Bounds for the Betti Numbers of Shellable Simplicial Complexes and Polytopes

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INTRODUCTION

In this paper we give upper bounds (Theorem 2.1 and Proposition 3.4) for the Betti numbers of shellable simplicial complexes with a given number of vertices and facets, and of the boundary complex of certain classes of simplicial polytopes. For shellable simplicial complexes our bound is attained when these complexes are $(d-1)$-trees, and for the class of polytopes which we are considering the given bound is attained when the polytope is stacked, that is, when it admits a triangulation which is a $(d-1)$-tree.

Recall that a $(d-1)$- dimensional shellable simplicial complex Δ is called a $(d-1)$-tree if in the shelling of Δ each facet intersects the previous facets in only one subfacet. It follows at once that the h-vector of a $(d-1)$-tree is of the form $(1, h, 0, \dots, 0)$.

The Betti numbers of stacked simplicial polytopes have first been computed by Hibi and Terai [7]. In this paper we give a different proof of their result, see 3.3. It has been shown by Terai [9] that for 3-polytopes with a given number of vertices the boundary complex of a stacked polytope has the maximal Betti numbers. One could hope that this is true in all dimensions. With the methods developed in this paper we can only give an upper bound for the Betti numbers of a d-polytope, if this polytope admits a proper triangulation, that is, a shellable triangulation with no interior vertices.

1. Basic concepts

We first recall some basic definitions on simplicial complexes. The reader is referred to [2], [6] or [8] for further details.

A simplicial complex on a vertex set $V = \{v_1, \dots, v_n\}$ is a collection Δ of subsets of V such that

(1) $\{v_i\} \in \Delta$;
(2) if $F, G \subset V$ with $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

The elements of Δ are called the *faces* of Δ. The *dimension* of a face F of Δ, denoted by $\dim F$, is the number $|F| - 1$. Faces of dimension 0 are called *vertices*, those of dimension 1 *edges* of Δ. The maximal faces under inclusion are called *facets*. The dimension of Δ is defined to be

$$\dim \Delta = \max\{\dim F \colon F \in \Delta\}.$$

A simplicial complex is called *pure* if all facets have the same dimension.

Let Δ be a $(d-1)$-dimensional simplicial complex. We denote by f_i the number of i-dimensional faces of Δ, and set $f_{-1} = 1$. The vector of integers $(f_{-1}, f_0, \dots, f_{d-1})$ is called the f-vector of Δ.

Given a collection $F_1, \dots, F_m$ of subsets of a vertex set $V = \{v_1, \dots, v_n\}$, there exists a unique smallest simplicial complex $\Delta = \langle F_1, \dots, F_m\rangle$ on the vertex set $\bigcup_{i=1}^m F_i$ containing all F_i as faces. Indeed,

$$\Delta = \{G \subset V : G \subset F_i \text{ for some } i, \dots, m\}.$$

We say that Δ is *spanned by* $F_1, \dots, F_m$.

Recall that a $(d-1)$-dimensional simplicial complex is called *shellable*, if Δ is pure, and if there exists an order of the facets of Δ, say, $F_1, \dots, F_m$, such that

$$\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i\rangle$$

is spanned by $(d-2)$-simplices. For $i = 2, \dots, m$ we denote by k_i the number of the $(d-2)$-simplices spanning these intersections, and set $k_1 = 0$. We call $(k_1, \dots, k_m)$ the k-vector of the shelling.

Definition 1.1. A $(d-1)$-dimensional shellable simplicial complex Δ with shelling $F_1, \dots, F_m$ is called a $(d-1)$-tree, if $k_i = 1$ for $i = 2, \dots, m$.

Let K be a field, and Δ a simplicial complex on the vertex set $V = \{v_1, \dots, v_n\}$. The *Stanley-Reisner ring* of Δ over K is the factor ring of the polynomial ring

$$K[\Delta] = K[x_1, \dots, x_n]/I_\Delta,$$

where I_Δ is the ideal generated by all monomials

$$x_F = \prod_{v_i \in F} x_i \quad \text{with} \quad F \notin \Delta.$$

The polynomial ring $K[x_1, \dots, x_n]$ is multigraded. The homogeneous elements are the terms λx^a with $\lambda \in K$ and $a \in \mathbb{Z}^n$. The multidegree of such a term is $a \in \mathbb{Z}^n$. Since the defining ideal of a simplicial complex Δ is defined by monomials, the Stanley-Reisner ring $K[\Delta]$ inherits a multigraded structure.

We set

$$H_{K[\Delta]}(\mathbf{t}) = \sum_{a \in \mathbb{Z}} \dim_K K[\Delta]_a \mathbf{t}^a,$$

and call it the *multigraded Hilbert function of* $K[\Delta]$. Here $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{t}^a = t_1^{a_1} \dots t_n^{a_n}$ for $a \in \mathbb{Z}^n$.

One has

$$H_{K[\Delta]}(\mathbf{t}) = \sum_{F \in \Delta} \prod_{v_i \in F} \frac{t_i}{1 - t_i}. \tag{1}$$

Of course, since I_Δ is a graded ideal in the ordinary sense, too, the algebra $K[\Delta]$ is homogeneous, that is, $K[\Delta]$ is a finitely generated K-algebra which is generated over K by elements of degree 1.

Recall that an arbitrary homogeneous K-algebra R has a Hilbert function of the form

$$H_R(t) = \frac{Q(t)}{(1-t)^d}, \tag{2}$$

where $d = \dim R$ and $Q(t) = \sum_{i=0}^m h_i t^i$ is a polynomial with $Q(1) \neq 0$; see [2, Lemma 4.1.7(b)]. The vector $(h_0, \dots, h_m)$ is called the h-vector of R, and $Q(1) = \sum_{i=0}^m h_i$ is called the *multiplicity of* R. We denote the multiplicity of R by $e(R)$.

The Hilbert function of $K[\Delta]$ is obtained from (1) by replacing all t_i by t. Therefore,

$$H_{K[\Delta]}(t) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}. \tag{3}$$

A comparison of (1) and (2) yields the identity

$$\sum_i h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} \tag{4}$$

which shows that the h-vector of a $(d-1)$-dimensional simplicial complex has length at most d. Moreover, using (4), one can compute the h-vector from the f-vector, and vice versa.

For a shellable simplicial complex Δ there is a result of McMullen (cf. [2, Corollary 5.1.14]) which gives the h-vector from the shelling.

Theorem 1.2. *Let Δ be a shellable $(d-1)$-dimensional simplicial complex with a shelling whose k-vector is $(k_1, \dots, k_m)$. Then the h-vector $(h_0, \dots, h_d)$ of Δ is given by*

$$h_j = |\{i : k_i = j\}| \quad \textit{for} \quad j = 0, \dots, d.$$

As an immediate consequence of the McMullen formulas one obtains

Corollary 1.3. *Let Δ be a shellable $(d-1)$-dimensional simplicial complex. The following conditions are equivalent:*

(a) *Δ is a $(d-1)$-tree;*
(b) *$h_i = 0$ for $i > 1$*

It follows from (4) that $h_1 = f_0 - d$. We also conclude from (4) that the multiplicity of $K[\Delta]$ equals

$$e(K[\Delta]) = \sum_i h_i = f_{d-1}.$$

In other words, $e(K[\Delta])$ is equal to the number of facets of Δ. It follows from 1.3 that for a $(d-1)$-tree one has $e(K[\Delta]) = f_{d-1} = f_0 - d + 1$.

For an arbitrary homogeneous K-algebra R with h-vector $(h_0, \dots, h_m)$ it is known that $h_1 = \operatorname{emb\,dim} R - \dim R$, and that $h_i \geq 0$ if R is Cohen-Macaulay; see [2, Proposition 4.3.1]. Here $\operatorname{emb\,dim} R$ denotes the embedding dimension of R. Therefore

$$e(R) \geq \operatorname{emb\,dim} R - \dim R + 1.$$

if R is Cohen-Macaulay. This is the inequality of Abhyankar. If equality holds, then R is said to have *minimal multiplicity.* It is clear that R has minimal multiplicity if and only if $h_i = 0$ for $i \geq 2$.

Since shellable simplicial complexes are always Cohen-Macaulay (see [2]), it follows that for a shellable simplicial complex $f_{d-1} \geq f_0 - d + 1$, with equality if and only if it is a $(d-1)$-tree. Of course, the inequality follows also directly by induction on the number of facets. More precisely, one has: if $(k_1, \dots, k_m)$ is the k-vector of the shelling, then

$$f_{d-1} - f_0 + d - 1 = |\{i : k_i > 1\}|.$$

2. Upper bound for the Betti numbers of a shellable simplicial complex

In this section we will study the minimal free resolution $\mathbb{F}$ over the polynomial ring $P = K[x_1, \dots, x_n]$ of the Stanley-Reisner ring $K[\Delta] = P/I_\Delta$ of a simplicial complex Δ. Since $K[\Delta]$ is multigraded, the resolution is multigraded as well, that is,

$$\mathbb{F}: \quad 0 \to F_p \to F_{p-1} \to \dots \to F_1 \to F_0 \to K[\Delta] \to 0,$$

where each $F_i = \bigoplus_{a \in \mathbb{Z}^n} P(-a)^{\beta_{ia}}$, and where all maps in this complex are homogeneous (of degree 0). Here we denote, as usual, for any $b \in \mathbb{Z}^n$ by $P(b)$ the shifted rank 1 free graded P-module with $P(b)_a = P_{a+b}$ for all $b \in \mathbb{Z}^n$. The numbers β_{ia} are called the *multigraded Betti numbers* of $K[\Delta]$. For each i there are only finitely many $\beta_{ia} \neq 0$. The *graded Betti numbers* of $K[\Delta]$ are defined to be $\beta_{ij}(K[\Delta]) = \sum_{a \in \mathbb{Z}^n, |a| = j} \beta_{ia}$, where for $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we set $|a| = \sum_i a_i$, and the numbers $\beta_i(K[\Delta]) = \sum_{j \in \mathbb{Z}} \beta_{ij}$ are simply called the *Betti numbers of* $K[\Delta]$. Finally, the series

$$\operatorname{Poin}(\mathbf{t}, s) = \sum_{i \in \mathbb{Z}, a \in \mathbb{Z}^n} \beta_{ia} \mathbf{t}^a s^i$$

is called the *multigraded Poincaré series of* $K[\Delta]$. Similarly one defines the (graded) Poincaré series.

We will prove the following

Theorem 2.1. *Let Δ be a shellable $(d-1)$-dimensional simplicial complex with n vertices and m facets. Then*

$$\beta_i(K[\Delta]) \leq \sum_{j=1}^{m-1} j \binom{j-1}{i-1} - \sum_{k=1}^{i} \binom{n-d}{i-k} \binom{m+d-1-n}{k}.$$

Moreover, the bound is reached if and only if Δ *is a* $(d-1)$*-tree.*

We first show that for a $(d-1)$-tree Δ the given bound is attained. As observed in the previous section, one has $n = m + d - 1$ when Δ is a $(d-1)$-tree. Hence the second sum in this upper bound is 0, and we have to show that

$$\beta_i(K[\Delta]) = \sum_{j=1}^{m-1} j\binom{j-1}{i-1}. \tag{5}$$

More generally we note the following well-known fact, whose proof we outline for the convenience of the reader.

Lemma 2.2. *Let* R *be a Cohen-Macaulay ring with minimal multiplicity of embedding dimension* n *and dimension* d. *Then* $\beta_{i i+j}(R) = \sum_{k=1}^{n-d} k\binom{k-1}{i-1}$ *if* $j = 1$, *and* 0 *otherwise.*

Proof. We may assume that K is infinite, because otherwise we may choose an infinite base field extension without changing the Betti numbers. Then there exists (c.f. [2, Proposition 1.5.12]) a regular sequence $f_1, \dots, f_d$ of forms of degree 1. If $\mathbb{F}$ is a minimal graded free $P = K[x_1, \dots, x_n]$ resolution of R, then $\mathbb{F}/(f_1, \dots, f_d)\mathbb{F}$ is a graded minimal free $\bar{P} = P/(f_1, \dots, f_d)P$ resolution of $\bar{R} = R/(f_1, \dots, f_d)R$.

It follows that

$$\beta_{ij}^{P}(R) = \beta_{ij}^{\bar{P}}(\bar{R}).$$

Since $f_1, \dots, f_d$ is a regular sequence of 1-forms, $\bar{P}$ is isomorphic to a polynomial ring in $n-d$ variables, say $\bar{P} = K[y_1, \dots, y_{n-d}]$, and $\bar{R}$ is a quotient ring of $\bar{P}$ with Hilbert function $1 + h_1 t = 1 + (n-d)t$.

Therefore,

$$\bar{R} \cong \bar{P}/(y_1, \dots, y_{n-d})^2.$$

The ideal $(y_1, \dots, y_{n-d})^2$ is strongly stable, hence its resolution over $\bar{P}$ is given by Eliahou-Kervaire ([4]). The explicit formulas for the graded Betti numbers in [4] (see also [1]) applied to this situation yield the desired result.

(Alternatively one may view $(y_1, \dots, y_{n-d})^2$ as an ideal of maximal minors of the matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-d} & 0 \\ 0 & y_1 & y_2 & \cdots & y_{n-d} \end{pmatrix},$$

whose resolution is a special case of the Eagon-Northcott resolution; see [3]). □

Proof. [*Proof of Theorem* 2.1] We prove the asserted inequalities for an arbitrary shellable simplicial complex Δ of dimension $d-1$ and with n vertices by induction on the number m of facets. We choose a $(d-1)$-tree Γ with m facets, and let $V = \{v_1, \dots, v_r\}$ be the vertex set of Γ. Then $n \le r = m + d - 1$. Hence may assume that the subset $W = \{v_1, \dots, v_n\}$ of V is the vertex set of Δ. For any integer $i \le r$ we set $P_i = K[x_1, \dots, x_i]$. Then $\beta_i(K[\Delta]) = \beta_i^{P_n}(K[\Delta])$.

We may view $K[\Delta]$ also as a P_r-module, since

$$K[\Delta] = P_r/(I_\Delta, x_{n+1}, \dots, x_r).$$

Let $\mathbb{F}$ be a minimal free P_n-resolution of $K[\Delta]$. Since $x_{n+1}, \dots, x_r$ is a regular sequence on P_r/I_Δ, we see that

$$(\mathbb{F} \otimes_{P_n} P_r) \otimes_{P_r} K(x_{n+1}, \dots, x_r; P_r)$$

is a minimal free P_r-resolution of $K[\Delta]$. Here $K(x_{n+1}, \ldots, x_r; P_r)$ is the Koszul complex of the sequence $x_{n+1}, \ldots, x_r$.

From this we deduce the following equations

$$\beta_i^{P_r}(K[\Delta]) = \sum_{j=0}^{i} \beta_j^{P_n}(K[\Delta]) \binom{r-n}{i-j}.$$

So

$$\beta_i^{P_n}(K[\Delta]) = \beta_i^{P_r}(K[\Delta]) - \sum_{j=0}^{i-1} \beta_j^{P_n}(K[\Delta]) \binom{r-n}{i-j}.$$

The Betti numbers $\beta_i(K[\Delta])$ are bounded below by $\binom{\text{height}\, I_\Delta}{j}$; see for example [5]. Therefore, since height $I_\Delta = n - d$, and $r = m + d - 1$, we get

$$\beta_i^{P_n}(K[\Delta]) \leq \beta_i^{P_r}(K[\Delta]) - \sum_{j=0}^{i-1} \binom{n-d}{j} \binom{m+d-1-n}{i-j}.$$

Thus if we can show that

$$\beta_i^{P_{m+d-1}}(K[\Delta]) \leq \beta_i^{P_{m+d-1}}(K[\Gamma])$$

for all i, then the theorem follows from (5).

We prove this inequality by induction on m. For $m = 1$ the assertion is trivial. Now we assume that the assertion is proved for $m \geq 1$, and prove it for $m + 1$.

Let

$$\Delta' = \Delta \cup \langle F_{m+1} \rangle \quad \text{and} \quad \Sigma = \Delta \cap \langle F_{m+1} \rangle.$$

In order to simplify notation we set

$$T_i^{m+d}(M) = \operatorname{Tor}_i^{P_{m+d}}(K, M)$$

for any P_{m+d}-module M, and if $M = K[\Pi]$, we set $T_i^{m+d}(\Pi) = T_i^{m+d}(K[\Pi])$. Recall that $\beta_i(M) = \dim_K \operatorname{Tor}_i^{P_{m+d}}(K, M)$.

There is a short exact sequence of P_{m+d}-modules (c.f. [2, Sequence (3), page 210])

$$0 \to K[\Delta'] \to K[\Delta] \oplus K[\langle F_{m+1} \rangle] \to K[\Sigma] \to 0$$

which gives rise to the long exact homology sequence

$$\begin{aligned} \ldots \quad & \to T_i^{m+d}(\Delta') \to T_i^{m+d}(\Delta) \oplus T_i^{m+d}(\langle F_{m+1} \rangle) \to T_i^{m+d}(\Sigma) \\ & \to T_{i-1}^{m+d}(\Delta') \to \ldots \end{aligned}$$

We first show that for all i the map

$$T_i^{m+d}(\langle F_{m+d} \rangle) \to T_i^{m+d}(\Sigma)$$

is injective.

Since Δ is shellable, Σ is the union of $(d-2)$-simplices, say, of the simplices $\langle F_{m+1} \setminus \{v_{r_l}\} \rangle$, $l = 1, \ldots, k$. Then we have

$$K[\langle F_{m+1} \rangle] = P_{m+d}/(x_i \colon v_i \notin F_{m+1}),$$

and

$$K[\Sigma] = P_{m+d}/(x_i \colon v_i \notin F_{m+1}) + (\prod_{l=1}^{k} x_{r_l}).$$

In both cases the defining ideals are generated by regular sequences, and hence the Koszul complexes

$$K(x_i; P_{m+d})_{v_i \notin F_{m+1}} \quad \text{and} \quad K(x_i, \prod_{l=1}^{k} x_{r_l}; P_{m+d})_{v_i \notin F_{m+1}}$$

are the multigraded free P_{m+d}-resolutions of $K[\langle F_{m+1}\rangle]$ and $K[\Sigma]$, respectively.

The natural inclusion map

$$K(x_i; P_{m+d})_{v_i \notin F_{m+1}} \xrightarrow{\alpha} K(x_i, \prod_{l=1}^{k} x_{r_l}; P_{m+d})_{v_i \notin F_{m+1}} \tag{6}$$

is a lifting of the epimorphism

$$K[\langle F_{m+1}\rangle] = P_{m+d}/(x_i \colon v_i \notin F_{m+1}) \to P_{m+d}/(x_i \colon v_i \notin F_{m+1}) + (\prod_{l=1}^{k} x_{r_l}) = K[\Sigma].$$

Since α is split exact, it induces for all i an injective map $T_i^{m+d}(\langle F_{m+1}\rangle) \to T_i^{m+d}(\Sigma)$.

For all i we now set $M_i = T_i^{m+d}(\Sigma)/T_i^{m+d}(\langle F_{m+1}\rangle)$. Then we get the exact sequence

$$\dots \xrightarrow{\alpha_{i+1}} M_{i+1} \to T_i^{m+d}(\Delta') \to T_i^{m+d}(\Delta) \xrightarrow{\alpha_i} M_i \to T_{i-1}^{m+d}(\Delta') \to \dots$$

From this we deduce

$$\begin{aligned} \dim_K T_i^{m+d}(\Delta') &= \dim_K T_i^{m+d}(\Delta) + \dim_K M_{i+1} \\ &- \dim_K \operatorname{Im}\alpha_i - \dim_K \operatorname{Im}\alpha_{i+1}. \end{aligned} \tag{7}$$

Similarly for Γ we get

$$\begin{aligned} \dim_K T_i^{m+d}(\Gamma') &= \dim_K T_i^{m+d}(\Gamma) + \dim_K M'_{i+1} \\ &- \dim_K \operatorname{Im}\beta_i - \dim_K \operatorname{Im}\beta_{i+1}, \end{aligned} \tag{8}$$

where the M'_i and β_i are defined for Γ as the M_i and α_i for Δ.

We note that M_i is $\mathbb{Z}^r$-graded, set $M = \oplus M_i$ and

$$H_M(t_1, \dots, t_r, s) = \sum_i H_{M_i}(t_1, \dots, t_r)s^i.$$

Then it follows from the definition of M and (6) that

$$H_M(t_1, \dots, t_r, s) = \prod_{v_i \notin F_{m+1}} (1 + t_i s)(\prod_{l=1}^{k} t_{r_l})s.$$

Similarly, if $\Gamma' = \Gamma \cup \langle F'_{m+1}\rangle$, with $\Gamma \cap \langle F'_{m+1}\rangle = F'_{m+1} \setminus \{v_{m+d}\}$ (since we assume that Γ' is a $(d-1)$-tree) we get for $M' = \oplus M'_i$ the formula

$$H_{M'}(t_1, \dots, t_r, s) = \prod_{v_i \notin F'_{m+1}} (1 + t_i s)t_{m+d}s.$$

It follows that

$$\dim_K M_{i+1} = \binom{r-d}{i} = \dim_K M'_{i+1}$$

for all i. Therefore, the equations (7) and (8) together with our induction hypothesis imply that $\beta_i^{P_{m+d}}(\Delta') \leq \beta_i^{P_{m+d}}(\Gamma')$ if

$$\dim_K \operatorname{Im} \alpha_i + \dim_K \operatorname{Im} \alpha_{i+1} \geq \dim_K \operatorname{Im} \beta_i + \dim_K \operatorname{Im} \beta_{i+1} \tag{9}$$

for all i.

We claim that $\beta_i = 0$ for $i > 1$. Indeed, the Poincaré series $\operatorname{Poin}_{m+d}(\mathbf{t}, s)$ of $K[\Gamma]$ as a P_{m+d}-module, and Poincaré series $\operatorname{Poin}_{m+d}(\mathbf{t}, s)$ of $K[\Gamma]$ as a P_{m+d-1}-module, are related by the equation

$$\operatorname{Poin}_{m+d}(\mathbf{t}, s) = \operatorname{Poin}_{m+d-1}(\mathbf{t}, s)(1 + t_{m+d}s).$$

It follows from 2.2 that

$$\operatorname{Poin}_{m+d-1}(\mathbf{t}, s) = 1 + \sum_{i \geq 1} P_i(\mathbf{t}) s^i$$

where $P_i(\mathbf{t}) \in K[t_1, \dots, t_{d+m-1}]$ is homogeneous of degree $i + 1$.

Therefore,

$$\operatorname{Poin}_{m+d}(\mathbf{t}, s) = 1 + t_{m+d}s + P_1(\mathbf{t})s + \sum_{i \geq 2} (P_i(\mathbf{t}) + P_{i-1}(\mathbf{t}) t_{m+d}) s^i.$$

On the other hand,

$$\begin{aligned} H_{M'}(\mathbf{t}) &= \prod_{v_j \notin F_{m+1}} (1 + t_j s) t_{d+m} s \\ &= t_{d+m} s + \sum_{i \geq 2} Q_{i-1}(\mathbf{t}) t_{m+d} s^i \end{aligned}$$

where $Q_{i-1}(\mathbf{t})$ is a homogeneous polynomial of degree $i - 1$ over K. Assume that $\beta_i \neq 0$ for $i \geq 2$, then $P_i(\mathbf{t}) + P_{i-1}(\mathbf{t}) t_{m+d}$ and $Q_i(\mathbf{t}) t_{m+d}$ must have a common monomial in the variables t_j, $v_j \notin F'_{m+1}$. But this is impossible by degree reasons.

Since we now know that $\beta_i = 0$ for $i > 1$, it follows that $\dim_K \operatorname{Im} \alpha_i \geq \dim \operatorname{Im} \beta_i$ for $i > 1$.

We finally show that $\dim_K \operatorname{Im} \alpha_1 = \dim_K \operatorname{Im} \beta_1 = 1$, then the desired inequalities (9) follow for all i, and the theorem is proved.

Notice that $\dim_K M_1 = \dim_K M'_1 = 1$. Therefore since

$$K[\Delta \cap \langle F_{m+1} \rangle] = P_{m+d}/(x_i \colon v_i \notin F_{m+1}) + (\prod_{l=1}^{k} x_{r_l})$$

and

$$K[\Gamma \cap \langle F'_{m+1} \rangle] = P_{m+d}/(x_i \colon v_i \notin F'_{m+1}) + (x_{m+d}),$$

we see from the definition of M_1 and M'_1, that the generator of M_1 corresponds to $\prod_{l=1}^{k} x_{r_l}$ and the generator of M'_1 to x_{m+d}. Thus in order to prove that $\dim_K \operatorname{Im} \alpha_1 = \dim_K \operatorname{Im} \beta_1$, we must show that the defining ideal (I_Γ, x_{m+d}) of the P_{m+d}-module $K[\Gamma]$ contains the element x_{m+d} as a minimal generator (which is trivial), and that the defining ideal $(I_\Delta, x_{n+1}, \dots, x_{m+d})$ of the P_{m+d}-module $K[\Delta]$ contains the element $\prod_{l=1}^{k} x_{r_l}$ as a minimal generator. The second assertion is seen by noting that F_{m+1} is a non-face of Δ (which we add in order to obtain Δ'). However, it is not a minimal non-face of Δ. The minimal non-face of Δ which is contained in F_{m+1} is the face $G = \{v_{r_1} \dots, v_{r_k})$. Since the minimal non-faces of Δ correspond to minimal generators of I_Δ, we conclude that $\prod_{v_i \in G} x_i = \prod_{l=1}^{k} x_{r_l}$ is a minimal generator of I_Δ and hence of $(I_\Delta, x_{n+1}, \dots, x_{m+d})$, as desired. □

3. Betti numbers of the boundary complex of polytopes

In this section we give a short proof of a result of Hibi and Terai [7] who computed the Betti numbers of the boundary complex of a stacked polytope, and give an upper bound for the Betti numbers of the boundary complex of a polytope with a proper triangulation.

Recall that a *triangulation of a simplicial d-polytope P* is a d-dimensional simplicial complex Γ whose geometric realization is P. A simplicial d-polytope P is called *stacked* if it admits a triangulation which is a $(d-1)$-tree. In other words, starting with a d-simplex, one adds new vertices by building shallow pyramides over facets to obtain P. Let $P(n,d)$ be a such stacked d-polytope with n vertices. We denote by $\Delta P(n,d)$ the boundary complex of $P(n,d)$, that is, the simplicial complex whose facets are the boundary faces of $P(n,d)$. Note that $\dim \Delta P(n,d) = d-1$.

It is well-known that the boundary complex $\Delta(P)$ of any simplicial d-polytope is Gorenstein; see [2, Corollary 5.5.6]. In other words, for any field K the Stanley-Reisner ring $K[\Delta(P)]$ is Gorenstein. In particular, P has a symmetric h-vector. More precisely, one has $h_i = h_{d-i}$ for $0 \le i \le d$. These are the famous Sommerville equations. (Here we follow the common convention to define the h-vector of P to be the h-vector of $\Delta(P)$.) For stacked polytopes the Sommerville equations are more special.

Proposition 3.1. *The h-vector of $P(n,d)$ is the vector $(1, n-d, n-d, \dots, n-d, 1)$ of length d.*

The proof could be easily done by induction on the number of vertices. Instead we will use the following theorem of Hochster (see [2, Theorem 5.6.2]), since this theorem will be crucial for other arguments of this section as well.

We denote by $\omega_{K[\Delta]}$ the canonical module of a Cohen-Macaulay simplicial complex Δ over a field K.

Theorem 3.2 (Hochster). *Let K be a field, and Γ a Cohen–Macaulay complex of dimension d over K whose geometric realization $X = |\Gamma|$ is a manifold with a non-empty boundary ∂X. Further let Δ be the subcomplex of Γ whose geometric realization is ∂X, and J the ideal in $K[\Gamma]$ generated by the monomials $x^F = \prod_{v_i \in F} x_i$, $F \in \Gamma \setminus \Delta$. Then the following conditions are equivalent:*

(a) *$\omega_{K[\Gamma]} \cong J$ as a $\mathbb{Z}^n$-graded $K[\Gamma]$-module;*
(b) *Δ is a Gorenstein complex over K;*

We apply this theorem to a d-polytope P with triangulation Γ and boundary complex Δ. Since $\omega_{K[\Gamma]} \cong J$ it clear that

$$K[\Gamma]/\omega_{K[\Gamma]} \cong K[\Delta]. \tag{10}$$

It follows then from [2, 3.3.18(b)] that Δ is Gorenstein. Moreover, if

$$H_{K[\Gamma]}(t) = \frac{\sum_{i=0}^{d+1} h_i t^i}{(1-t)^{d+1}}$$

is the Hilbert function of $K[\Gamma]$, then, by a result of Stanley [8],

$$H_{\omega_{K[\Gamma]}}(t) = \frac{\sum_{i=0}^{d+1} h_{d-i}t^i}{(1-t)^{d+1}}$$

is the Hilbert function of $\omega_{K[\Gamma]}$; see [2, Corollary 4.3.8]. Therefore (10) yields

$$\begin{aligned} H_{K[\Delta]}(t) &= H_{K[\Gamma]}(t) - H_{\omega_{K[\Gamma]}}(t) \\ &= \frac{\sum_{i=0}^{d+1} h_i t^i - \sum_{i=0}^{d+1} h_{d-i}t^i}{(1-t)^{d+1}} \\ &= \frac{\sum_{i=0}^{d} g_i t^i}{(1-t)^d}, \end{aligned}$$

where

$$g_i = \sum_{j=0}^{i}(h_j - h_{d+1-j}) \quad \text{for} \quad j = 0, \dots, i. \tag{11}$$

Now let us apply this to prove Proposition 3.1: Let Γ be the stacked shelling of $P(n,d)$. We know from 1.3 that the h-vector of Γ is $(1, n-d-1, 0, \dots, 0)$, so that from (11) we obtain for $\Delta(P(n,d))$ the h-vector, as asserted in 3.1.

Next we give a short proof for the Hibi-Terai formulas [7].

Theorem 3.3 (Hibi-Terai). *Let $P(n,d)$ be a stacked simplicial d-polytope with n vertices, and $\Delta P(n,d)$ its boundary complex. Then the minimal graded free resolution of the Stanley-Reisner ring has only a 2-linear and a d-linear strand. More precisely, one has*

$$\beta_{ii+j}(K[\Delta P(n,d)]) = \begin{cases} \sum_{k=i}^{n-d-1} k\binom{k-1}{i-1} & \text{for } 0 \le i \le n-d-1 \text{ and } j = 1, \\ \sum_{k=n-d-i}^{n-d-1} k\binom{k-1}{n-d-i-2} & \text{for } 1 \le i \le n-d \text{ and } j = d-1. \end{cases}$$

Proof. We apply Hochster's theorem 3.2 to $P(n,d)$. Let $\Gamma(n,d)$ be the stacked shelling of $P(n,d)$. By (10) and 3.2 we have

$$K[\Delta P(n,d)] = K[\Gamma(n,d)]/\omega_{K[\Gamma(n,d)]}.$$

Furthermore, $\omega_{K[\Gamma(n,d)]}$ is generated by all monomials $x^F \in K[\Gamma(n,d)]$ where $F \in \Gamma(n,d) \setminus \Delta P(n,d)$. Since the shelling of $P(n,d)$ is stacked all such faces are of dimension $d-1$. Therefore all generators of $\omega_{K[\Gamma(n,d)]}$ are of degree d.

By 2.2, $K[\Gamma(n,d)]$ has the graded minimal free $S = K[x_1, \dots, x_n]$-resolution

$$\mathbb{F} : 0 \to F_{n-d-1} \to \dots \to F_1 \to F_0 \to K[\Gamma(n,d)] \to 0,$$

where $F_i = S(-i-1)^{b_i}$ for $i = 1, \dots, n-d-1$ with $b_i = \sum_{k=i}^{n-d-1} k\binom{k-1}{i-1}$.

Let $\mathbb{G}$ be the minimal free graded S-resolution of $\omega_{K[\Gamma(n,d)]}$. The resolution $\mathbb{G}$ is obtained from $\mathbb{F}$ by dualizing $\mathbb{F}$ into S and shifting it suitably, see [2, Exercise 3.3.25]. Since in our case all generators of $\omega_{K[\Gamma(n,d)]}$ are of degree d, we see that $G_i \cong F^*_{n-d-1-i}(-n)$ for $i = 0, \dots, n-d-1$.

The inclusion $\omega_{K[\Gamma(n,d)]} \to K[\Gamma(n,d)]$ lifts to a map of complexes $\alpha : \mathbb{G} \to \mathbb{F}$. The mapping cone of α gives the resolution of $K[\Delta P(n,d)]$. It has the form

$$0 \to F_0^*(-n) \to F_1^*(-n) \oplus F_{n-d-1} \to \dots \to F_{n-d-1}^*(-n) \oplus F_1 \to F_0 \to K[\Delta P(n,d)] \to 0.$$

This yields the assertion of the theorem. □

The Sommerville equations for a 3-polytope P with $n = h+3$ vertices imply that P has the h-vector $(1, h, h, 1)$. Here we see that the combitorial type of $\Delta(P)$ is not reflected by the h-vector since it only depends on the number of vertices of P. In particular, a stacked 3-polytope with the same number of vertices as P has the same h-vector. Nevertheless the stacked 3-polytope is distinguished by the fact that it has the largest Betti numbers among all 3-polytopes with the same number of vertices. This is the theorem of Terai [9].

Let Γ be a triangulation of a d-polytope P. The boundary complex ΔP of P is generated by all $d-1$-dimensional faces F of Γ which belong to exactly one facet of Γ. The faces of Γ which do not belong to ΔP are called the interior faces of Γ. We say that Γ is a *proper triangulation of P* if Γ is shellable and has no interior vertices.

We conclude this paper with the following

Proposition 3.4. *Let P be a simplicial d-polytope admitting a proper triangulation with m facets. Then*

$$\beta_i(K[\Delta P]) \leq b_i + b_{n-d-i}$$

for all $i = 0, \dots, n-d$, *where* $b_i = \sum_{j=1}^{m-1} j\binom{j-1}{i-1}$.

Proof. Let Γ be the proper triangulation of P. Since Γ has no interior vertices it follows that $\omega_{K[\Gamma]}$ is contained in the square of the graded maximal ideal of $K[\Gamma]$. Therefore $K[\Gamma]$ and $K[\Delta P] = K[\Gamma]/\omega_{K[\Gamma]}$ are defined over the same polynomial ring S, and a free S-resolution of $K[\Delta P]$ is obtained as a mapping cone of the S-resolution of $K[\Gamma]$ and the S-resolution of $\omega_{K[\Gamma]}$, cf. the proof of 3.3. This resolution need not to be minimal. Thus, if β_i denotes the i-th Betti number of $K[\Gamma]$ we get $\beta_i(K[\Delta P]) \leq \beta_i + \beta_{n-d-i}$. Hence together with 2.1 the desired inequality follows. □

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