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A Simple and Relatively Efficient Triangulation of the n-Cube*

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Abstract. The only previously published triangulation of the n-cube using $o(n!)$ simplices, due to Sallee, uses $O(n^{-2}n!)$ simplices. We point out a very simple method of achieving $O(\rho^n n!)$ simplices, where $\rho < 1$ is a constant.

1. Introduction

This short note is intended to point out a simple method of triangulating the n-cube I^n using significantly fewer simplices than in previous constructions.

Various authors [2], [3], [5]–[8] have considered the problem of triangulating I^n into fewer than the easily achievable maximum of $n!$ simplices. Since the volume of a simplex with vertices in $\mathbf{Z}^n$ is an integral multiple of $1/n!$, it is clear that $n!$ is in fact the maximum number of simplices in any triangulation. A lower bound can also be derived from volume considerations as follows. I^n can be inscribed in a sphere of diameter $\sqrt{n}$. The maximum volume of a simplex contained in this sphere is $(n+1)^{(n+1)/2}/2^n n!$, attained by the equilateral simplex. This shows that any triangulation of I^n uses at least $2^n(n+1)^{-(n+1)/2}n!$ simplices. This lower bound is very much less than $n!$, being $O(c^n(n!)^{1/2})$. By replacing the cube with an "ideal cube" in hyperbolic space and using hyperbolic instead of Euclidean volume it is possible to improve the lower bound [8], but only by a factor of $O((3/2)^{n/2})$.

In view of the large gap between the lower and upper bounds it is perhaps surprising that all triangulations of I^n proposed so far use nearly $n!$ simplices. In fact, only Sallee [7] achieves $o(n!)$ simplices. Sallee's triangulation, however, uses more than $2A(n-1, \lfloor (n-1)/2 \rfloor)$ simplices, where the *Eulerian number* $A(n, k)$ is the number of permutations of n having k "descents" [10]. Since $A(n, k)$ is unimodal as a function of k, $A(n-1, \lfloor (n-1)/2 \rfloor) \geq (n-1)!/(n-1) = (n-2)!$

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and hence Sallee uses at least $O(n^{-2}n!)$ simplices. (Readers of [7] should observe that the function $g(n, m)$ of that paper is actually $A(n-1, m-1)$, a fact not noted there.)

The construction given below triangulates I^n with $O(\rho^n n!)$ simplices, where $\rho < 1$. In fact, given a triangulation of I^n into $T(n)$ simplices for any particular value of n, we can take $\rho = (T(n)/n!)^{1/n}$. The smallest value for ρ obtainable from triangulations published to date is $\rho = (13{,}248/40{,}320)^{1/8} \approx 0.870$ from Sallee's triangulation of I^8. We remark that Todd [11] proposed the quantity $R(n) = (T(n)/n!)^{1/n}$ as a measure of a triangulation's efficiency. Previous constructions have $\lim_{n\to\infty} R(n) = 1$, whereas our results show that any value of $R(n)$ achievable for one triangulation is achievable asymptotically. This is still far from $R(n) = o(1)$, let alone the lower bound of $R(n) = O(n^{-1/2})$.

2. Construction

Definition. A *polyhedral decomposition* of an n-dimensional polytope P is a union $P = T_1 \cup T_2 \cup \cdots \cup T_k$ of n-dimensional polytopes T_i such that for all i, j the vertices of T_i are vertices of P and $T_i \cap T_j$ is a (possibly empty) face of both T_i and T_j. If each T_i is a simplex, then $\{T_i\}$ is a *triangulation* of P.

Lemma 1. *Every polyhedral decomposition of P can be refined to a triangulation.*

Proof. We require triangulations θ_i of the T_i such that θ_i and θ_j induce the same triangulation on $T_i \cap T_j$ considered as a face of T_i and as a face of T_j. Now there are well-known constructions [4], [6], [9], [12] whereby we associate to a total ordering α of the vertices of a polytope T a triangulation θ in such a way that the triangulation induced on each face $F \subseteq T$ is the one associated to the restriction of α to the vertices of F. Hence fixing any total ordering α_0 of all the vertices of P and triangulating each T_i in accordance with α_0 we obtain compatible triangulations θ_i as required. □

Lemma 2 [1]. *Every triangulation of $\Delta_k \times \Delta_l$ uses exactly $(k+l)!/k!\,l!$ simplices, where Δ_n denotes an n-dimensional simplex.*

Proof. Realize Δ_k in $\mathbf{R}^k$ as the convex hull of 0 and the unit coordinate vectors $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$. Likewise Δ_l. Then $\Delta_k \times \Delta_l \subseteq \mathbf{R}^{k+l}$ has vertices 0, e_i $(1 \le i \le k+l)$, and $e_i + e_j$ $(1 \le i \le k < j \le k+l)$. Its volume is $v(\Delta_k)v(\Delta_l) = 1/k!\,l!$. We claim every nondegenerate $(k+l)$-simplex Δ spanned by vertices of $\Delta_k \times \Delta_l$ has volume $1/(k+l)!$. Note that there are affine-linear symmetries of $\Delta_k \times \Delta_l$ acting transitively on the vertices. These preserve volume, so we can assume 0 is a vertex of Δ. Then $\pm(k+l)!\,v(\Delta)$ is the determinant of the matrix M whose rows are the coordinates of the other vertices; we are to show that this determinant is ± 1. If some e_i is a vertex of Δ, then expanding by minors on the corresponding row gives the result by induction. If not, then M is the edge-vertex incidence matrix of a (k, l)-bipartite graph with $k + l$ edges. Having one too many edges to be a tree, this graph must contain a cycle, necessarily even. But then M is singular, contrary to hypothesis. □

Theorem 1. *Given a triangulation* $\{S_1, \dots, S_s\}$ *of a k-dimensional polytope P and a triangulation* $\{T_1, \dots, T_t\}$ *of an l-dimensional polytope Q, there exists a triangulation of* $P \times Q$ *using* $s \cdot t \cdot (k+l)!/k!\, l!$ *simplices.*

Proof. It is easy to see that $\{S_i \times T_j\}$ is a polyhedral decomposition of $P \times Q$. Refine it to a triangulation by Lemma 1. Each $S_i \times T_j$ will then contain $(k+l)!/k!\, l!$ simplices by Lemma 2, establishing the result. □

Corollary 1. *If* I^n *can be triangulated into* $T(n)$ *simplices, then* I^{kn} *can be triangulated into* $[(kn)!/(n!)^k]T(n)^k = \rho^{kn}(kn)!$ *simplices, where* $\rho = (T(n)/n!)^{1/n}$.

Proof. Immediate from Theorem 1 by induction on k. □

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