A GEOMETRIC PROOF OF THE "n!" AND MACDONALD POSITIVITY CONJECTURES.

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Notice

This is a preliminary draft only. It gives in detail the proof referred to in the title. It is missing background and motivating material and references, which will be given in a future, full version, along with improved exposition (I hope) and some further results.

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1. Introduction

Let $E = \mathbf{A}^2(k)$ be the affine plane over an algebraically closed field k of characteristic zero. All schemes considered will be quasi-projective over k. Let H_n denote the Hilbert scheme $\mathrm{Hilb}^n(E)$ parametrizing zero-dimensional subschemes $S \subseteq E$ of total length n. By definition such subschemes S are in one-to-one correspondence with ideals $I \subseteq k[x, y]$ such that $\dim_k k[x, y]/I = n$. The (closed) points of H_n will be referred to by their corresponding ideals I.

If $S \subseteq E$ is zero-dimensional we can assign to each point $P \in S$ a multiplicity, defined as the length of the local ring $\mathcal{O}_{S,P}$. For $I(S) \in H_n$ these multiplicities sum to n, and taking each point with its multiplicity gives an n-element multiset, or unordered n-tuple, of points in E, which can be identified with a point of E^n/S_n, where S_n is the symmetric group permuting the factors of E^n.

The map τ associating with $I(S)$ the underlying multiset of S is a projective morphism

$$\tau \colon H_n \to E^n/S_n,$$

called the Chow morphism.

The *isospectral Hilbert scheme* X_n is defined to be the reduced fiber product

$$\begin{array}{ccc} X_n & \longrightarrow & E^n \\ \downarrow & & \downarrow \\ H_n & \xrightarrow{\ \tau\ } & E^n/S_n. \end{array}$$

Note that the scheme-theoretic product here is not reduced; X_n is its underlying reduced subscheme. A (closed) point of $X_n \subseteq H_n \times E^n$ is a pair $(I(S), (P_1, \ldots, P_n))$, where P_1 through P_n are the points of S, each repeated with its multiplicity in S, in some order. The term *isospectral* alludes to the fact that the coordinates of the points P_i are the joint eigenvalues of the operators of multiplication by x and y on $k[x, y]/I(S)$.

In an earlier paper [MSRI] (M. Haiman, *Macdonald polynomials and geometry*, New Perspectives in Geometric Combinatorics, MSRI Publications **37** (1999) 207–254) we obtained the following results.

Theorem 1. *The isospectral Hilbert scheme X_n is Cohen-Macaulay if and only if the $n!$ conjecture of Garsia and the author holds for all partitions μ of n.*

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Theorem 2. *The Cohen-Macaulay property of X_n implies the Macdonald positivity conjecture, namely, that the Macdonald-Kostka coefficients $K_{\lambda\mu}(q,t)$ are polynomials with non-negative integer coefficients.*

In fact it implies the conjecture of Garsia and the author that the coefficient of $q^h t^k$ in $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,1/t)$ is the multiplicity of the irreducible character χ^λ in the character of the (h,k)-graded component of the doubly graded S_n module D_μ, the space of all derivatives of the polynomial Δ_μ figuring in the $n!$ conjecture.

Theorem 3. *Let $J_n \subseteq k[x_1, y_1, \ldots, x_n, y_n]$ denote the ideal generated by all S_n-alternating polynomials in $k[\mathbf{x}, \mathbf{y}]$, where S_n acts by $\sigma x_i = x_{\sigma(i)}, \sigma y_i = y_{\sigma(i)}$. Assume that for all n and m, J_n^m is a free module over $k[\mathbf{y}] = k[y_1, \ldots, y_n]$. Then X_n is Cohen-Macaulay for all n.*

In section 2 we prove this freeness hypothesis for J_n^m.

Since we are using the result, we must rectify an omission in the proof of the implication (X_n Cohen-Macaulay) $\Rightarrow$ ($n!$ conjecture) in [MSRI]. This is done in section 3.

2. Freeness results

The freeness result for the ideals J_n^m will be derived from corresponding results for certain subspace arrangements in $E^n \times E^l$ which we now define.

We denote the coordinates on $E^n \times E^l$ by $x_1, y_1, \ldots, x_n, y_n, a_1, b_1, \ldots, a_l, b_l$, so $E^n \times E^l = \operatorname{Spec} R$, where $R = k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]$. We write $[n]$ for the set of integers $\{1, 2, \ldots, n\}$. Given a function $f\colon [l] \to [n]$, let $W_f \subseteq E^n \times E^l$ be the linear affine subspace defined by the equations $a_i = x_{f(i)}$, $b_i = y_{f(i)}$ for all $i \in [l]$. Note that W_f is the graph of a linear map $\pi_f\colon E^n \to E^l$ determined by f.

The union of the subspaces W_f over all $f\colon [l] \to [n]$ will be called a *polygraph* and denoted $Z((n^l), \emptyset)$.

More generally, given a subset $T \subseteq [l]$ we define $W_{f,T}$ to be the subspace of W_f defined by the equations

$$\begin{aligned} a_i = x_{f(i)}, b_i = y_{f(i)} \quad &\text{for all } i \in [l], \\ a_i = a_j \quad &\text{for all } i, j \in T. \end{aligned} \tag{2.1}$$

Equivalently, $W_{f,T}$ is the graph of the restriction of π_f to the subspace of E^n defined by the equations $x_{f(i)} = x_{f(j)}$ for all $i, j \in T$. Since the coordinates $\mathbf{y}$ remain independent on $W_{f,T}$, the coordinate ring of $W_{f,T}$ is a free $k[\mathbf{y}]$ module.

Given a sequence $(\nu_1, \ldots, \nu_l)$ of integers $0 \leq \nu_i \leq n$, we define the scheme $Z(\nu, T) \subseteq E^n \times E^l$ to be the union of the subspaces $W_{f,T}$ over all $f\colon [l] \to [n]$ satisfying the conditions:

$$\begin{aligned} f(i) \leq \nu_i \quad &\text{for all } i \in [l], \\ f(i) \neq f(j) \quad &\text{for } i < j,\ i \in T,\ j \notin T. \end{aligned} \tag{2.2}$$

It may happen that no f satisfies these conditions (e.g. if $\nu_i = 0$ for some i), in which case $Z(\nu, T)$ is empty. If $l = 0$, the empty function f satisfies (2.2) vacuously, and $Z(\emptyset, \emptyset) = E^n$. Note that the notation $Z((n^l), \emptyset)$ for polygraphs is consistent with the definition of $Z(\nu, T)$. When we want to specify n and l explicitly we write $Z(n, l, \nu, T)$ instead of $Z(\nu, T)$.

The ring $R = k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]$ is doubly graded by degrees (r, s) in the variables $\{\mathbf{x}, \mathbf{a}\}$ and $\{\mathbf{y}, \mathbf{b}\}$ respectively. These degrees will be referred to as $\mathbf{x}$-degree and $\mathbf{y}$-degree. The ideals of $W_{f,T}$ and therefore of $Z(\nu, T)$ are doubly homogeneous. Moreover, for these ideals I, R/I is a finitely generated module over $k[\mathbf{x}, \mathbf{y}]$, and hence each of its $\mathbf{x}$-degree homogeneous components $(R/I)_{(r,-)}$ is a finitely generated $k[\mathbf{y}]$ module, graded by $\mathbf{y}$-degree. Throughout, we will make use of the principle that the theory of finitely generated graded $k[\mathbf{y}]$ modules may be applied to the rings R/I via their $\mathbf{x}$-homogeneous components.

Theorem 4. *The coordinate ring $R/I(Z(\nu, T))$ of $Z(\nu, T)$ is a free $k[\mathbf{y}]$ module if $\nu_i = n$ for all $i \notin T$. In particular this holds for the polygraph $Z((n^l), \emptyset)$.*

We believe the theorem should hold without the proviso on ν, though we need it for the inductive proof given here. Before proving Theorem 4 we show how it implies the freeness hypothesis for J_n^m, and hence the $n!$ and Macdonald positivity conjectures.

Theorem 5. *Assume the coordinate ring of the polygraph $Z((n^l),\emptyset)$ is a free $k[\mathbf{y}]$ module for all l. Then the ideal $J_n^m \subseteq k[\mathbf{x},\mathbf{y}]$ is a free $k[\mathbf{y}]$ module for all m.*

Proof. Fix m and take $l = mn$. Let $G = S_n^m$ be the cartesian product of m symmetric groups S_n, acting on $E^n \times E^{mn}$ by permuting the factors of E^{mn} within m consecutive blocks of length n. Thus each element $\sigma \in G$ fixes the coordinates $\mathbf{x},\mathbf{y}$, and for each $d = 0,\dots,m-1$ permutes the coordinate pairs (a_{dn+i}, b_{dn+i}) for $i = 1,\dots,n$ among themselves.

Let $I \subseteq R$ be the ideal of $Z((n^l),\emptyset)$. Clearly I is a G-invariant ideal. We claim that the space $(R/I)^\epsilon$ of G-alternating elements of R/I is isomorphic to J_n^m as a $k[\mathbf{y}]$ module. Since $(R/I)^\epsilon$ is a direct summand of R/I it is a free $k[\mathbf{y}]$ module. (It is elementary that finitely generated graded projective $k[\mathbf{y}]$ modules are free.)

Let $\psi : R \to k[\mathbf{x},\mathbf{y}]$ be the ring homomorphism mapping a_{dn+i}, b_{dn+i} to x_i, y_i, and $\mathbf{x},\mathbf{y}$ to themselves. By definition, J_n^m is generated as a $k[\mathbf{x},\mathbf{y}]$ module by products $\Delta_1 \cdots \Delta_m$ of m alternating polynomials. Therefore ψ maps R^ϵ surjectively onto J_n^m.

Let p be an arbitrary element of R^ϵ. Since p is G-alternating, p vanishes on W_f if $f(dn+i) = f(dn+j)$ for any $i,j \in [n]$ with $i \neq j$. Thus the regular function defined by p on $Z((n^l),\emptyset)$ is determined by its restriction to those W_f with the property that for each $d = 0,\dots,m-1$, the sequence $f(dn+1),\dots,f(dn+n)$ is a permutation of n. Furthermore, since all such W_f are conjugate by elements of G, p is determined by its restriction to W_{f_0}, where $f_0(dn+i) = i$. But W_{f_0} is defined by the equations $a_{dn+i} = x_i, b_{dn+i} = y_i$ for all d, i, so the restriction of p to W_{f_0} is given by $\psi(p)$. This shows that p vanishes on $Z((n^l),\emptyset)$ if and only if $\psi(p) = 0$, that is, the kernel of the map $\psi\colon R^\epsilon \to J_n^m$ is $I \cap R^\epsilon$, and hence $(R/I)^\epsilon = R^\epsilon/I^\epsilon \cong J_n^m$. □

The proof of Theorem 4 will be given following a number of lemmas.

We first record certain elementary algebraic facts concerning $k[\mathbf{y}]$ modules M. The first of these hold for arbitrary M, not necessarily graded or finitely generated.

(1) Given an exact sequence
$$0 \to M \xrightarrow[\phi]{} N \to P \to 0,$$
with M and P free, N is free. If B_1 and B_2 are free module bases of M and P, respectively, and $B_2' \subseteq N$ maps bijectively on B_2, then $\phi(B_1) \cup B_2'$ is a free module basis of N.

(2) Let $\tilde{M}$ be the quasi-coherent sheaf on Spec $k[\mathbf{y}]$ associated to M. If $U \subseteq$ Spec $k[\mathbf{y}]$ is an open set whose complement has codimension d, and M has a free resolution
$$0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to M \to 0$$
of length $k < d$, the canonical map $M \to H^0(U,\tilde{M})$ is injective.

The remaining facts hold for finitely generated graded $k[\mathbf{y}]$ modules. We write depth M for $\operatorname{depth}_{\mathfrak{m}} M$, where $\mathfrak{m} = (\mathbf{y})$ is the homogeneous maximal ideal. The zero module has infinite depth.

(3) We have depth $M = n$ if and only if M is free.

(4) Given an exact sequence
$$0 \to M \to N \to P \to 0,$$
we have have depth $M \geq \min(\operatorname{depth} N, 1 + \operatorname{depth} P)$. In particular if N is free and depth $P \geq n-1$, then M is free.

(5) If M is free over a subring $k[y_{j_1},\dots,y_{j_k}]$ then depth $M \geq k$.

(6) Let P denote a maximal ideal of $k[\mathbf{y}]$, i.e., a closed point of $\operatorname{Spec} k[\mathbf{y}]$. The integer $\dim_k M/PM$ is constant for all P in a dense open set, and is maximized for $P = (\mathbf{y})$. If the maximum value is equal to the generic value, then M is free.

Throughout we denote by U the open subset of $\operatorname{Spec} k[\mathbf{y}]$, or more generally of $\operatorname{Spec} R/I$, on which at most two of the coordinates y_i coincide. More precisely for $1 \leq p < q \leq n$, we define U_{pq} to be the complement of the closed union

$$\bigcup_{\substack{i<j \\ \{i,j\} \neq \{p,q\}}} V(y_i - y_j),$$

and U to be the union $\bigcup_{p<q} U_{pq}$. The complement of U is the union of the subspaces $V(y_i - y_j, y_k - y_l)$ where i, j, k, l are distinct and $V(y_i - y_j, y_i - y_k)$ where i, j, k are distinct. In particular it has codimension 2.

Lemma 2.1. *Let M be a torsion-free $k[\mathbf{y}]$ module and let B be a subset of M. Assume that for all $P \in U$, M_P is a free $k[\mathbf{y}]_P$ module with basis B. Then M is a free $k[\mathbf{y}]$ module with basis B.*

Proof. Let F be a free $k[\mathbf{y}]$ module with basis $\{e_b : b \in B\}$, and let $\phi\colon F \to M$ be the homomorphism defined by $\phi(e_b) = b$ for all $b \in B$. By hypothesis the induced homomorphism $\tilde{\phi}\colon \tilde{F} \to \tilde{M}$ of quasi-coherent sheaves on $\operatorname{Spec} k[\mathbf{y}]$ restricts to an isomorphism on U.

If s is in the kernel of ϕ the corresponding section $\tilde{s} \in \tilde{F}(U)$ is zero. Then s is zero because F is free and hence torsion-free. This shows ϕ is injective.

Given any element $t \in M$ let $\tilde{t} \in \tilde{M}(U)$ be the corresponding section, and let $\sigma = \tilde{\phi}^{-1}(\tilde{t}) \in \tilde{F}(U)$. If Y denotes the complement of U, the local cohomology module $H^1_Y(F)$ is zero, since F is free, $k[\mathbf{y}]$ is Cohen-Macaulay, and $\operatorname{codim} Y = 2$. This implies that the canonical map $F \to H^0(U, \tilde{F})$ is surjective, so $\sigma = \tilde{s}$ for some $s \in F$. Then $\phi(s) - t$ induces the zero section on U, and since M is torsion-free, $\phi(s) = t$. This shows ϕ is surjective. □

Lemma 2.2. *Let R be a $k[\mathbf{y}]$ algebra, $I, J \subseteq R$ ideals, and $f \subseteq R$ an arbitrary element. Assume that*

(1) *R/I is a torsion-free $k[\mathbf{y}]$ module,*
(2) *$R/I : (f)$ and R/J are free $k[\mathbf{y}]$ modules, and*
(3) *$V(I + (f)) \cap U = V(J) \cap U$ scheme-theoretically, i.e., we have $I + (f) = J$ locally on U.*

Then $I + (f) = J$, and R/I is a free $k[\mathbf{y}]$ module.

Proof. Since R/J is free, the canonical map $R/J \to H^0(U, (R/J)\tilde{})$ is injective. With (3) this implies $I + (f) \subseteq J$.

Let B_1 and B_2 be free $k[\mathbf{y}]$ module bases of R/J and $R/I : (f)$, respectively. From the exact sequence

$$0 \to R/I : (f) \xrightarrow[\cdot f]{} R/I \to R/I + (f) \to 0, \tag{2.3}$$

we see that for $P \in U$, $(R/I)_P$ is free with basis $B_1 \cup fB_2$. By Lemma 2.1 this implies that R/I is free. The exact sequence (2.3) is now a free resolution of $R/I + (f)$. Hence the canonical map $R/I + (f) \to H^0(U, (R/I + (f))\tilde{})$ is injective, which implies $J \subseteq I + (f)$ and therefore $J = I + (f)$, by hypothesis (3). □

In the application of Lemma 2.2 we will always have $J = \sqrt{I + (f)}$, though this is not necessary to the proof.

Lemma 2.3. *Let M be a graded $k[\mathbf{y}]$ module, and let $I, J, K \subseteq M$ be homogeneous submodules. Assume that*

(1) *M/I, M/J and M/K are finitely generated,*
(2) *M/I and M/J are free,*
(3) depth $M/K \geq n - 1$, *and*

(4) *we have* $I + J = K$ *locally on* U.

Then $I + J = K$ *and* $M/I \cap J$ *is a free* $k[\mathbf{y}]$ *module.*

Proof. Since depth $M/K \geq n-1$, the canonical map $M/K \subseteq H^0(U, (M/K)\tilde{})$ is injective. Together with (4), this implies $I + J \subseteq K$.

From the exact sequence

$$0 \to K/I \to M/I \to M/K \to 0$$

we see that K/I is a free $k[\mathbf{y}]$ module. We have further exact sequences

$$0 \to M/I \cap J \to M/I \oplus M/J \to M/I + J \to 0 \tag{2.4}$$

and

$$0 \to (I+J)/I \cong J/I \cap J \to M/I \cap J \to M/J \to 0. \tag{2.5}$$

Let B_1 and B_2 be free module bases of K/I and M/J, respectively. Localizing (2.5) at $P \in U$ and using $(I+J)_P = K_P$, we see that $B_1 \cup B_2$ is a free module basis of $(M/I \cap J)_P$. Since M/I and M/J are torsion-free, so is $M/I \cap J$. By Lemma 2.1, $M/I \cap J$ is free. Then (2.4) is a free resolution of $M/I + J$, which implies that the canonical map $M/I + J \to H^0(U, (M/I+J)\tilde{})$ is injective. By (4) this implies $K \subseteq I + J$, hence $I + J = K$. □

Lemma 2.3 will be applied with M an $\mathbf{x}$-degree homogeneous component of R, and I, J, K the corresponding components of doubly homogeneous ideals, where R/I, R/J and R/K are finitely generated $k[\mathbf{x}, \mathbf{y}]$ modules, so (1) holds.

In the next Lemma and in the proofs of results to follow we will need to consider subschemes similar to $Z(\nu, T)$ but defined by equations in subsets of the variables $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}$. For this we need some further notation.

Let $N \subseteq [n]$ and $L \subseteq [l]$ be given. Let the elements of N and L in increasing order be

$$N = \{j_1 < j_2 < \cdots < j_{n'}\}, \quad L = \{i_1 < i_2 < \cdots < i_{l'}\}.$$

We define a linear morphism $\psi_{N,L} \colon E^n \times E^l \to E^{n'} \times E^{l'}$ by

$$\psi_{N,L}(x_1, y_1, \ldots, x_n, y_n, a_1, b_1, \ldots, a_l, b_l) = (x_{j_1}, y_{j_1}, \ldots, x_{j_{n'}}, y_{j_{n'}}, a_{i_1}, b_{i_1}, \ldots, a_{i_{l'}}, b_{j_{l'}}).$$

Equivalently, the underlying ring homomorphism $\psi^\sharp_{N,L} \colon R' \to R$ is given by $\psi^\sharp_{N,L}(x_k) = x_{j_k}$, $\psi^\sharp_{N,L}(y_k) = y_{j_k}$, $\psi^\sharp_{N,L}(a_k) = a_{i_k}$, $\psi^\sharp_{N,L}(b_k) = b_{i_k}$, where $R' = k[x_1, y_1, \ldots, x_{n'}, y_{n'}, a_1, b_1, \ldots, a_{l'}, b_{l'}]$.

We denote by $I(n, l, \nu, T) \subseteq R$ the ideal of $Z(\nu, T)$, abbreviating this to $I(\nu, T)$ when n and l are understood. With N, L as above we set

$$I(N, L, \nu', T') = I(\psi^{-1}_{N,L}(Z(n', l', \nu', T'))) = R\psi^\sharp_{N,L}(I(n', l', \nu', T')).$$

Note that $I(N, L, \nu', T')$ is generated by polynomials in the variables x_j, y_j for $j \in N$ and a_i, b_i for $i \in L$ only. Let $Y \in E^n \times E^l$ be a linear subspace on which these variables are independent coordinates. Then it is clear that the subscheme $\psi^{-1}_{N,L}(Z(\nu', T')) \cap Y$ defined by the ideal $I(N, L, \nu', T') + I(Y)$ is reduced and isomorphic to the product

$$Z(n', l', \nu', T') \times A,$$

where A is a linear affine space. It follows that if $R'/I(n', l', \nu', T')$ is a free $k[y_1, \ldots, y_{n'}]$ module, then $R/I(N, L, \nu', T') + I(Y)$ is a free $k[\mathbf{y}']$ module for any set of coordinates $\mathbf{y}' \subseteq \mathbf{y}$ independent on Y.

Lemma 2.4. *With* U_{pq} *and* $W_{f,T}$ *as defined earlier, we have*

(1) $U_{pq} \cap W_{f,T} \cap W_{g,T} = \emptyset$ *unless* $f^{-1}(\{p,q\}) = g^{-1}(\{p,q\})$ *and* $f(i) = g(i)$ *for all* $i \notin f^{-1}(\{p,q\})$. *In particular, all* $W_{f,T}$ *containing a given point* $P \in U_{pq}$ *have a common value of* $L = f^{-1}(\{p,q\})$ *and* $h = f|_{[l]\setminus L}$.

(2) *Fix a subset* $L = \{i_1 < \cdots < i_{l'}\} \subseteq [l]$ *and a function* $h\colon [l] \setminus L \to [n] \setminus \{p, q\}$ *satisfying* (2.2) *for indices* $i, j \notin L$. *Let* Z *be the union of the irreducible components* $W_{f,T} \subseteq Z(n, l, \nu, T)$ *for which* $f^{-1}(\{p, q\}) = L$ *and* $f|_{[l]\setminus L} = h$. *Then with* $N = \{p, q\}$, *we have* $Z = \psi_{N,L}^{-1}(Z(2, l', \nu', T')) \cap Y$, *where* $\nu'_j = |[\nu_{i_j}] \cap \{p, q\}|$, $T' = \{j : i_j \in T\}$, *and* Y *is the affine subspace defined by the equations*

$$a_i = x_{h(i)}, b_i = y_{h(i)} \quad \text{for} \quad i \notin L, \quad x_{h(i)} = x_{h(j)} \quad \text{for} \quad i, j \in T \setminus L,$$

and, if $T \cap L \neq \emptyset$,

$$x_{h(i)} = a_t \quad \text{for} \quad i \in T \setminus L,$$

where t *is an arbitrary element of* $T \cap L$.

Proof. (1) If for some i we have $f(i) \neq g(i)$ and $\{f(i), g(i)\} \neq \{p, q\}$, then $U_{pq} \cap W_{f,T} \cap W_{g,T} = \emptyset$, since $W_{f,T} \subseteq V(b_i - y_{f(i)})$, $W_{g,T} \subseteq V(b_i - y_{g(i)})$, and $V(y_{f(i)} - y_{g(i)}) \cap U_{pq} = \emptyset$. The condition that for all $i \in [l]$, either $f(i) = g(i)$ or $\{f(i), g(i)\} = \{p, q\}$ is equivalent to the condition that $f^{-1}(\{p, q\}) = g^{-1}(\{p, q\})$ and $f(i) = g(i)$ for all $i \notin f^{-1}(\{p, q\})$.

(2) Let f satisfy $f^{-1}(\{p, q\}) = L$ and $f(i) = h(i)$ for all $i \notin L$. Given the assumed conditions on h, the condition that $f(i) \leq \nu_i$ for all $i \in [l]$ is equivalent to $f'(i) \leq \nu'_i$ for all $i \in L$, where

$$f'(i) = \begin{cases} 1, & f(i) = p \\ 2, & f(i) = q. \end{cases}$$

The condition that $f(i) \neq f(j)$ for $i < j$, $i \in T$, $j \notin T$ holds if and only if it holds for $i, j \in L$, since in all other cases it is implied by the assumed conditions on f and h. Therefore, if we define $g\colon [l'] \to [2]$ by $g(j) = f'(i_j)$, we see that $W_{g,T'} \subseteq Z(2, l', \nu', T')$ if and only if $W_{f,T} \subseteq Z(\nu, T)$.

Since $W_{f,T} = \psi_{N,L}^{-1}(W_{g,T}) \cap Y$ we have $Z = \psi_{N,L}^{-1}(Z(2, l', \nu', T')) \cap Y$, set-theoretically. The coordinates x_p, y_p, x_q, y_q and a_i, b_i for $i \in L$ are independent on Y, so $\psi_{N,L}^{-1}(Z(2, l', \nu', T')) \cap Y$ is scheme-theoretically reduced. □

Corollary 2.5. *For each scheme* Z *of the form specified in Lemma 2.4 (2), if* $R'/I(2, l', \nu', T')$ *is a free* $k[y_1, y_2]$ *module then* $R/I(Z)$ *is a free* $k[\mathbf{y}]$ *module.*

For $l = 0$, $Z(\nu, T)$ is tautologically equal to E^n. For $l > 0$, $Z(\nu, \emptyset) = Z(\nu, \{l\})$. To analyze the schemes $Z(\nu, T)$, therefore, we may assume without loss of generality that $T \neq \emptyset$. With this assumption we let t_1 denote the least element of T.

Lemma 2.6. *For* $n = 2$, *if* $Z(2, l, \nu, T)$ *is not empty, there is a subset* $L \subseteq [l]$ *such that* $\psi_{[2],L}$ *induces an isomorphism of* $Z(2, l, \nu, T)$ *onto some* $Z(\nu', T') = Z(2, l', (2^{l'}), [t', l'])$ *or onto* $Z(\nu', T') \cap V(a_{t'} - x_1)$ *or* $Z(\nu', T') \cap V(a_{t'} - x_2)$. *(The reduced intersection is meant here, but in any event the next lemma shows that the scheme-theoretic intersection is reduced).*

Proof. If $l = 0$ the result is trivial. Otherwise we can assume $T \neq \emptyset$, and let t_1 be the least element of T. If $T \neq [l]$, let s_1 be the greatest element of $[l] \setminus T$. Otherwise, for simplicity of notation, let $s_1 = 0$.

For all i in the (possibly empty) interval $[t_1 + 1, s_1]$, and for all components $W_{f,T} \subseteq Z(\nu, T)$, $f(t_1)$ determines and is determined by $f(i)$. For $i \notin T$ this follows because $f(i) \neq f(t_1)$ and $n = 2$. In particular, if $s_1 > t_1$, $f(s_1) \neq f(t_1)$. But then for $i \in T$, $f(i) \neq f(s_1)$, so $f(i) = f(t_1)$.

Conversely, if we choose $f(i) \in 1, 2$ arbitrarily for indices $i \notin [t_1 + 1, s_1]$, and then define $f(i)$ for $i \in [t_1 + 1, s_1]$ by the rule $f(i) = f(t_1)$ if and only if $i \in T$, then f automatically satisfies the condition $f(i) \neq f(j)$ for all $i < j$, $i \in T$, $j \notin T$, since for any such i, j we must have $t_1 \leq i < j \leq s_1$.

If $\nu_i = 0$ for some i, $Z(\nu, T)$ is empty. If there are indices $i, j \in [t_1, s_1]$ with $i \in T$, $j \notin T$, and $\nu_i = \nu_j = 1$, $Z(\nu, T)$ is again empty. We need only consider the remaining cases.

Let $L = \{i_1 < \cdots < i_{l'}\}$ be the set of indices $i \notin [t_1, s_1]$ such that $\nu_i = 2$, together with t_1 provided that $\nu_i = 2$ for all $i \in [t_1, s_1]$. Let $f'\colon L \to [2]$ be arbitrary. By the analysis above, since $f(i)$ must equal 1 for indices such that $\nu_i = 1$, there is a unique $f\colon [l] \to [2]$ such that $f(i) = f'(i)$ for $i \in L$ and $W_{f,T} \subseteq Z(\nu, T)$. Every component $W_{f,T} \subseteq Z(\nu, T)$ arises this way.

Let $T' = \{j : i_j \in T\}$. Since $[t_1 + 1, s_1] \cap L = \emptyset$, T' is a final segment $[t', l']$ of $[l']$ (we take $t' = l' + 1$ if $T' = \emptyset$). If we define $g\colon [l'] \to [2]$ by $g(j) = f(i_j)$, then $\psi_{[2],L}$ maps $W_{f,T}$ isomorphically onto $W_{g,T'}$, or, if $t_1 \notin L$ and $T' \neq \emptyset$, onto $W_{g,T'} \cap V(a_{t'} - x_1)$ or $W_{g,T'} \cap V(a_{t'} - x_2)$. Which of the latter we get depends on the value of $f(t_1)$, which is fixed when $t_1 \notin L$. This shows that $\psi_{[2],L}$ maps $Z(\nu, T)$ isomorphically onto $Z((2^{l'}), T')$ or its intersection with $V(a_{t'} - x_1)$ or $V(a_{t'} - x_2)$. □

Lemma 2.7. *The ideal I of $Z(2, l, (2^l), [t, l])$ is generated by the following elements:*

1. $(a_i - x_1, b_i - y_1)(a_i - x_2, b_i - y_2)$ *for each* $i \in [l]$,
2. $a_i - a_j$ *for* $i, j \in [t, l]$.
3. *the determinants*

$$\det \begin{bmatrix} 1 & a_i & b_i \\ 1 & a_j & b_j \\ 1 & x_k & y_k \end{bmatrix}$$

for $i, j \in [l]$, $k \in [2]$.

Moreover the ideals $I + (a_t - x_1)$ and $I + (a_t - x_2)$ are radical, i.e. the scheme-theoretic intersection of $Z(2, l, (2^l), [t, l])$ with $V(a_t - x_1)$ or $V(a_t - x_2)$ is reduced.

Proof. It is clear that the specified generators do vanish on $Z((2^l), [t, l])$. The determinants vanish because two of the rows must be equal. The vanishing of generators (1)–(2) is already enough to define $Z((2^l), [t, l])$ set-theoretically. Let J be the ideal generated by the elements (1)–(3). We have $V(J) = Z((2^l), [t, l])$ set-theoretically.

We claim that R/J is generated as a $k[y_1, y_2, x_1]$ module by the images of monomials of the form

$$x_2^d b_{i_1} \cdots b_{i_r} a_{j_1} \cdots a_{j_s}, \tag{2.6}$$

where $i_1 < \cdots < i_r < j_1 < \cdots j_s$, $j_s \leq t$ if $s \neq 0$, and $d = 0$ if $r \neq 0$.

To see this we order the monomials in $R/(y_1, y_2, x_1) = k[x_2, \mathbf{a}, \mathbf{b}]$ lexicographically, with the ordering of variables $x_2 < a_1 < \cdots < a_l < b_1 < \cdots < b_l$. Reducing the given generators modulo (y_1, y_2, x_1) we obtain polynomials with the following leading terms. From the generators (1) we get a_i^2, $a_i b_i$, and b_i^2 for all i, and also $x_2 b_i$ (which belongs to $(a_i, b_i)(a_i - x_2, b_i)$). From the determinants

$$\det \begin{bmatrix} 1 & a_i & b_i \\ 1 & a_j & b_j \\ 1 & x_1 & y_1 \end{bmatrix} \equiv a_i b_j - a_j b_i \pmod{(y_1, y_2, x_1)}$$

we get $a_i b_j$ for $i < j$. From generators (2) we get a_i for $i > t$. The only monomials not divisible by any of these leading terms are those of the form (2.6).

The ideal I is doubly homogeneous. For any closed point $P = (y_1 - \alpha, y_2 - \beta) \in \operatorname{Spec} k[\mathbf{y}]$, the coordinate ring of the fiber of $Z((2^l), [t, l])$ over P is graded by $\mathbf{x}$-degree. For $\alpha \neq \beta$, the components $W_{f,T}$ of $Z((2^l), [t, l])$ have disjoint fibers over P. There are 2^l of them, of which 2^t (those with $f(t) = f(t+1) = \cdots = f(l)$) project isomorphically onto $\operatorname{Spec} k[x_1, x_2]$ and the rest onto the subspace $V(x_1 - x_2)$. It follows that the Hilbert series of the fiber over a generic point P is equal to

$$H(q) = 2^t \frac{1}{(1-q)^2} + (2^l - 2^t) \frac{1}{1-q}.$$

The ideal J is also doubly homogeneous, and since it is contained in I, the Hilbert series of $R/J + (y_1, y_2)$ must be coefficientwise greater than or equal to $H(q)$.

Let B be the set of monomials $x_1^e m$, where m is of the form (2.6) and e is arbitrary. Then B spans $R/J + (y_1, y_2)$ as a vector space over k. The generating function enumerating elements of B by their degree in the variables $\mathbf{x}, \mathbf{a}$ is equal to $H(q)$. This implies that B is a basis of $R/J + (y_1, y_2)$, that $J + (y_1, y_2) = I + (y_1, y_2)$, and that R/I is a free $k[y_1, y_2]$ module with basis B. Since B generates R/J as a $k[y_1, y_2]$ module, the canonical surjection $R/J \to R/I$ is injective, and $I = J$.

The case of $I + (a_t - x_1)$ or $I + (a_t - x_2)$ is handled similarly. The extra generator allows us to impose the extra restriction $j_s < t$ if $s \neq 0$ on the monomials in (2.6). The Hilbert series of a generic fiber of the reduced intersection $Z((2^l), [t, l]) \cap V(a_t - x_k)$ becomes

$$H_1(q) = 2^{t-1}\frac{1}{(1-q)^2} + (2^l - 2^{t-1})\frac{1}{1-q},$$

since we have $x_1 = x_2$ on $W_{f,T} \cap V(a_t - x_k)$ unless $f(t) = f(t+1) = \cdots = f(l) = k$. The generating function for the restricted monomial basis B_1 is equal to $H_1(q)$ and the rest of the argument proceeds as before. □

Corollary 2.8. *For $n = 2$, $R/I(2, l, \nu, T)$ is a free $k[y_1, y_2]$ module for all ν and T.*

Proof. For $\nu = (2^l)$, $T = [t, l]$ this follows from the proof of Lemma 2.7, and all other cases reduce to these by Lemma 2.4. □

Corollary 2.9. *For $n = 2$ the ideal of $Z(\nu, T)$ is generated by 1, if $Z(\nu, T)$ is empty, or else by the elements*
(1) $a_i - x_1, b_i - y_1$, *for* $\nu_i = 1$,
(2) $a_i + a_j - (x_1 + x_2), b_i + b_j - (y_1 + y_2)$, *for* $i < j$, $i \in T$, $j \notin T$,
(3) $(a_i - x_1, b_i - y_1)(a_i - x_2, b_i - y_2)$ *for* $\nu_i = 2$,
(4) $a_i - a_j$ *for* $i, j \in T$, *and*
(5) *the determinants in Lemma 2.7 (3).*

Proof. Let $Y \subseteq E^2 \times E^l$ be the affine subspace defined by (1)–(2). With L, l', t' as in the proof of Lemma 2.6, the map $\psi_{[2],L}$ is an isomorphism of Y onto $E^2 \times E^{l'}$, and the proof of Lemma 2.6 shows that $Z(\nu, T) = \psi_{[2],L}^{-1}(Z) \cap Y$, where Z is equal to $Z((2^{l'}), [t', l'])$, or to its intersection with $V(a_{t'} - x_1)$ or $V(a_{t'} - x_2)$ if $t_1 \notin L$ and $T \cap L \neq \emptyset$. The images under $\psi_{[2],L}^{\sharp}$ of the generators given by Lemma 2.7 for $I((2^{l'}), [t', l'])$ are among (3)–(5). In the case $t_1 \notin L$, $T \cap L \neq \emptyset$, the image of the extra generator $a_{t'} - x_1$ or $a_{t'} - x_2$ reduces to zero modulo (1)–(2) and (4).

This shows that the ideal generated by (1)–(5) contains $I(\nu, T)$, and the reverse containment is clear. □

The next two lemmas provide reductions that we will use to prove Theorem 4 by induction. As above we consider only $Z(\nu, T)$ for $l > 0$ and $T \neq \emptyset$, and let t_1 denote the least element of T.

Lemma 2.10. *Assume $t_1 > 1$. Let $\theta\nu = (\nu_2, \ldots, \nu_l, \nu_1)$ and $\theta T = \{t - 1 : t \in T\}$. If $R/I(\theta\nu, \theta T)$ and $R/I(\theta\nu, \theta T \cup \{l\})$ are free $k[\mathbf{y}]$ modules, then so is $R/I(\nu, T)$.*

Lemma 2.11. *Assume $t_1 = 1$, $\nu_1 > 0$, and $\nu_i \geq \nu_1$ for all $i \notin T$. Let $S = \{i \in T \setminus \{1\} : \nu_i \geq \nu_1\}$. Suppose that $R'/I(n', l', \nu', T')$ is a free $k[y_1, \ldots, y_{n'}]$ module in the following cases:*
(1) $n' = n$, $l' = l$, $\nu' = (\nu_1 - 1, \nu_2, \ldots, \nu_l)$, $T' = T$;
(2) *for each subset $S' \subseteq S$, with $L = [l] \setminus (S' \cup \{1\}) = \{i_1 < \cdots < i_{l'}\}$, the case $n' = n - 1$, $l' = |L|$, $T' = \{j : i_j \in T\}$, and $\nu' = (\hat{\nu_{i_1}}, \ldots, \hat{\nu_{i_l}})$, where*

$$\hat{k} = \begin{cases} k - 1 & \text{if } k \geq \nu_1, \\ k & \text{otherwise;} \end{cases}$$

(3) *for each subset $S' \subseteq S$, with $L = [l] \setminus S' = \{i_1 < \cdots < i_{l'}\}$, the case $n' = n - 1$, $l' = |L|$, $T' = \{j - 1 : i_j \in T \setminus \{1\}\} \cup \{l'\}$, $\nu' = (\hat{\nu_{i_2}}, \ldots, \hat{\nu_{i_{l'}}}, \hat{\nu_{i_1}})$.*

Then $R/I(\nu, T)$ is a free $k[\mathbf{y}]$ module.

Proof of Lemma 2.10. By analogy with the definition of $\theta\nu$ and θT in the statement of the lemma, define $\theta\colon E^n \times E^l \to E^n \times E^l$ to be the "cyclage" automorphism

$$\theta(\mathbf{x}, \mathbf{y}, a_1, b_1, \dots, a_l, b_l) = (\mathbf{x}, \mathbf{y}, a_2, b_2, \dots, a_l, b_l, a_1, b_1).$$

Let $I = I(\nu, T)$.

Set theoretically, we have

$$\theta V(I + (a_1 - a_{t_1})) = Z(\theta\nu, \theta T \cup \{l\}) \tag{2.7}$$

$$\theta V(I : (a_1 - a_{t_1})) = Z(\theta\nu, \theta T). \tag{2.8}$$

Equation (2.7) is obvious. For (2.8) observe that the components $W_{f,T} \subseteq Z(\nu, T)$ on which $a_1 - a_{t_1}$ does not vanish identically are those with $f(1) \neq f(t)$ for all $t \in T$. Applying θ to these gives exactly the components of $Z(\theta\nu, \theta T)$. Equation (2.8) holds scheme-theoretically, since for a radical ideal I, $I : J$ is always radical.

We claim that (2.7) holds scheme-theoretically, locally on U. If $Z(\theta\nu, \theta T \cup \{l\})$ is empty there is nothing to prove. Otherwise fix a point $P \in U_{pq} \cap Z(\theta\nu, \theta T \cup \{l\})$. By Lemma 2.4 (1), all components $W_{f,\theta T \cup \{l\}} \subseteq Z(\theta\nu, \theta T \cup \{l\})$ containing P have a common value of $L = f^{-1}(\{p, q\})$ and $h = f|_{[l]\setminus L}$. The same holds for the components $W_{f,\theta T} = W_{\theta g, \theta T} = \theta W_{g,T}$ of $\theta Z(\nu, T)$. These components correspond to the same functions f in both cases. Now let Z be the union of the components $W_{f,\theta T \cup \{l\}}$ for these f and let Z_1 be the union of the corresponding components $W_{f,\theta T}$, so $Z(\theta\nu, \theta T \cup \{l\})$ is locally equal to Z and $\theta Z(\nu, T)$ is locally equal to Z_1. We are to show that $Z = Z_1 \cap V(a_l - a_{t_1 - 1})$ scheme-theoretically.

Now we apply Lemma 2.4 (2) to Z and Z_1. More accurately, for Z_1 we apply Lemma 2.4 (2) to $\theta^{-1} Z_1$, which is locally isomorphic to $Z(\nu, T)$, and then apply θ to the result. This yields $Z = \psi_{N,L}^{-1}(Z') \cap Y$ and $Z_1 = \psi_{N,L}^{-1}(Z_1') \cap Y_1$, where $N = \{p, q\}$, $Z' = \psi_{N,L}(Z)$, $Z_1' = \psi_{N,L}(Z_1)$, Y is defined as in Lemma 2.4 (2) with $\theta T \cup \{l\}$ in place of T, and Y_1 is defined similarly but with θT in place of T.

For $l \notin L$, we have $Y = Y_1 \cap V(a_l - a_{t_1 - 1})$, as can be seen from the defining equations of Y and Y_1 by considering the cases $t_1 - 1 \in L$ and $t_1 - 1 \notin L$ separately, and taking the arbitrary element t in Lemma 2.4 (2) to be $t_1 - 1$ in the case $t_1 - 1 \in L$. For $l \in L$ and $\theta T \cap L = \emptyset$, since $(\theta T \cup \{l\}) \cap L = \{l\}$, Y is defined by the equations of Y_1 together with $a_l = x_{h(i)}$ for $i \in \theta T$, which is equivalent to $a_l = a_{t_1 - 1}$, and we again have $Y = Y_1 \cap V(a_l - a_{t_1 - 1})$. In these cases, $\psi_{N,L}(W_{f,\theta T}) = \psi_{N,L}(W_{f,\theta T \cup \{l\}})$ for all relevant f, so $\psi_{N,L}(Z) = \psi_{N,L}(Z_1)$. As the identities $Z = \psi_{N,L}^{-1}(Z') \cap Y$, $Z_1 = \psi_{N,L}^{-1}(Z_1') \cap Y_1$ hold scheme-theoretically, so therefore does $Z = Z_1 \cap V(a_l - a_{t_1 - 1})$.

For $l \in L$ and $\theta T \cap L \neq \emptyset$ we have $Y = Y_1$ by inspection. Note that $i_{l'} = l$ in this case. Let $t = i_{t'}$ be an arbitrary element of $\theta T \cap L$. On Z_1 we have $a_t = a_{t_1 - 1}$ identically, so $Z_1 \cap V(a_l - a_{t_1 - 1}) = Z_1 \cap V(a_l - a_t) = \psi_{N,L}^{-1}(Z_1' \cap V(a_{l'} - a_{t'})) \cap Y$, all intersections being scheme-theoretic.

It remains to prove that the scheme-theoretic intersection $\psi_{N,L}(Z_1) \cap V(a_{l'} - a_{t'})$ is reduced. But $\psi_{N,L}(Z_1) = \theta' Z(2, l', \nu', T')$ for a suitably defined ν' and T' with $1 \notin T'$, $t' \in \theta' T'$, where θ' is the cyclage automorphism of $E^2 \times E^{l'}$. Using Corollary 2.9 we see that $\theta' Z(2, l', \nu', T') \cap V(a_{l'} - a_{t'}) = Z(2, l', \theta'\nu', \theta' T' \cup \{l'\})$ scheme-theoretically.

By hypothesis, $R/I(\theta\nu, \theta T \cup \{l\})$ and $R/I(\theta\nu, \theta T)$ are free $k[\mathbf{y}]$ modules. Since we have shown that (2.7) holds scheme-theoretically on U, we may apply Lemma 2.2, with the present I, $f = a_1 - a_{t_1}$, and $J = \sqrt{I + (a_1 - a_{t_1})}$, to conclude that (2.7) holds scheme-theoretically everywhere, and R/I is a free $k[\mathbf{y}]$ module. □

Proof of Lemma 2.11. Let Z_1 denote the union of the components $W_{f,T} \subseteq Z(\nu, T)$ for which $f(1) = \nu_1$ and let Z_2 be the union of those for which $f(1) < \nu_1$ (either may possibly be empty). Then $Z_2 = Z(\nu', T)$, where $\nu' = (\nu_1 - 1, \nu_2, \dots, \nu_l)$, and therefore $R/I(Z_2)$ is a free $k[\mathbf{y}]$ module by hypothesis (1).

Since $1 \in T$, we have $f(j) \neq \nu_1$ for all $j \notin T$, for all components $W_{f,T} \subseteq Z_1$. Thus the set S contains all indices $s \neq 1$ which we could possibly have $f(s) = \nu_1$. To analyze Z_1 we further classify its components $W_{f,T}$ according to the subset $f^{-1}(\{\nu_1\}) = S' \cup \{1\}$, where $S' \subseteq S$.

Let the elements of S be $\{s_1, s_2, \dots, s_k\}$, fix a subset $S' \subseteq S$, and define the ideal

$$I' = I(Z_1) +/\!: (b_{s_1} - y_{\nu_1}) +/\!: (b_{s_2} - y_{\nu_1}) +/\!: \cdots +/\!: (b_{s_k} - y_{\nu_1}), \tag{2.9}$$

where at each step we apply the ideal-theoretic operation $+(b_{s_i} - y_{\nu_1})$ if $s_i \in S'$, or $: (b_{s_i} - y_{\nu_1})$, otherwise. Also define

$$I'_0 = I(Z_1) \;:\; \prod_{s \in S \setminus S'} (b_s - y_{\nu_1}) + \sum_{s \in S'} (b_s - y_{\nu_1})$$

and

$$I'_1 = \sqrt{I(Z_1) + \sum_{s \in S'} (b_s - y_{\nu_1})} \;:\; \prod_{s \in S \setminus S'} (b_s - y_{\nu_1}).$$

We have $I'_0 \subseteq I' \subseteq I'_1$, by the general relation $(I : J) + K \subseteq (I + K) : J$ valid for arbitrary ideals I, J, K.

Note that on Z_1 we have identically $a_t = x_{\nu_1}$ for all $t \in T$, since $1 \in T$, and every component $W_{f,T} \subseteq Z_1$ has $f(1) = \nu_1$. Given any such component $W_{f,T}$, let $g\colon [l] \to [n]$ agree with f except on indices $s \in S'$. For these set $g(s) = \nu_1$. The defining equations (2.1) of $W_{g,T}$ hold also on $W_{f,T}$, except for those of the form $b_s = y_{\nu_1}$ for $s \in S'$. Therefore we have

$$W_{f,T} \cap V(\sum_{s \in S'} (b_s - y_{\nu_1})) \subseteq W_{g,T}. \tag{2.10}$$

We also have $g(i) \le \nu_i$ for all i by the definition of S, and $g(i) \ne g(j)$ for $i < j$, $i \in T$, $j \notin T$, since $g(j) = f(j) \ne \nu_1$, and either $g(i) = f(i) \ne f(j)$ or $g(i) = \nu_1$. Therefore $W_{g,T}$ is a component of Z_1. Moreover $W_{f,T}$ and $W_{g,T}$ are the graphs of the maps $\pi_f, \pi_g\colon E^n \to E^l$, restricted respectively to the subspaces $V(x_i - x_j : i, j \in f(T))$ and $V(x_i - x_j : i, j \in g(T))$. Since $\nu_1 \in f(T)$ we have $g(T) \subseteq f(T)$ and $V(x_i - x_j : i, j \in f(T)) \subseteq V(x_i - x_j : i, j \in g(T))$. It follows that the projection of $W_{f,T}$ on the coordinates $\mathbf{x}, \mathbf{y}$ and a_i, b_i for $i \notin S'$ is contained in that of $W_{g,T}$.

The containment (2.10) implies that $Z_1 \cap V(\sum_{s \in S'} (b_s - y_{\nu_1}))$ is, set-theoretically, the union of those components $W_{f,T} \subseteq Z_1$ with $f(s) = \nu_1$ for all $s \in S'$. Since $b_s - y_{\nu_1}$ vanishes on $W_{f,T}$ if and only if $f(s) = \nu_1$, it follows that $V(I'_1)$ is the union of those components $W_{f,T} \subseteq Z_1$ with $f^{-1}(\{\nu_1\}) = S' \cup \{1\}$. Note that $I'_1 = \sqrt{I'_1}$ by construction. Similarly, $I(Z_1) : \prod_{s \in S \setminus S'} (b_s - y_{\nu_1})$ is the ideal of the union of those components $W_{f,T} \subseteq Z_1$ with $f(s) \ne \nu_1$ for $s \in S \setminus S'$.

Now let $p \in I'_1$ be arbitrary, and let q be the polynomial resulting from the substitutions $(a_s \mapsto x_{\nu_1}, b_s \mapsto y_{\nu_1} : s \in S')$ in p. We have $q - p \in I'_0$ (recall that $a_s - x_{\nu_1} \in I(Z_1)$ for all $s \in T$), hence $q \in I'_1$, i.e., q vanishes on $W_{g,T} \subseteq Z_1$ if $g^{-1}(\{\nu_1\}) = S' \cup \{1\}$. Since q does not depend on a_i, b_i for $i \in S'$ it follows from the remarks after (2.10) that q vanishes on every $W_{f,T} \subseteq Z_1$ satisfying $f(s) \ne \nu_1$ for all $s \in S \setminus S'$, since the g corresponding to such an f has $g^{-1}(\{\nu_1\}) = S' \cup \{1\}$. This proves $q \in I(Z_1) : \prod_{s \in S \setminus S'} (b_s - y_{\nu_1})$ and hence $p \in I'_0$. We have now proved $I'_1 \subseteq I'_0$ and hence $I'_0 = I' = I'_1$.

Given f satisfying $f^{-1}(\{\nu_1\}) = S' \cup \{1\}$, the condition for $W_{f,T}$ to be a component of Z_1 is merely that (2.2) should hold for indices $i, j \notin S' \cup \{1\}$. The conditions hold automatically for other indices because for $i \in S' \cup \{1\}$ and $j \notin T$ we have $f(i) = \nu_1 \ne f(j)$. Set $N = [n] \setminus \{\nu_1\}$, $L = [l] \setminus (S' \cup \{1\}) = \{i_1 < \cdots < i_{l'}\}$, and let ν', T' be as in hypothesis (2) of the lemma. We then have $I' = I(N, L, \nu', T') + I(Y)$ where

$$I(Y) = (a_s - x_{\nu_1}, b_s - y_{\nu_1} : s \in S' \cup \{1\}) + \begin{cases} a_t - x_{\nu_1} & \text{if } T' \ne \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

with $t = i_j$ for an arbirtrary $j \in T'$. Set-theoretically, this is clear, and $\psi_{N,L}^{-1}(Z(n-1, l', \nu', T')) \cap Y$ is scheme-theoretically reduced because the coordinates x_j, y_j for $j \in N$ and a_i, b_i for $i \in L$ are independent on Y. Since all the $\mathbf{y}$ coordinates are independent on Y we further have that R/I' is a free $k[\mathbf{y}]$ module.

For any ideal $I \subseteq R$ and element $f \in R$ we have the exact sequence

$$0 \to R/I : (f) \underset{\cdot f}{\to} R/I \to R/I + (f) \to 0,$$

in which the middle term is a free $k[\mathbf{y}]$ module if the outer terms are. Iterating this and using the freeness of R/I' for every $S' \subseteq S$ we deduce that $R/I(Z_1)$ is a free $k[\mathbf{y}]$ module.

Now let $J = I(Z_1) + I(Z_2)$. Since $Z(\nu, T) = Z_1 \cup Z_2$ we have the exact sequence

$$0 \to R/I(\nu, T) \to R/I(Z_1) \oplus R/I(Z_2) \to R/J \to 0. \tag{2.12}$$

We might hope to apply Lemma 2.3 at this point, but an obstacle to doing this is that $V(J) = Z_1 \cap Z_2$ is not scheme-theoretically reduced. To evade this difficulty we consider the ideals

$$J' = J +/\!: (b_{s_1} - y_{\nu_1}) +/\!: \cdots +/\!: (b_{s_k} - y_{\nu_1}),$$

where again we apply $+(b_{s_i} - y_{\nu_1})$ if $s_i \in S'$, or $: (b_{s_i} - y_{\nu_1})$, otherwise, with $S' \subseteq S$ as before. We will show that R/J' is a free $k[\mathbf{y}']$ module, where $\mathbf{y}' = \mathbf{y} \setminus \{y_{\nu_1}\}$. Since this holds for all S', R/J is then a free $k[\mathbf{y}']$ module, which implies that $R/I(\nu, T)$ is a free $k[\mathbf{y}]$ module, by (2.12).

Let Z_3 be the union of the subspaces $W_{f,T} \cap V(x_{\nu_1} - a_1, y_{\nu_1} - b_1)$ over functions $f \colon [l] \to [n]$ satisfying

$$\begin{array}{ll} f^{-1}(\{\nu_1\}) = S' & \text{(in particular } \nu_1 \neq 1\text{)}, \\ f(i) \leq \nu_i & \text{for all } i \in [l], \text{ and} \\ f(i) \neq f(j) & \text{for } i < j,\ i \in T \setminus (S' \cup \{1\}),\ j \notin T. \end{array} \tag{2.13}$$

Let $K = I(Z_3) \subseteq R$ be the ideal of Z_3.

Given the condition $f^{-1}(\{\nu_1\}) = S'$, the remaining conditions in (2.13) are equivalent to their apparently weaker variants in which the indices i and j which appear are restricted to $[l] \setminus S'$. This is the case because for $i \in S'$ we have $f(i) = \nu_1 \leq \nu_i$, and $j \notin T$ implies $j \notin S'$. Note that the conditions (2.13) do *not* include $f(1) \neq f(j)$ for $j \notin T$.

Set $N = [n] \setminus \{\nu_1\}$, $L = [l] \setminus S' = \{i_1 < \cdots < i_{l'}\}$, $\nu' = (\hat{\nu_{i_2}}, \ldots, \hat{\nu_{i_{l'}}}, \hat{\nu_{i_1}})$, $T' = \{j - 1 : i_j \in T \setminus \{1\}\} \cup \{l'\}$. Note that this use of L, ν', T' differs from our preceding one, and that we now have $1 \in L$, so $i_1 = 1$. Let $\theta' \colon E^{n-1} \times E^{l'} \to E^{n-1} \times E^{l'}$ be the cyclage automorphism $\theta'(\mathbf{x}, \mathbf{y}, a_1, b_1, \ldots, a_{l'}, b_{l'}) = (\mathbf{x}, \mathbf{y}, a_2, b_2, \ldots, a_{l'}, b_{l'}, a_2, b_2)$. Then we have

$$Z_3 = \psi_{N,L}^{-1}(\theta^{-1} Z(n-1, l', \nu', T')) \cap Y,$$

where Y is defined by the equations $a_s = x_{\nu_1}, b_s = y_{\nu_1}$ for $s \in S' \cup \{1\}$.

The coordinates x_j, y_j for $j \in N$ and a_i, b_i for $i \in L$ are independent, and in fact are a basis of coordinates on Y, which implies that Z_3 is isomorphic to $Z(n-1, l', \nu', T')$. By hypothesis (3) of the lemma it follows that R/K is a free $k[\mathbf{y}']$ module.

We will next show that $V(I') \cap Z_2 \cap U = V(J') \cap U = Z_3 \cap U$, scheme-theoretically. It then follows from Lemma 2.3 that $I' + I(Z_2) = K$. In particular $V(I') \cap Z_2$ is reduced and equal to the closure of its intersection with U. It is clear that $I' + I(Z_2) \subseteq J'$, so $V(J')$ is contained (scheme-theoretically) in $V(I') \cap Z_2$. Since these subschemes agree on U it follows that $J' = I' + I(Z_2) = K$, and R/J' is a free $k[\mathbf{y}']$ module, as was to be shown.

Now fix $p < q$ and a point $P \in U_{pq}$. By Lemma 2.4 all spaces $W_{f,T}$ containing P have a common value of $L = f^{-1}(\{p, q\})$ and $h = f|_{[l] \setminus L}$. Fix L and h, and let $Y \subseteq E^n \times E^l$ be the affine subspace defined by the equations

$$\begin{array}{rl} x_p = x_q, y_p = y_q & \\ a_i = x_q, b_i = y_q & \text{for } i \in L, \\ a_i = x_{h(i)}, b_i = y_{h(i)} & \text{for } i \notin L, \text{ and} \\ x_{h(i)} = x_q & \text{for } i \in T \setminus L. \end{array} \tag{2.14}$$

For all f such that $f^{-1}(\{p, q\}) = L$ and $f|_{[l] \setminus L} = h$ we have from the definition of $W_{f,T}$ that $W_{f,T} \cap V(x_p - x_q, y_p - y_q) = Y$, provided $T \cap L \neq \emptyset$.

On every component $W_{f,T} \cap V(x_{\nu_1} - a_1, y_{\nu_1} - b_1)$ of Z_3 we have $x_{\nu_1} = x_{f(1)}$, $y_{\nu_1} = y_{f(1)}$, with $f(1) < \nu_1$. Such a component is disjoint from U_{pq} unless we have $p = f(1)$, $q = \nu_1$. In particular Z_3 is locally empty at P unless $q = \nu_1$ and $1 \in L$. On Z_1 we have identically $a_1 = x_{\nu_1}$, $b_1 = y_{\nu_1}$, and every component $W_{f,T} \subseteq Z_2$ has $f(1) < \nu_1$. Hence $Z_1 \cap Z_2$ is also locally empty at P unless $q = \nu_1$ and $1 \in L$, as is $V(J') \subseteq V(I') \cap Z_2 \subseteq Z_1 \cap Z_2$. Therefore we may assume without loss of generality that $q = \nu_1$ and $1 \in L$. In particular, $T \cap L \neq \emptyset$.

Locally, every component $W_{f,T} \cap V(x_{\nu_1} - a_1, y_{\nu_1} - b_1)$ of Z_3 has $f(1) = p$, so $W_{f,T} \cap V(x_{\nu_1} - a_1, y_{\nu_1} - b_1) = W_{f,T} \cap V(x_p - x_q, y_p - y_q) = Y$. This shows that locally at P, Z_3 is either empty or equal to Y (scheme-theoretically, since Z_3 is reduced).

Only one possible component $W_{g,T} \subseteq V(I')$ could contain P, since g must satisfy $g(i) = h(i)$ for $i \notin L$, while for $i \in L$ we must have $g(i) = \nu_1 = q$ for $i \in S' \cup \{1\}$ and $g(i) = p$ for $i \notin S' \cup \{1\}$. In particular $g(1) = q$, so we have $x_q - a_1, y_q - b_1 \in I'_P$. Every component $W_{f,T} \subseteq Z_2$ containing P must have $f(1) = p$, since $1 \in L$ and $f(1) \neq \nu_1 = q$. Therefore we have $x_p - a_1, y_p - a_1 \in I(Z_2)_P$. It follows that $V(I') \cap Z_2$ is locally contained (scheme-theoretically) in Y, since we have $W_{g,T} \cap V(x_p - x_q, y_p - y_q) = Y$ for the unique possible local component $W_{g,T}$ of $V(I')$. Since every $W_{f,T}$ containing P also contains Y, $V(I') \cap Z_2$ is locally either empty or equal to Y. In particular it is locally reduced.

Let $f\colon [l] \to [n]$ satisfy (2.13), and define f' to agree with f, except $f'(1) = \nu_1$. Since $f^{-1}(\{\nu_1\}) = S' \subseteq T$, we have $f'(j) = f(j) \neq \nu_1$ for $j \notin T$, and therefore $f'(i) \neq f'(j)$ for all $i < j$, $i \in T$, $j \notin T$. We clearly also have $f'(i) \leq \nu_i$ for all i, so $W_{f',T} \subseteq Z_1$. Since $(f')^{-1}(\{\nu_1\}) = S' \cup \{1\}$, we have $W_{f',T} \subseteq V(I')$. Moreover, $W_{f,T} \cap V(x_{\nu_1} - a_1, y_{\nu_1} - b_1) \subseteq W_{f',T}$. This shows that $Z_3 \subseteq V(I')$ (not just locally at P, but everywhere). Hence if $V(I')$ is locally empty at P then so are Z_3 and $V(J') \subseteq V(I')$, and we have $V(I') \cap Z_2 = V(J') = Z_3$ locally, as desired.

If $V(I')$ is not locally empty at P, the function g giving the unique local component $W_{g,T} \subseteq V(I')$ must satisfy $g^{-1}(\{\nu_1\}) = S' \cup \{1\}$, so we must have $S' \subseteq L$. Also, g must satisfy (2.2). Conversely, if g satisfies these conditions then we do have $W_{g,T} \subseteq V(I')$, and hence locally $Y \subseteq V(I')$. Now define f to agree with g except set $f(i) = p$ for all $i \in T \cap L$, and set $f(j) = \nu_1 = q$ for all $j \in L \setminus T$. Note that we still have $f^{-1}(\{p, q\}) = L$ and $f|_{[l]\setminus L} = h$. Since $g(i) \leq \nu_i$ for all i, we have $\nu_i \geq p$ for all $i \in L$, and by the hypothesis of the lemma we have $\nu_i \geq \nu_1 = q$ for all $i \notin T$. Therefore $f(i) \leq \nu_i$ for all i. Suppose $i < j$, $i \in T$, $j \notin T$. If $i, j \notin L$, then $f(i) = g(i) \neq g(j) = f(j)$. If $i, j \in L$, then $f(i) = p$, $f(j) = q$. If one of i, j belongs to L and the other does not, then $f(i) \neq f(j)$ a fortiori. Thus $W_{f,T} \subseteq Z_2$, so $Y \subseteq Z_2$. Hence $V(I') \cap Z_2 = Y$ locally at P.

Now, still assuming $V(I')$ locally non-empty, consider a new f defined to agree with g except for $f(1) = p$. Then f satisfies (2.13), so $W_{f,T} \cap V(x_{\nu_1} - a_1, y_{\nu_1} - b_1) \subseteq Z_3$, and hence $Z_3 = Y$ locally at P. In particular we have $V(I') \cap Z_2 = Z_3$ locally at P in every case. (It may still be that Y itself is locally empty at P, but there is no harm in this).

It remains to prove that $Y \subseteq V(J')$ locally at P in the case where we have locally $V(I') \cap Z_2 = Z_3 = Y$. In other words we are to prove (locally) that

$$I(Z_1) + I(Z_2) +/\!:(b_{s_1} - y_{\nu_1}) +/\!: \cdots +/\!:(b_{s_k} - y_{\nu_1}) \subseteq I(Y), \tag{2.15}$$

where we may assume that $q = \nu_1$, $S' \cup \{1\} \subseteq L$, and g as defined above satisfies (2.2).

Let $\hat{Y}_1 \subseteq E^n \times E^l$ be the non-reduced subscheme whose ideal is generated by the following elements:

$$(2.16)\qquad \begin{array}{ll} x_p - x_q, y_p - y_q, & \\ a_i - x_q & \text{for } i \in L, \\ b_i - b_1 & \text{for } i \in S', \\ (b_i - y_q)^2 & \text{for } i \in L \cap (S \cup \{1\}) \setminus S', \\ b_i - y_q & \text{for } i \in L \cap T \setminus (S \cup \{1\}), \\ b_i + b_1 - 2y_q & \text{for } i \in L \setminus T, \\ a_i - x_{h(i)}, b_i - y_{h(i)} & \text{for } i \notin L, \text{ and} \\ x_{h(i)} - x_q & \text{for } i \in T \setminus L. \end{array}$$

Let $\hat{Y} = \hat{Y}_1 + (b_1 - y_q)$. Note that generators of $\hat{Y}$ are given by (2.16) with $S \cup \{1\}$ replaced by S. Since the linear forms in the variables $\mathbf{y}, \mathbf{b}$ that appear, possibly squared, in (2.16) are independent, we have

$$(2.17)\qquad I(\hat{Y}) : \prod_{s \in L \cap (S \setminus S')} (b_s - y_q) = I(Y).$$

For $s \notin L$ we have $b_s - y_q \notin I(Y)$, which implies

$$(2.18)\qquad I(Y) : \prod_{s \in S \setminus L} (b_s - y_q) = I(Y),$$

since $I(Y)$ is prime. Combining (2.17) and (2.18), and recalling that $S' \subseteq L$, we obtain

$$I(\hat{Y}) : \prod_{s \in S \setminus S'} (b_s - y_q) = I(Y).$$

The left-hand side of (2.15) is contained in $I(Z_1) + I(Z_2) + \sum_{s \in S'} (b_s - y_{\nu_1}) : \prod_{s \in S \setminus S'} (b_s - y_{\nu_1})$, so to establish (2.15) it suffices to prove

$$I(Z_1) + I(Z_2) + \sum_{s \in S'} (b_s - y_q) \subseteq I(\hat{Y}).$$

As the last summand on the left-hand side is contained in $I(\hat{Y})$ by definition, we only have to prove $I(Z_1) + I(Z_2) \subseteq I(\hat{Y})$.

By Lemma 2.4, part (2), $Z(\nu, T)$ is locally equal to the union Z of those components $W_{f,T} \subseteq Z(\nu, T)$ with $f^{-1}(\{p, q\}) = L$ and $f|_{[l] \setminus L} = h$. The ideal of Z is given by

$$(2.19)\qquad I(Z) = I(N, L, \nu', T') + (a_i - x_{h(i)}, b_i - y_{h(i)} : i \notin L) + (x_{h(i)} - a_1 : i \in T \setminus L),$$

where $N = \{p, q\}$, and if $L = \{i_1 < \cdots < i_{l'}\}$, we have $\nu'_j = |[\nu_{i_j}] \cap \{p, q\}|$ and $T' = \{j : i_j \in T\}$. Since $f(1) \in \{p, q\}$ for all components $W_{f,T} \subseteq Z$, we also have locally $I(Z_1) = I(Z) : (b_1 - y_p)$ and $I(Z_2) = I(Z) : (b_1 - y_q)$. Hence to prove $I(Z_1), I(Z_2) \subseteq I(\hat{Y})$ it suffices to prove $I(Z) \subseteq I(\hat{Y}_1)$, since $I(\hat{Y}_1) : (b_1 - y_q) = I(\hat{Y}_1) : (b_1 - y_p) = I(\hat{Y})$.

By definition we have $a_i - x_{h(i)}, b_i - y_{h(i)} \in I(\hat{Y}_1)$ for $i \notin L$, and $x_{h(i)} - a_1 \in I(\hat{Y}_1)$ for $i \in T \setminus L$ (note $a_1 - x_q \in I(\hat{Y}_1)$ because $1 \in L$). Thus we only have to show $I(N, L, \nu', T') \subseteq I(\hat{Y}_1)$. By the definition of $I(N, L, \nu', T')$ this reduces to showing that $I(2, l', \nu', T') \subseteq \hat{I}$, where

$$(2.20)\qquad \hat{I} = (x_1 - x_2, y_1 - y_2) + \sum_{i \in [l']} (a_i - x_2) + \sum_{i_j \in S'} (b_j - b_1) + \sum_{i_j \in (S \cup \{1\}) \setminus S'} (b_j - y_2)^2 + \sum_{i_j \in T \setminus (S \cup \{1\})} (b_j - y_2) + \sum_{i_j \notin T} (b_j + b_1 - 2y_2).$$

Note that if $\nu'_j = 1$, so $\nu_{i_j} < q$, we must have $i_j \in T \setminus (S \cup \{1\})$ and therefore $b_j - y_2 \in \hat{I}$ (here we again use the hypothesis $\nu_i \geq \nu_1$ for $i \notin T$). If $j < k$ are such that $i_j \in T$, $i_k \notin T$, then the function g giving the unique local component $W_{g,T} \subseteq V(I')$ satisfies $g(i_j), g(1) \neq g(i_k)$ and therefore $g(i_j) = g(1) = q$, so $i_j \in S'$. Hence I contains $b_j - b_1$, $b_k + b_1 - 2y_2$, and consequently $b_j + b_k - 2y_2$. For all j, we have $(b_j - y_2)^2 \in \hat{I}$. To see this in the cases $i_j \in S'$ or $i_j \notin T$ where it is not obvious, note that we have $(b_1 - y_2)^2 \in \hat{I}$ and $b_j - y_2 \equiv \pm(b_1 - y_2) \pmod{\hat{I}}$. Having made these observations, we now easily see that the generators of $I(2, l', \nu', T')$ given by Corollary 2.9 belong to $\hat{I}$. □

Proof of Theorem 4. Without loss of generality we may assume $l > 0$, $T \neq \emptyset$, and let t_1 be the least element of T. The proof is by double induction on t_1 and $|\nu|$, where $|\nu| = \nu_1 + \cdots + \nu_l$.

If $t_1 > 1$ the result follows by induction using Lemma 2.10. Note that the cases which must be assumed for Lemma 2.10 to apply have the same value of $|\nu|$, smaller t_1, and retain the hypothesis $\nu_i = n$ for $i \notin T$.

If $t_1 = 1$, the result is trivial if $\nu_1 = 0$, since then $Z(\nu, T)$ is empty, and otherwise follows by induction from Lemma 2.11. The hypothesis $\nu_i \geq \nu_1$ for $i \notin T$ required by Lemma 2.11 holds because we are assuming $\nu_i = n$ for $i \notin T$. Note that all cases which must be assumed for Lemma 2.11 to apply have smaller $|\nu|$, since $\nu'_i \leq \nu_i$, with strict inequality for ν_1. These cases also retain the hypothesis $\nu'_i = n'$ for $i \notin T'$. □

Given the intricacy of the proofs, the reader might reasonably question whether some small point might not have been overlooked which would invalidate the reduction given by Lemma 2.10 or 2.11. Some "experimental" reassurance can be given on this score. Let $H_{\nu,T}(q,t)$ denote the doubly graded Hilbert series whose coefficient of $t^r q^s$ is the dimension of $(R/I(\nu,T))_{(r,s)}$, i.e., t keeps track of $\mathbf{x}$-degree and q keeps track of $\mathbf{y}$-degree.

The Hilbert series $H_{\nu,T}(q,t)$ depends on n, but its "numerator" $H'_{\nu,T}(q,t) = (1-q)^n(1-t)^n H_{\nu,T}(q,t)$ depends only on ν and T. Since $R/I(\nu,T)$ is a finitely generated $k[\mathbf{x},\mathbf{y}]$ module, $H'_{\nu,T}(q,t)$ is a polynomial. From the inductive proof of Theorem 4 we can extract the following recursive procedure for computing $H'_{\nu,T}(q,t)$.

(1) If $l = 0$, $H'_{\nu,T} = 1$. Otherwise assume $l > 0$.
(2) If $\nu_i = 0$ for some i, $H'_{\nu,T} = 0$. Otherwise assume $\nu_i > 0$ for all i.
(3) For $T = \emptyset$, $H'_{\nu,\emptyset} = H'_{\nu,\{l\}}$. Assume $T \neq \emptyset$ and let t_1 be the least element of T.
(4) If $t_1 > 1$, $H'_{\nu,T} = tH'_{\theta\nu,\theta T} + H'_{\theta\nu,\theta T \cup \{l\}}$, where $\theta\nu$, θT are as in Lemma 2.10.
(5) If $t_1 = 1$,

$$H'_{\nu,T} = H'_{(\nu_1-1,\nu_2,\ldots,\nu_l),T} + \sum_{S' \subseteq S} q^{|S \setminus S'|}\left((1-t)^e H'_{\nu',T'} - (1-t)(1-q)H'_{\nu'',T''}\right),$$

where S is as in Lemma 2.11, ν', T' are given by Lemma 2.11 (2), ν'', T'' are given by Lemma 2.11 (3), and $e = 1$ if $T' \neq \emptyset$, otherwise $e = 0$.

The definition of $H_{(n^l),\emptyset}(q,t)$ as the Hilbert series of the polygraph $Z((n^l),\emptyset)$ is symmetric in q, t, although the recurrence for computing it is not. If the induction were not valid it would seem most improbable that the recurrence should give symmetric results for $H_{(n^l),\emptyset}(q,t)$. We have verified this symmetry by computer for all n, l with $n + l \leq 10$.

3. A correction to [MSRI]

We adopt the notation and terminology of [MSRI]. All equation and theorem numbers below refer to [MSRI]. In [MSRI] we took the ground field to be $\mathbb{C}$, and accordingly we do so here. The results are valid over any algebraically closed field k of characteristic zero, as these are the only properties of $\mathbb{C}$ we use.

The following sentence occurs in the paragraph preceding Theorem 4.5 (page 226).

> Conversely, if X_n is flat over $\mathrm{Hilb}^n(\mathbb{C}^2)$ at I_μ, then $B^{\otimes n}$ and $B^{\otimes n}/\mathcal{J}$ are both locally free, which implies that $\mathcal{J}$ is locally free and ϕ is a homomorphism of vector bundles.

Unfortunately the conclusion that ϕ is a homomorphism of vector bundles, i.e., that ϕ induces maps of constant rank on the fibers, does not follow. This conclusion is used to deduce the $n!$ conjecture if X_n

is Cohen-Macaulay, and again later, in the proof of Lemma 6.18. The set of points where ϕ is locally a homomorphism of vector bundles is open and $\mathbb{T}$-invariant. By the remarks following Proposition 4.4, this set contains I_μ if the $n!$ conjecture holds for μ. Below we prove directly that if X_n is Cohen-Macaulay, the $n!$ conjecture holds for all partitions μ of n. It then follows from Lemma 6.7 that ϕ is a vector bundle homomorphism everywhere, justifying our subsequent uses of this conclusion.

Assume now that X_n is Cohen-Macaulay. We first show that X_n is Gorenstein. We cannot use Lemma 6.18 for this, since its proof depends on the faulty proof of Theorem 4.5, but we can give a correct alternate proof. Let $P = \sigma_* \mathcal{O}_{X_n}$, the image of the sheaf homomorphism $\phi\colon B^{\otimes n} \to (B^{\otimes n})^* \otimes \mathcal{O}(1)$. Since we are assuming X_n Cohen-Macaulay, P is locally free. Let $W \subseteq H_n$ be an open set on which the map ϕ is a homomorphism of vector bundles. By the definition of ϕ, the pairing $B^{\otimes n} \otimes B^{\otimes n} \to \mathcal{O}(1)$ given by multiplication followed by alternation induces a non-degenerate pairing $P \otimes P \to \mathcal{O}(1)$ and thus an isomorphism $P^* \cong P \otimes \mathcal{O}(-1)$ on W.

The $n!$ conjecture holds for $\mu = (n)$ and $\mu = (1^n)$, where it reduces to the well-known theorem that the Vandermonde determinant in n variables has $n!$ linearly independent partial derivatives. Every ideal in the open set W_x (see the proof of Lemma 6.11) has $I_{(1^n)}$ in the closure of its $\mathbb{T}$ orbit, so ϕ is locally a homomorphism of vector bundles on W_x. This also holds on on W_y, for similar reasons.

Now, W_x is the Ellingsrud-Strömme cell $C_{(1^n)}$. There is a unique cell $C_{(2,1^{n-2})}$ of codimension 1, and its intersection with W_y is non-empty. Hence the complement of $W = W_x \cup W_y$ has codimension at least 2. For any locally free sheaf F on a normal scheme H and any open set W whose complement Z has codimension at least 2, we have $j_* F = F$, where $j\colon W \to H$ is the open embedding. This follows from the Serre criterion for normality and the exact sequence for local cohomology sheaves

$$0 = \mathcal{H}^0_Z(F) \to F \to j_* F \to \mathcal{H}^1_Z(F) = 0.$$

Applying this to the open set $W = W_x \cup W_y \subseteq H_n$ and the locally free sheaves P^* and $P \otimes \mathcal{O}(-1)$, which are isomorphic on W, we have $P^* \cong P \otimes \mathcal{O}(-1)$ globally. This implies X_n is Gorenstein with dualizing sheaf $\omega \cong \mathcal{O}(-1)$, by the last part of the proof of Lemma 6.18.

Since H_n is non-singular, the maximal ideal of the point I_μ is generated by a regular sequence, and the scheme-theoretic fiber of X_n over I_μ is therefore a local complete intersection in X_n and hence Gorenstein also. The coordinate ring of the fiber is $B^{\otimes n}(I_\mu)/\mathcal{J}(I_\mu)$. By Proposition 4.4, this ring is $R = \mathbb{C}[\mathbf{x}, \mathbf{y}]/J$, where J is a doubly homogeneous S_n-invariant ideal contained in J_μ.

Every non-constant homogeneous S_n-invariant polynomial in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ belongs to J, because the function it defines on X_n is pulled back from a function on H_n which vanishes at I_μ. Thus the only S_n invariants in R are constants. The socle of the Gorenstein graded ring R is a 1-dimensional S_n module, which must therefore afford the alternating representation. The ideals $I_\mu(x_i, y_i)$ are contained in J because R is a quotient of $B^{\otimes n}(I_\mu)$. Hence the alternating polynomials Δ_D belong to J, except for Δ_μ. It follows that the image of Δ_μ in R spans $\operatorname{soc}(R)$.

If J_μ properly contains J, then $\operatorname{soc}(R) \subseteq J_\mu/J$, contradicting the fact that $\Delta_\mu \notin J_\mu$. Hence $J_\mu = J$. Since we are assuming X_n is Cohen-Macaulay and therefore flat over H_n, each scheme-theoretic fiber has length $n!$. In particular $\dim_k R_\mu = \dim_k R = n!$, establishing the $n!$ conjecture for μ.

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