Non-constructible simplicial balls and a way of testing constructibility

(Final version)

Masahiro Hachimori *
Department of Systems Science,
Graduate School of Arts and Sciences,
University of Tokyo,
3–8–1, Komaba, Meguro, Tokyo 153-8902, Japan
E-mail : hachi@klee.c.u-tokyo.ac.jp

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Abstract

Constructibility of simplicial complexes is a notion weaker than shellability. It is known that shellable pseudomanifolds are homeomorphic to balls or spheres but simplicial complexes homeomorphic to balls or spheres need not be shellable in general. Constructible pseudomanifolds are also homeomorphic to balls or spheres, but the existence of non-constructible balls was not known. Here in this paper we study the constructibility of some non-shellable balls and show that some of them are not constructible, either. Moreover, we give a necessary and sufficient condition for the constructibility of 3-dimensional simplicial balls all of whose vertices are on the boundary.

1 Introduction

In the study of combinatorics of complexes, shellability has played an important role since Bruggesser and Mani showed the shellability of convex polytopes [4]. One of the most famous applications is the proof of the Upper Bound Theorem for convex polytopes by McMullen [8]. Though shellability has nice properties, such as the fact that a shellable pseudomanifold is always homeomorphic to a ball or a sphere, it is in general a very difficult problem to show whether a complex in interest is shellable or not. The problem is that a complex which is homeomorphic to a ball or a sphere is not always shellable, which is shown by several authors [1], [7], [11], [13, Lect. 8], and [14].

The notion of constructibility is known as a weaker one than that of shellability. This notion for simplicial complexes appears in [2], [5], and [10]. As in the case of shellability, constructible pseudomanifolds are always balls or spheres. So it is natural to ask whether the converse is true or not, i.e., whether there

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exist simplicial balls which are not constructible. As is mentioned in [2], constructibility is strictly weaker than shellability, so it may be thought that every simplicial complex which is homeomorphic to a ball or a sphere is at least constructible. So in Section 3 and Section 5 we study whether the known examples of non-shellable balls are constructible or not, and show that some of them are not constructible.

In Section 4, we study some operations for simplicial balls which preserve constructibility, and by using these operations, we give a necessary and sufficient condition for the constructibility of 3-dimensional simplicial balls all of whose vertices are on the boundary. This result gives an efficient algorithm to test the constructibility for complexes in this class.

2 Preliminaries

In this section, we review some terminology on simplicial complexes and the definition of constructibility.

A *simplicial complex* C is a finite set of simplices σ in some Euclidean space such that (i) if $\sigma \in C$, then all the faces of σ (including the empty set) are contained in C, and (ii) if $\sigma, \sigma' \in C$, then $\sigma \cap \sigma'$ is a face of both σ and σ'. The *underlying space* $\|C\|$ of a simplicial complex C is the set $\cup_{\sigma \in C} \sigma$. The simplices in a simplicial complex are called *faces*. The empty set is considered to be a (-1)-dimensional face. 0-dimensional faces are *vertices*, 1-dimensional faces are *edges*, and the maximal faces (concerning the inclusion relation) of a simplicial complex are *facets* of C. The *dimension* of a simplicial complex is the maximum dimension of its facets. If all the facets have the same dimension, the simplicial complex is called *pure*.

A d-dimensional pure simplicial complex is *strongly connected* if for any 2 of its facets F and F', there is a sequence of facets $F = F_1, F_2, \ldots, F_k = F'$ such that F_i and F_{i+1} have a common face of dimension $d-1$, for $1 \leq i \leq k-1$. A *pseudomanifold* is a d-dimensional pure strongly connected simplicial complex in which each $(d-1)$-dimensional face belongs to at most 2 facets. The boundary complex ∂C of a pseudomanifold C is a subcomplex generated by the $(d-1)$-dimensional faces belonging to only one facet. The interior $\mathring{C}$ is the set $\|C\| - \|\partial C\|$. A pseudomanifold whose underlying space is homeomorphic to a ball or a sphere is also called a *ball* or a *sphere*, respectively. A d-dimensional ball and a d-dimensional sphere are called a *d-ball* and a *d-sphere*.

For the definition of shellability, see [3], [13, Ch. 8] etc. As is mentioned in [2], constructibility is *strictly* weaker than shellability. Constructibility appears in [2], [5] and [10]. It is defined recursively as follows.

Definition 1. A pure d-dimensional simplicial complex C is said to be *constructible* if

(i) C is a simplex, or

(ii) there exist d-dimensional constructible subcomplexes C_1 and C_2 such that $C = C_1 \cup C_2$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible complex.

It is known that constructible pseudomanifolds are homeomorphic to balls or spheres ([2, Th. 11.4]) as same as the case of shellability. Counterexamples

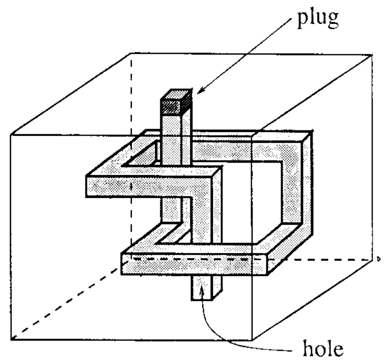

Figure 1: Furch's "knotted hole ball"

to the converse, i.e., the existence of non-constructible balls will be shown in Section 3 and Section 5.

3 Some examples of non-shellable balls

To explore whether there are non-constructible balls or not, it is a shortcut to study whether known examples of non-shellable balls are constructible or not.

Example 1. Ziegler [14, Sec. 4] made an example of a non-shellable simplicial 3-ball which has 10 vertices and 21 facets. The ball has all its vertices on the boundary. This ball has vertices $\{1, 2, \ldots, 9, 0\}$ and facets:

a: 1234	b: 1256	f: 1569	k: 2560	p: 3678	s: 4578
	c: 2367	g: 1629	l: 2670	q: 3248	t: 4137
	d: 3478	h: 1249	m: 2730	r: 3268	u: 4157
	e: 4185	i: 1489	n: 2310		
		j: 1859	o: 2150		

In the paper, he gave a way to realize this complex in $\mathbb{R}^3$ by attributing coordinates to each vertex. This complex is indeed non-shellable, but we can divide this complex into two 3-balls C_1 and C_2, where C_1 is a complex induced by the facets $\{\mathrm{d}, \mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s}, \mathrm{t}, \mathrm{u}\}$ and C_2 is a complex induced by the other facets. Here, both C_1 and C_2 are shellable. Because every 2-ball is known to be shellable, $C_1 \cap C_2$ is also shellable. So this complex is constructible.

Example 2. Next example is "Furch's knotted hole ball". This is a pile of cubes with a plugged knotted hole as Figure 1 and each cube is triangulated so that the edges of the cubes are also the edges of the triangulation. This object appears in [1] and [14] as an example of a non-shellable ball. The critical fact for the proof of its non-shellability is that it contains a non-trivial knot which is made up of one edge in the interior of the ball (an edge of the plug cube) and an arc on its boundary. In the following theorem, we show that this knot also causes non-constructibility. The idea of the proof given here is essentially the same as the proof of its non-shellability.

Theorem 1. *Let C be a triangulated 3-ball which has a tame knot K such that*

K is a non-trivial knot in C and is contained in ∂C except one edge e of C contained in $\mathring{C}$. (K need not be a subcomplex of C.)

Then C is not constructible.

Proof. First note that each arc in ∂C joining two endpoint of e always produces the same type of knot.

Let C be constructible. Then C can be divided into two constructible balls A and B, where $A \cap B$ is a 2-ball. Without loss of generality, we can assume that A contains the edge e. One can get a knot K' in A by joining two ends of e by an arc in $\partial A \cap \partial C$. The observation above shows that K' has the same type as K. If e is on ∂A, then the whole knot K' is embedded in S^2, which is impossible. So e must be in the interior of A, and K' satisfies the condition of the statement.

Because A is constructible again, it can be divided into two constructible 3-balls, and one of them has a knot of the same type as K with the same property. Continuing this process, we finally reach at the situation that the ball is divided into two simplices, which is a contradiction because there is no knot of the property in a simplex. □

4 Testing constructibility of 3-balls

As is shown in Theorem 1, there exist non-constructible balls, so we want to test whether a given ball is constructible or not. Here in this section, we show a necessary and sufficient condition for the constructibility of simplicial 3-balls all of whose vertices are on the boundary.

First, in the following two lemmas, we give two operations for simplicial 3-balls which preserve constructibility.

Lemma 1. *Let a simplicial 3-ball C have a 2-simplex T whose 3 edges are all on the boundary and T itself is in the interior of C, so C can be divided into two 3-balls C_1 and C_2 such that $C_1 \cap C_2 = T$. Then C is constructible if and only if both of C_1 and C_2 are constructible.*

Proof. Obvious. □

Lemma 2. *Let a simplicial 3-ball C have a 2-simplex T exactly 2 of whose edges e_1 and e_2 are on the boundary and the remaining edge e_3 is in the interior of C, and let C' be a simplicial complex made by splitting C along this 2-simplex T as Figure 2. Then C is constructible if and only if C' is constructible.*

Proof. Let us assume that C' is constructible. Then C' must be divided into two constructible balls C'_1 and C'_2. The constructibility of C can be easily shown by dividing C into C_1 and C_2 such that C_1 and C'_1, C_2 and C'_2 have the same sets of facets and using an induction on the size.

The converse can be also shown in the same way. The only difficulty arises when C is divided into C_1 and C_2 such that $C_1 \cap C_2$ does not contain T but contains e_3. In this case, the constructibility of C' can be shown as follows. Assume that C can be divided into two constructible balls C_1 and C_2 such that C_1 contains T. Here the constructible 3-ball C_1 contains T whose 3 edges are

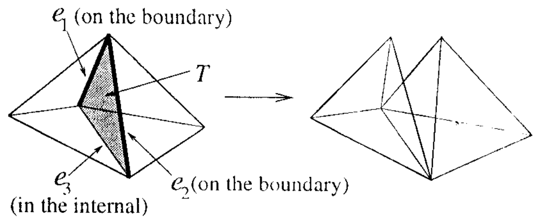

Figure 2: Split along T

all on the boundary of C_1, so if we divide C_1 into C_{11} and C_{12} by T, by Lemma 1, both C_{11} and C_{12} are constructible. On the other hand, the 2-ball $C_1 \cap C_2$ contains e_3 in the interior and both ends of e_3 on the boundary, so this ball is divided by e_3 into 2-balls $(C_1 \cap C_2) \cap C_{11}$ and $(C_1 \cap C_2) \cap C_{12}$. Now we can join C_2 and C_{11} by $(C_1 \cap C_2) \cap C_{11}$, and then add C_{12} by $(C_1 \cap C_2) \cap C_{12}$. The resulting complex is exactly the same as C', and is constructible by the construction.

□

Now let a simplicial 3-ball C have all its vertices on the boundary. We successively transform C as follows: if there is a 2-simplex T whose 3 edges are on the boundary, then divide the ball into two balls by T, and if there is a 2-simplex T exactly 2 of whose edges are on the boundary, then split along T. If at last we reach the situation that the complex is divided into a disjoint set of 3-simplices, we conclude that the original complex C is constructible. The converse is shown in the following lemma.

Lemma 3. *Let a simplicial 3-ball C has all its vertices on the boundary. If some successive operations described above get stuck, then C is not constructible.*

Proof. When these operations get stuck, we have a simplicial 3-ball C' whose 2-simplices are only of the following three types: (a)whole the faces are on the boundary (b)only 1 vertex and 1 edge is on the boundary (c)only 3 vertices are on the boundary. Let us assume that C' is constructible. Then it can be divided into two 3-balls whose intersection is a 2-ball D. Now because D meets $\partial C'$ only with ∂D, D must be consist of the 2-simplices of type (b) and (c), which is impossible. □

Now we have shown the following theorem which gives a necessary and sufficient condition for the constructibility of simplicial 3-balls whose vertices are all on the boundary.

Theorem 2. *Let a simplicial 3-ball C has all its vertices on the boundary. Do successive operations such that:*

> *(i) if there is a 2-simplex T whose 3 edges are on the boundary, then divide the ball into two 3-balls by T, and*
> *(ii) if there is a 2-simplex T exactly 2 of whose edges are on the boundary, then split along T.*

Then we have C divided into a disjoint set of 3-simplices if and only if the original ball C is constructible.

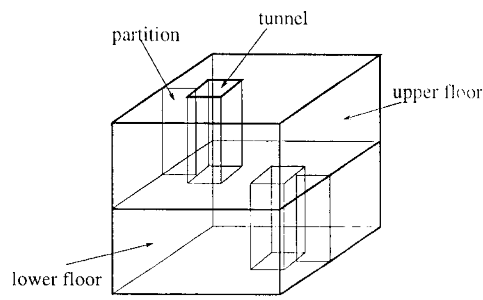

Figure 3: A ball with 2 rooms

Remark. The algorithm given in the theorem can be reduced to the following form:
(1)Mark the edges which are on the boundary.
(2)Repeat: If just 2 edges on a 2-face are marked, then mark the third edge.
(3)All the edges are marked if and only if the ball is constructible.
By a careful treatment, the testing can be done in $O(f)$ time, where f is the number of facets. Using the Euler-Poincaré equation, f can be estimated to be $O(v^2)$, where v is the number of vertices, so the time bound is $O(v^2)$. Thus in this special case, the testing problem of constructibility is polynomially solvable. Currently, it is not known whether the problem in general is polynomially solvable or not.

5 Some more examples

Example 3. Before Ziegler's non-shellable ball was found, the smallest number of vertices for non-shellable balls was 14, given by Rudin [9] (with 41 facets) and Grünbaum (with 29 facets). These two balls also have all their vertices on the boundary. The lists of facets of these balls are given in [5]. By a computer calculation using the result of the previous section, both of these balls were shown to be constructible.

Example 4. Another example of a non-shellable ball is "Bing's house with 2 rooms"[1]. This is a house with 2 rooms as Figure 3, the walls are made out of one layer of cubes, one enters to the lower floor through a tunnel from the roof and to the upper floor through a tunnel from below. After constructing such an object C with cubes, we triangulate the cubes as follows. Let us order the vertices as follows. First list such vertices v that there is a cube D such that v is a connected component of $D \cap \partial C$. (The vertices on the inside corners of C.) Next list the vertices which is not listed yet and is on an edge that is a connected component of $D \cap \partial C$, for some cube D. Last list the remainder. Then we triangulate each 2-face such that the first vertex in the list is contained in the added diagonal. Finally we triangulate each cube into six simplices by taking cones from the first vertex to the six triangles contained in the 2-faces of the cube which do not contain the vertex. (This triangulation is a "pulling triangulation", made by pulling the vertices in the listed order. The concept "pulling" is described in [6, Sec 14.2].)

In this systematic triangulation, we can see that each facet intersects with ∂C in a disconnected set, and this is the reason why C is not shellable. Moreover, we can also see that there is no triangle in the interior of C such that 2 or 3 of its edges are on ∂C. So by Theorem 2, C is not constructible either. (Because all of the vertices of C are on ∂C, the condition for Theorem 2 is satisfied.)

Remark. There is a typo in the list of Grünbaum's ball in [5]. The ninth facet "1 7 8 9" must be "1 7 8 10". This typo was found during the computer calculation, and was checked by Prof. Dr. Ziegler and Prof. Dr. Grünbaum. Prof. Dr. Grünbaum taught me a set of coordinates for the vertices as follows in order to realize the ball in $\mathbb{R}^3$. (In the following, the name of the vertices are the same as the ones used in [5].)

Vertices:

$1 = (0, 1, -1)$, $2 = (1, 0, 1)$, $3 = (0, 1, 1)$, $4 = (1, 0, -1)$,
$5 = (-1, 0, 1)$, $6 = (0, -1, -1)$, $7 = (0, 0.5, -0.3)$, $8 = (0.5, 0, 0.3)$,
$9 = (0, 0.5, 1)$, $10 = (0.5, 0, -1)$, $11 = (0.25, 0, -1)$, $12 = (0, 0.25, 1)$,
$13 = (-0.5, 0, 0.3)$, $14 = (0, -0.5, -0.3)$.

Facets:

$\{1,2,3,7\}$, $\{1,2,4,8\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,8,10\}$,
$\{1,5,6,13\}$, $\{1,5,7,13\}$, $\{1,6,11,13\}$, $\{1,7,8,10\}$, $\{1,7,11,13\}$,
$\{2,3,7,9\}$, $\{2,4,6,8\}$, $\{2,5,6,14\}$, $\{2,5,12,14\}$, $\{2,6,8,14\}$,
$\{2,7,8,9\}$, $\{2,8,12,14\}$, $\{3,5,7,9\}$, $\{4,6,8,10\}$, $\{5,6,13,14\}$,
$\{5,7,9,13\}$, $\{5,12,13,14\}$, $\{6,8,10,14\}$, $\{6,11,13,14\}$, $\{7,8,9,13\}$,
$\{7,8,10,14\}$, $\{7,8,13,14\}$, $\{7,11,13,14\}$, $\{8,12,13,14\}$.

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