular curve C of degree d.

It is well known that $\mathbb{C}C$ is an orientable surface of genus $\frac{(d-1)(d-2)}{2}$. Similarly the double of $F(K)$ along its boundary is a surface of genus $\frac{(d-1)(d-2)}{2}$. Indeed to double it we just have to let the (flat) thick-Y's become "hollow-Y's", glue them without worrying about the twists, and close the open ends on $\partial\mathbf{T}$ by disks (see fig. 12). The number of handles of this surface is equal to the number of interior integer points of $\mathbf{T}$, and this is precisely $\frac{(d-1)(d-2)}{2}$ (see fig. 12). Moreover the number of connected components of the boundary is the number of connected components of K. Therefore $F(K)$ is homeomorphic to the quotient of $\mathbb{C}C$ by the complex conjugation.

The analogy goes further. The surface quotient of $\mathbb{C}C$ by the complex conjugation allows to define the orientability and, if it is orientable, an orientation of C. Similarly a notion of orientability and orientation of K emerges naturally from $F(K)$ (see section 3.3.3).

Remark 40 *If the T-filling of a T-surface is to be homeomorphic to the quotient of the complexification of a real algebraic surface, it is of dimension four. It is not clear how to construct this four dimensional variety in a similar way than the T-filling of a T-curve.*

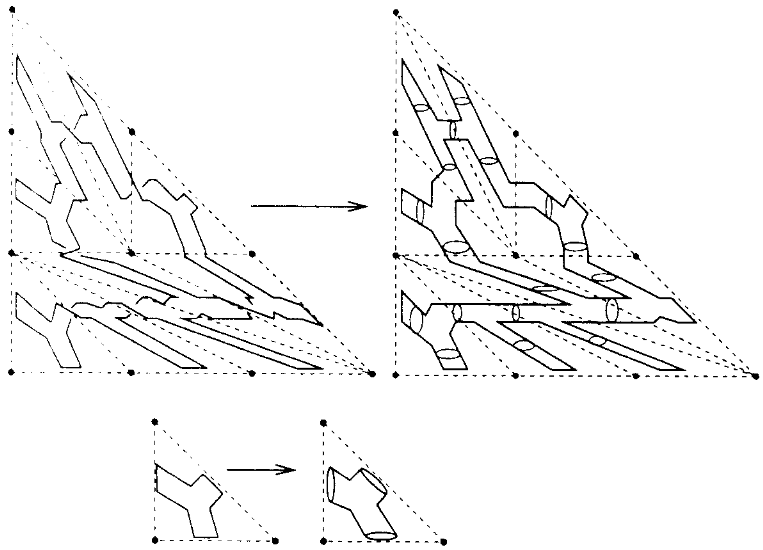

Fig. 12. The double of this T-curve on $\mathbb{R}P^2 = S(\mathbf{T}_3)$ is a surface of genus $\frac{(3-1)(3-2)}{2} = 1$.

3.3 Applications

3.3.1 First Application: The Harnack Theorem for T-Curves

Theorem 41 ("Harnack" for T-curves) *(1) The number of connected components of a T-curve is no more than the number of integer points in the interior of its Newton polygon, plus one.*
(2) There are T-curves with arbitrary Newton polygons which achieve the upper bound above.

PROOF. Let us prove just **(1)** for the moment. **(2)** will be proved in prop. 48 where we will construct T-curves with arbitrary Newton polygon which achieve the upper bound.

Let $K(\Pi, \mathcal{T}, \delta)$ be a T-curve, and let i be the number of integer points in the interior of Π. Let us glue disks along $\partial F(K)$ to obtain from $F(K)$ a closed surface $\Sigma = \Sigma(K)$. It is clear from the construction of $F(K)$ that the incidence graph $G(\Pi)$ of the triangulation $\mathcal{T}$ is a retraction of $F(K)$, so let us consider Σ as the cell complex obtained by gluing disks on $G(\Pi)$ along the cycles $\mu(\gamma)$.

Let D be the number of disks glued to $G(\Pi)$, and let E' and V' be respectively the number of edges and vertices of $G(\Pi)$. Let T, E and V be respectively the numbers of triangles, edges and vertices of the triangulation $\mathcal{T}$ of Π, and let L be the length of $\partial\Pi$. Each triangle of $\mathcal{T}$ contains three edges of $G(\Pi)$ so $E' = 3T$. The vertices of $G(\Pi)$ lie in each triangle and each edge of $\mathcal{T}$, so $V' = T + E$.

On one hand the Euler characteristic of Σ is

$$\begin{aligned}
\chi(\Sigma) &= D - E' + V' \\
&= D - 3T + (T + E) \\
&= D + T - E + L \qquad \mathcal{T} \text{ is a triangulation, therefore } 3T = 2E - L \\
&= D + 1 - V + L \qquad \text{Euler relation on } \mathcal{T} \text{ implies } T - E + V = 1
\end{aligned}$$

On the other hand $\chi(\Sigma)$ is equal to $2-2g$ where g is the genus of Σ. Therefore we get

$$\begin{aligned}
D &= (V - L) + 1 - 2g \\
&\leq i + 1 \qquad \text{since } V - L = i \text{ and } g \geq 0
\end{aligned}$$

This proves **(1)** since D is also the number of connected components of K. □

Definition 42 *A T-curve $K(\Pi, \mathcal{T}, \delta)$ with the number of connected components equal to the number of integer points in the interior of Π plus one is called a* maximal *T-curve.*

It is well known that a real projective curve is maximal if and only if the quotient of its complexification by the complex conjugation is homeomorphic to a sphere with a positive number of points removed. In particular it has no handle. We have here the analog statement in the case of T-curves:

Corollary 43 *A T-curve is maximal if and only if its T-filling is homeomorphic to a sphere with a positive number of points removed.*

PROOF. This statement follows from the proof of 41. Indeed the curve is maximal if the surface $\Sigma(K)$ has genus 0. Since a T-curve has at least one connected component, $F(K)$ is obtained from $\Sigma(K)$ by removing a positive number of disks. □

3.3.2 Harnack T-Curves

Let K be a T-curve with Newton polygon Π. Recall that the quadrants of the ambient surface $S(\Pi)$ are represented in $\mathbb{R}^2$ by the copies $(\sigma_{a,b} \cdot \Pi) = \{(x,y), ((-1)^a x, (-1)^b y) \in \Pi\}$, where $a, b \in \{0, 1\}$.

Definition 44 *Let $c, a, b \in \{0, 1\}$, and let δ be the distribution of signs defined on the points $(x, y) \in (\sigma_{a,b} \cdot \Pi)$ by:*

$$\delta(x, y) = \begin{cases} (-1)^c & \text{if } (x, y) \text{ is of even parity} \\ (-1)^{c+1} & \text{if } (x, y) \text{ is of odd parity} \end{cases}$$

The distribution of sign of a T-curve deduced from δ by the formula 1 page 14 will be called a Harnack sign distribution, *and (c, a, b) will be called the* type *of δ. A T-curve with a Harnack sign distribution will be called a* Harnack T-curve.

In order to write a unique formula to describe a Harnack distribution of sign, let $[P]$ be equal to 1 if the proposition P is true, and equal to 0 if the proposition P is false. Now let $g, h = 0$ or 1, let (x, y) be an integer point of $(\sigma_{g,h} \cdot \Pi)$, let (e, f) be the parity of (x, y), and let (c, a, b) be the type of a Harnack distribution of signs δ on Π. Then the formula of the above definition (44), is rewritten:

$$\delta(x, y) = (-1)^{c+[(e,f)\neq(0,0)]+\langle(e,f),(g+a,h+b)\rangle} \tag{3}$$

Definition 45 *Let Π be a polygon, and let a be an arc which splits Π into two connected components. We will say that a* surrounds an integer point P in Π *if P is the only integer point in the closure of one of the connected component of $\Pi \setminus a$.*

Lemma 46 *Let $K(\Pi, \mathcal{T}, \delta)$ be a Harnack curve, and let (c, a, b) be the type of δ. Then for every $g, h = 0$ or 1, the integer points of parity $(b + h, a + g)$ will be surrounded in the quadrant $(\sigma_{g,h} \cdot \Pi)$ by an oval of K if they belong to the interior of the quadrant, or by an arc of K if they are on the boundary of the quadrant.*

PROOF. According to formula 3 above, the sign of an integer point of parity (e, f) in quadrant (g, h) depend only on the value of the expression

$$[(e, f) \neq (0, 0)] + \langle(e, f), (g + a, h + b)\rangle \tag{4}$$

The following array displays the value of this expression for each parity (e, f) (one parity per line) in each quadrant $(g+a, h+b)$ (one quadrant per column):

	$(0,0)$	$(1,0)$	$(1,1)$	$(0,1)$
$(0,0)$	$0+0=0$	$0+0=0$	$0+0=0$	$0+0=0$
$(1,0)$	$1+0=1$	$1+1=0$	$1+1=0$	$1+0=1$
$(1,1)$	$1+0=1$	$1+1=0$	$1+0=1$	$1+1=0$
$(0,1)$	$1+0=1$	$1+0=1$	$1+1=0$	$1+1=0$

By looking at each row of the array we see that in a given quadrant, points of parity $(e, f) = (b+h, a+g)$ have opposite signs than their neighbors. This implies that they are separated from their neighbors by K (by an arc if the point is on the boundary of the quadrant, and by an oval otherwise). $\square$

Harnack T-curves are called so because on $\mathbb{R}P^2$ they are congruent to the well known curves constructed by Harnack [5] which have

- One one-sided component and $\frac{(d-1)(d-2)}{2}$ outermost empty ovals if they have odd degree d (see example fig. 13).
- One outermost oval containing $\frac{(k-1)(k-2)}{2}$ empty ovals and $\frac{3k(k-1)}{2}$ outermost empty ovals if they have even degree $d = 2k$ (see example fig. 14).

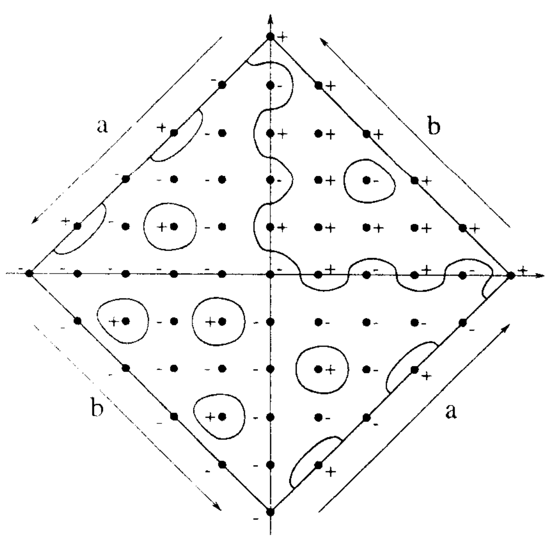

Fig. 13. A Harnack T-curve of degree 5.

The additive group $(\mathbb{Z}_2)^3 = \{0,1\} \times \{0,1\} \times \{0,1\}$ acts on the distributions of signs as follows: Let $\theta = (c, a, b) \in (\mathbb{Z}_2)^3$, and let δ be a distribution of signs

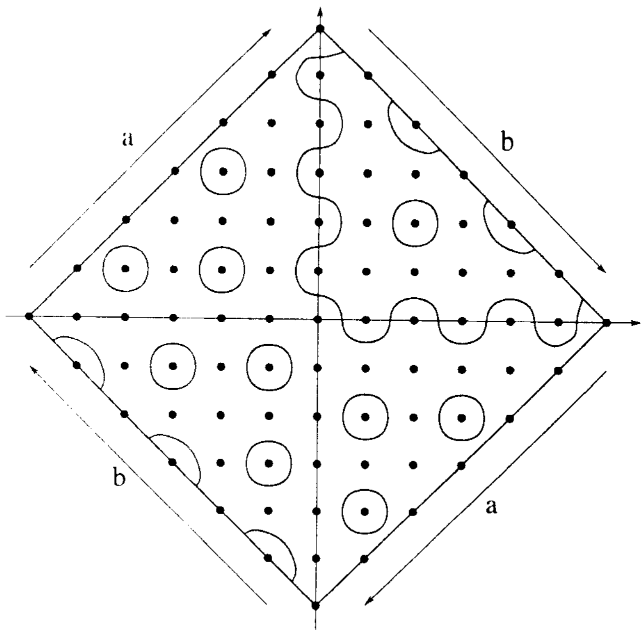

Fig. 14. A Harnack T-curve of degree 6.

on some integer points. Then

$$(\theta \cdot \delta)(x, y) = (-1)^{c+\langle(a,b),(x,y)\rangle}\delta(x, y) \tag{5}$$

Therefore $(\mathbb{Z}_2)^3$ acts also on the set of T-curves with a given Newton polygon:

$$\theta \cdot K(\Pi, \mathcal{T}, \delta) = K(\Pi, \mathcal{T}, \theta \cdot \delta)$$

Observe that $\theta{\cdot}\delta(x, y) = (-1)^c\delta(\sigma_{a,b}{\cdot}(x, y))$, and since $K(\Pi, \mathcal{T}, \delta) = K(\Pi, \mathcal{T}, -\delta)$, the group $(\mathbb{Z}_2)^3$ must be considered as $(\mathbb{Z}_2) \times ((\mathbb{Z}_2)^2)$, where the first factor $(\mathbb{Z}_2)$ acts trivially and the second factor $(\mathbb{Z}_2)^2 = \{(a, b),\ a, b = 0 \,\text{or}\, 1\}$ acts as the group of symmetries $\{\sigma_{a,b} : (x, y) \mapsto ((-1)^a x, (-1)^b y)\}$ on the four quadrants of $S(\Pi)$.

Lemma 47 *A Harnack T-curve $K'(\Pi, \mathcal{T}, \delta')$, with δ' of type $\theta' = (c, a, b)$, is the image by the symmetry $\sigma_{a,b}$ of the Harnack T-curve $K(\Pi, \mathcal{T}, \delta)$ with δ of type $(1, 0, 0)$.*

PROOF. Indeed from the definition 44 of the Harnack distributions and from formula 5 above we get that, for any $\theta \in (\mathbb{Z}_2)^3$, the distribution $\theta \cdot \delta'$ is a Harnack distribution of type the sum $\theta + \theta'$. So for $\theta = (c + 1, a, b)$ we get that $\theta \cdot K' = K$. This is equivalent to $K' = \theta \cdot K$. Since $\theta \cdot K = K(\Pi, \mathcal{T}, \theta \cdot \delta)$ we get from the observation above that $K' = \sigma_{a,b}(K)$. see fig. 15. $\square$

Proposition 48 *Let $K = K(\Pi, \mathcal{T}, \delta)$ be a Harnack T-curve with Harnack*

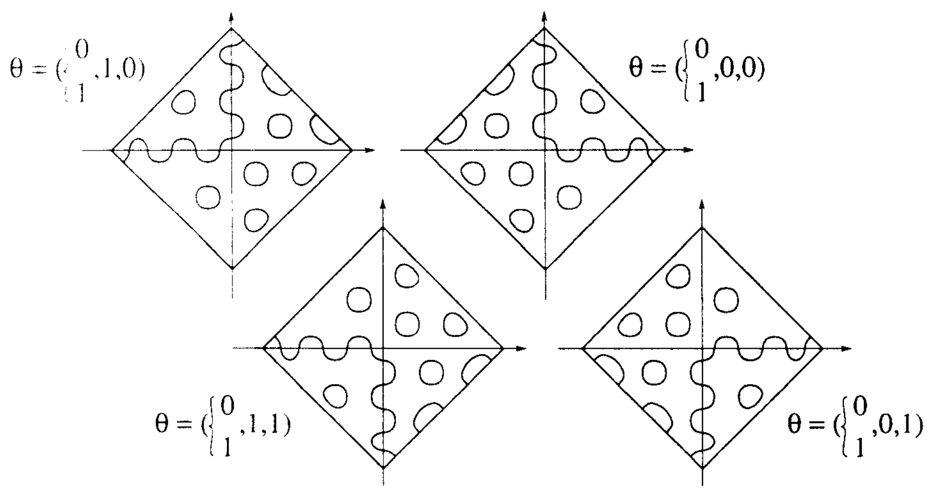

Fig. 15. Eight types of Harnack distributions for four Harnack curves symmetric to one another.

distribution of type (c, a, b). *Let* $g(s,t)$ *be the number of points of parity* (s,t) *in the interior of* Π. *Then*

(1) K is a maximal T-curve.
(2) The connected components of K are distributed on its ambient surface as follows:
  (a) $g(0,0)$ empty ovals of sign[2] $(-1)^c$ *in the quadrant* $(\sigma_{a,b} \cdot \Pi)$.
  (b) $g(s,t)$ empty ovals of sign $(-1)^{c-1}$ in the quadrant $\sigma_{t,s} \cdot \sigma_{a,b} \cdot \Pi = \sigma_{t+a,s+b} \cdot \Pi$, *for each odd parity* (s,t).
  (c) either a nontrivial connected component if Π has one broken edge of odd length.
  (d) or an oval surrounding exactly all the ovals of (a) if all the broken edges of Π are of even length.

PROOF. Thanks to lemma 47 we assume without loss of generality that $(c, a, b) = (1, 0, 0)$.

From lemma 46 we get that every point of even parity which lies in the interior of Π is surrounded by an oval of K. From the definition 44 of Harnack distribution of sign, we know that this oval is of sign -1. This proves assertion 2a.

From lemma 46 we get as well that for every point P of odd parity (s,t) which lies in the interior of Π, its copy $(\sigma_{t,s} \cdot P)$ is surrounded by an oval of K. From the definition 44 of Harnack distribution, and from formula 3 page 26 we get that this oval is of sign $+1$. This proves assertion 2b.

[2] recall the definition of the sign of an oval is def. 34

Let $g = \sum_{s,t} g(s,t)$. We just proved that K has g empty ovals. Moreover from lemma 46 we get that for every integer point $\partial\Pi$ one of its symmetric copy is surrounded by an arc of K, so there is at least one more connected component of K. From the part **(1)** of theorem 41 we deduce that K has exactly $g+1$ connected component. This proves assertion 1.

Let us call O the connected component of K which intersects (transversally) all the lifts of the broken edges of Π.

Assume first that Π has a broken edge ℓ of odd length. From lemma 36 we know that among the two lifts of an edge e of the triangulation $\mathcal{T}$ lying on ℓ one is of negative sign and the other one is of positive sign. Therefore the lift $\mu^{-1}(\ell)$ in $S(\Pi)$ is intersected transversally an odd number of times by O, which implies, since $\mu^{-1}(\ell)$ is an embedded circle, that O is not an oval of K (see fig. 16). This proves assertion 2c.

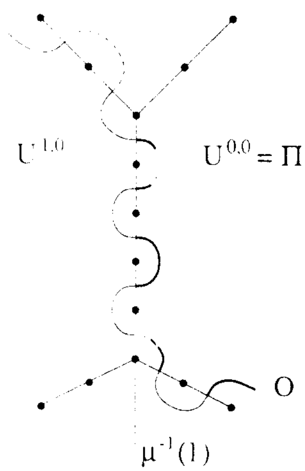

Fig. 16. If a broken edge of Π is of odd length, then the component O of K cuts the embedded circle $\mu^{-1}(\ell)$ an odd number of times.

Assume now that all the broken edges of Π are of even length. This implies that all the endpoints $\mu^{-1}(u_i)$ of the broken edges are of same parity. Thanks to prop. 28, we assume without loss of generality that they are of even parity. Therefore one integer point over two on $\partial\Pi$ is of even parity. Since δ is assumed to be of type $(1,0,0)$, all the points of even parity lying on $\partial\Pi$ are surrounded by arcs of Π in the quadrant Π, and all the other points lying on $\partial\Pi$ are surrounded in another quadrant.

By retracting each arc, which surrounds an integer point lying on $\partial\Pi$, toward this point, we shrink O onto the boundary of the quadrant Π. Since Π is homeomorphic to a disk, this boundary, and hence O, is an oval. It is clear that during the shrinking no crossing with another connected component of K happened, so O, like the boundary of Π, surrounds exactly the empty ovals lying within Π (see fig.17). This proves assertion 2d. $\square$

Notice that the assertion 1 of proposition 48 proves **(2)** of theorem 41, and therefore finishes the proof of 41. $\square$

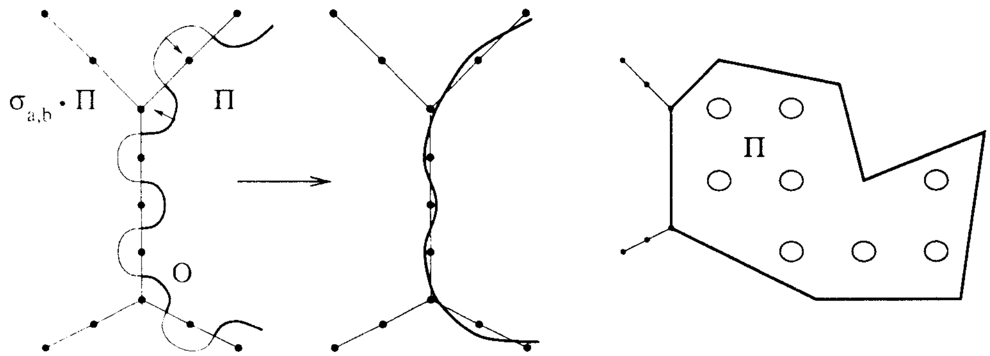

Fig. 17. When all the edges of Π are of even length, the component O of K can be shrank onto the boundary of Π.

Corollary 49 *Any two Harnack T-curves with same Newton polygon are congruent by an homeomorphism of $S(\Pi)$ which is the identity on the boundary of the quadrants of $S(\Pi)$.*

PROOF. Thanks to lemma 47 we restate without loss of generality this corollary for any two Harnack T-curves with sign distribution of type $(1,0,0)$. Now since the sign distributions on the integer points are the same, the sign of the segments on the boundary of the quadrants of $S(\Pi)$ are also the same. This fixes the T-curves on this boundary. Prop. 48 shows that the connected components of any two such Harnack T-curves, restricted to a given quadrant, are congruent. $\square$

3.3.3 Second Application: Orientation of T-Curves

Let C be a real plane projective nonsingular curve of degree d, let $\mathbb{C}C$ be its complexification, and let τ be the complex conjugation. Recall that $\mathbb{C}C$ is an orientable surface of genus $\frac{(d-1)(d-2)}{2}$ and that C is the set of fixed points of τ.

Definition 50 *C is called a* dividing curve *(or a curve of* type I*) if it divides its complexification into two connected components, and is called a* non-dividing curve *(or a curve of* type II*) if it does not divide its complexification.*

Lemma 51 *C is of type I if and only if the quotient of its complexification by the complex conjugation is orientable.*

PROOF. If C is of type I, then the quotient $\mathbb{C}C/\tau$ is homeomorphic to a connected component of $\mathbb{C}C \setminus C$, and therefore is orientable.

Assume now C is of type II. Let $P \in \mathbb{C}C$ a point not in C, and let γ be a path connecting P to its conjugate $\tau(P)$ which doesn't intersect C. Since τ reverses the orientation of $\mathbb{C}C$, the image of the path γ in $\mathbb{C}C/\tau$ is a closed path which reverses the orientation. □

By analogy we can now define a notion of orientability on T-curves via the T-filling in the following way.

Definition 52 *A T-curve is of* type I *if its T-filling is orientable, and of* type II *if its T-filling is non-orientable.*

Let C be a real plane projective nonsingular curve of type I. The curve C divides its complexification $\mathbb{C}C$ into two orientable halves. An orientation on $\mathbb{C}C$ induces an orientation on each of the halves. These orientations induce in turn two opposite orientations on C.

By analogy we can now define a notion of orientation on any T-curve $K = K(\Pi, \mathcal{T}, \delta)$ of type I via its T-filling $F(K)$ in the following way. Since $F(K)$ is orientable, an orientation on it induces an orientation on its boundary. By letting $F(K)$ retract to the incidence graph $G(\Pi)$, we get an orientation of the cycles $\gamma = \mu(\tilde{\gamma})$ which are the images of the connected components of K. An segment e of γ lifts to a segment $\sigma_{a,b} \cdot e$ of $\tilde{\gamma}$. If e is oriented from endpoint (x, y) to endpoint (x', y'), then segment $\sigma_{a,b} \cdot e$ will be oriented from $\sigma_{a,b} \cdot (x, y)$ to $\sigma_{a,b} \cdot (x', y')$. We get this way an orientation of all the connected components $\tilde{\gamma}$ of K (see fig. 18). The two opposite orientations on $F(K)$ induce then two opposite orientations on K.

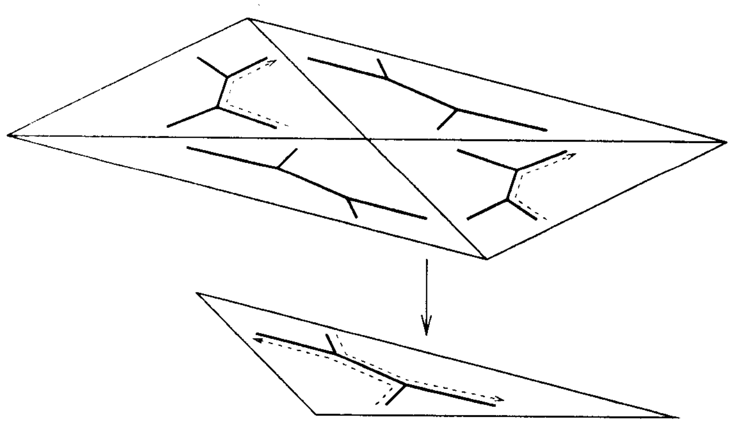

Fig. 18. The oriented cycles on the incidence graph on Π lift to oriented cycles on the incidence graph on $S(\Pi)$.

Definition 53 *The orientation described above on a T-curve of type I will be called a* (type I)-orientation *of the T-curve.*

Now we have defined type I and type II, and type-I-orientation for a T-curve, it would be interesting to prove for T-curves of given degree on $\mathbb{R}P^2$ theorems known for real plane projective nonsingular curves which take into account type and orientation (in particular Arnol'd congruence [1], and Rokhlin formula [10]). It would be a first step in order to find extensions of these theorems to T-curves on arbitrary ambient surfaces and to real algebraic curves on arbitrary toric varieties.

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