The Magic Square 4 × 4

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1 Description of the problem

The Magic Square is an arrangement of numbers in a square form. The sum of all numbers in a row, in a column and in the two diagonals must be the same. There are methods to write down some Magic Squares, but so far there exists no method to find all possible Magic Squares. This paper is limited to a square of size 4 times 4 and to the numbers from 1 to 16. There should be no problem (instead of calculation time) to expand this problem. This paper is nearly written in the order I discovered facts about Magic Squares. This not only makes it easier to read it, but also gives some inside on how to approach to this problem. Or simply: How to make a small problem bigger.

2 Elemental considerations

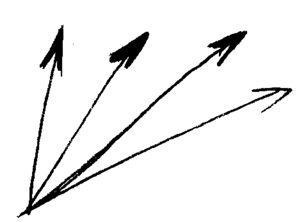
The first task is to find the row/column sum. This is possible by using the fact that the sum of all four row–sums must be the sum of all 16 numbers (136). Using this conclusion in the opposite direction one can derive that the sum of one row/column must be *34*.

Because in computer science it is easier to work with numbers starting at zero (at least in the programming language C) I will use the numbers 0 to 15 instead of 1 to 16. So the row/column sum is 30. This rearranging has no influence to the other properties. In general none of the following properties changes, if 16 numbers in sequence are taken.

3 The program

The first approach to get all Magic Squares will be to test each possible arrangement of 16 numbers in a square and than to test if it is a magic one. This leads to a very large number of squares which must be tested: 16! = 20.922.789.888.000.

Estimation



So another approach is needed. First the set of possible rows is generated. The sum of the 4 numbers of such a row must be 30 and so there are only 2064 possible rows. Using this set of rows and the fact that no number may occur twice, far less squares will be generated (3.121.348.608). Further there are combinations of two rows which are not possible. Using all this facts I can go through all possibilities. I need about 2 hours on my computer (Intel PPro–200). The number of Magic Squares is surprisingly high (=7040), so I started to think about this set.

4 Symmetric operations

Assuming a Magic square. If it is turned around by 90 degree, one gets another Magic Square. So some questions arises: How many of this symmetric operations are there? How do they influence each other? And how many total different Magic Squares are there?
The starting point will be the following Magic Square:

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

This square has the following properties:
The letters from a to p are standing for the numbers from 0 to 15.

$$a+b+c+d = e+f+g+h = i+j+k+l = m+n+o+p = 30 \qquad \text{(Sum of rows.)}$$

$$a+e+i+m = b+f+j+n = c+g+k+o = d+h+l+p = 30 \qquad \text{(Sum of columns.)}$$

$$a+f+k+p = d+g+j+m = 30 \qquad \text{(Sum of diagonals.)}$$

Now a pure listing of possible symmetric operations follows:

4.1 Outer symmetric operations

The operations in this chapter are obvious from the geometric point. Each symmetric operation gets a big latin letter in order to identify the operation later in the text.

4.1.1 Identity operation (I)

This operation is trivially a symmetric one.

4.1.2 Mirroring by the horizontal (H) and vertical (V) axis

a	b	c	d		m	n	o	p
e	f	g	h	⟷	i	j	k	l
i	j	k	l		e	f	g	h
m	n	o	p		a	b	c	d

a	b	c	d		d	c	b	a
e	f	g	h	⟷	h	g	f	e
i	j	k	l		l	k	j	i
m	n	o	p		p	o	n	m

As is could be seen at the first glance, these operations do not change the properties of a square. It is also valid for all numbers x: If x is replaced by y then y will be replaced by x. In short: Two numbers only exchange: $x \leftrightarrow y$. From this it follows: $HH = VV = I$.

4.1.3 Mirroring by the diagonal (D) and the cross diagonal ($\bar{D}$)

a	b	c	d		a	e	i	m
e	f	g	h	⟷	b	f	j	n
i	j	k	l		c	g	k	o
m	n	o	p		d	h	l	p

a	b	c	d		p	l	h	d
e	f	g	h	⟷	o	k	g	c
i	j	k	l		n	j	f	b
m	n	o	p		m	i	e	a

The same as with the horizontal/vertical axis. Also: $DD = \bar{D}\bar{D} = I$.

4.1.4 Rotation (R, L) and point–mirror (F)

a	b	c	d		m	i	e	a
e	f	g	h	↔	n	j	f	b
i	j	k	l		o	k	g	c
m	n	o	p		p	l	h	d

↕ ↕

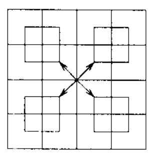
d	h	l	p		p	o	n	m
c	g	k	o	↔	l	k	j	i
b	f	j	n		h	g	f	e
a	e	i	m		d	c	b	a

R stands for a rotation to the right and L for a rotation to the left. F stands for a rotation by 180 degree or a point–mirror For F like for H, V, D and $\bar{D}$ there is $FF = I$. But for R and L there is: $RRRR = LLLL = I$. The operation F will be important later, because it is the only operation which commutes with all other operations.

4.2 Inner symmetric operations

4.2.1 Special operation (U)

This operation switches whole 2×2 sub–squaresand may be represented by the following diagram:



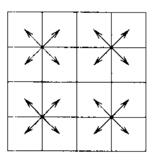
a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

⟷

k	l	i	j
o	p	m	n
c	d	a	b
g	h	e	f

4.2.2 Special operation ($\bar{U}$)

This operation may be represented by the following diagram:



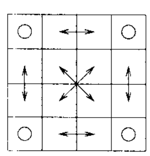
a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

⟷

f	e	h	g
b	a	d	c
n	m	p	o
j	i	l	k

4.2.3 Special operation (A)

This operation may be represented by the following diagram:



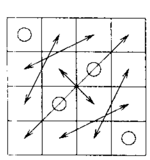
a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

⟷

a	c	b	d
i	k	j	l
e	g	f	h
m	o	n	p

4.2.4 Special operation (B)

This operation may be represented by the following diagram:



a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

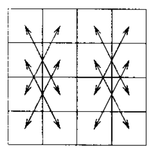
⟷

a	i	e	m
c	k	g	o
b	j	f	n
d	l	h	p

There exists a second version of this operation by mirroring the diagram by its horizontal axis.

4.2.5 Special operation (J)

This operation may be represented by the following diagram:



a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

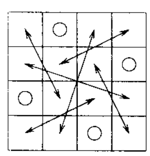
⟷

j	i	l	k
n	m	p	o
b	a	d	c
f	e	h	g

There exists also a second version of this operation by turning around the diagram by 90 degree.

4.2.6 Special operation (E)

This operation may be represented by the following diagram:



a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

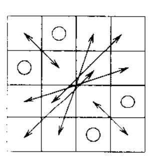
⟷

j	b	n	f
l	d	p	h
i	a	m	e
k	c	o	g

Second version by horizontal mirroring.

4.2.7 Special operation (G)

This operation may be represented by the following diagram:



a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

⟷

f	b	n	j
e	a	m	i
h	d	p	l
g	c	o	k

Second version by horizontal mirroring.

4.3 Symmetric operations of non geometric type

4.3.1 Special operation (T)

It is not possible to make a diagram for this operation. This operation uses a total different method. It does not switch positions as in the above operations but swit–

ches numbers in the following way: $0 \longleftrightarrow 15$, $1 \longleftrightarrow 14$, and so on. This also conserves the properties of a Magic Square. It may be possible to integrate this operation in the considerations below, but I have decided not to do so because the method is too different.

5 Working with the symmetric operations

This chapter is far from being ready.

After I have found more than the standard symmetric operations, I am looking for an ordering of them. So this chapter tries to bring some systematic approach to this operations.

5.1 Definition of operators

At first the question arises, if it is possible at all, to define such an operation in an exact way. I want a definition such that:

$$X = Y \Longleftrightarrow X\alpha = Y\alpha.$$

I will always use big latin letters for operators and small greek letters for Magic Squares. It is possible to describe an operator by using an ordered 16–tuple of numbers from the set of numbers from 0 to 15. Arranging them into a square, each number tells the new position of the element on that position. For example the operation H is given as:

12	13	14	15
8	9	10	11
4	5	6	7
0	1	2	3

Be aware that the first position is position 0. The double vertical line is a reminder for that this is a operator and not a Magic Square.

5.2 Combining of operators

Second it must be possible to combine two operations resulting in a new operation such that:

$$XY = Z \Longleftrightarrow XY\alpha = Z\alpha$$

If the two starting operations are symmetric ones, the new operation must also be a symmetric operation. This is definitely the case. After combining two operators, one can find it necessary to combine three operators. And so the next question arises: Associative law.

5.3 Associative law

Is the following true?

$$(XY)Z = X(YZ)$$

Yes, after letting my computer test all possibilities.

5.4 Commutative law

Do all symmetric operations commute?
The result is surprising: No. They commute in a seemingly random manner:

	F/I	$U/\bar{U}$	H/V	$D/\bar{D}$	R/L	J	G	A	E	B
F/I	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$U/\bar{U}$	Y	Y	Y	Y	Y	Y	Y	N	N	N
H/V	Y	Y	Y	N	N	Y	N	Y	N	N
$D/\bar{D}$	Y	Y	N	Y	N	N	Y	Y	N	Y
R/L	Y	Y	N	N	Y	N	N	Y	Y	N
J	Y	Y	Y	N	N	Y	N	N	Y	Y
G	Y	Y	N	Y	N	N	Y	N	Y	N
A	Y	N	Y	Y	Y	N	N	Y	N	Y
E	Y	N	N	N	Y	Y	Y	N	Y	Y
B	Y	N	N	Y	N	Y	N	Y	Y	Y

5.5 The inverse operation

Each operation must have an inverse, which also must be a symmetric operation. The inverse operation of X is written as X^{-1} and defined by $XX^{-1} = I$. Most operators are self inverse: $X = X^{-1}$. The only two which are not are R and L. They have $RL = LR = I$.

5.6 The point-mirror F

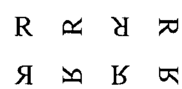
It is interesting to see that there are always two operations, which do commute, which give the operation F: $HV = D\bar{D} = RR = LL = F$.
This also holds true for the inner symmetric operations: $U\bar{U} = F$.
(This is the cause why I have chosen the name $\bar{U}$.)

5.7 Operators are self referential

Be aware that talking about operators is the next logical level. If you call the Magic Squares objects, operators are meta–objects and working with operators could be considered as a meta–meta–level. I do not like meta–metaobjects. Instead I think operators are already self–referential,because of the fact that they could be combined. So they could modify each other.

5.8 Reduction

It may be concluded that each symmetric operation reduces the number of Magic Squares by the half. This is true if only one operation is used but not if they are used all together. For example: Taking the 7 operations H, V, D, $\bar{D}$, R, L, F and making all possible combinations the number of different operations is not 128 (2 to the power of 7) but only 8. This fact could be easily understood by looking at the letter R:



But it is much more complicate to find such things about the inner symmetric operations. From this there follows some properties about the operators:
For example: $HV = F$.
This also holds true for the special operations:
For example: $U\bar{U} = F$.

6 Results and open questions

The total number of Magic Squares is 7040. Using all symmetric operations this number reduces to 220. This is the equivalent to a division by 32.

6.1 Questions

I am still working on this paper, so there are some open questions:
(This is also my TODO list.)
How many different symmetric operations are there?
Is there an easy way to find them all?
What is the smallest set of operations for constructing them all?
How could the strange commutative–tablebe explained.
Program (C) to find all 7040 Magic Squares:
`http://wwwi.wu-wien.ac.at`

`/home/ckarl/ckarl/logic/magic/magic.c`
Program (C) to reduce to 220 Magic Squares:
`http://wwwi.wu-wien.ac.at`
`/home/ckarl/ckarl/logic/magic/conv.c`

6.2 Problems with English

This is a list of word where I don't know the correct translation:

- Punktspiegelung: point–mirror