On the Stanley ring of a cubical complex

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Abstract

We investigate the properties of the Stanley ring of a cubical complex, a cubical analogue of the Stanley-Reisner ring of a simplicial complex. We compute its Hilbert-series in terms of the f-vector, and prove that by taking the initial ideal of the defining relations, with respect to the reverse lexicographic order, we obtain the defining relations of the Stanley-Reisner ring of the triangulation via "pulling the vertices" of the cubical complex. Applying an old idea of Hochster we see that this ring is Cohen-Macaulay when the complex is shellable, and we show that its defining ideal is generated by quadrics when the complex is also a subcomplex of the boundary complex of a convex cubical polytope. We present a cubical analogue of balanced Cohen-Macaulay simplicial complexes: the class of edge-orientable shellable cubical complexes. Using Stanley's results about balanced Cohen-Macaulay simplicial complexes and the degree two homogeneous generating system of the defining ideal, we obtain an infinite set of examples for a conjecture of Eisenbud, Green and Harris. This conjecture says that the h-vector of a polynomial ring in n variables modulo an ideal which has an n-element homogeneous system of parameters of degree two, is the f-vector of a simplicial complex.

Introduction

This paper is about some properties of the Stanley ring of a cubical complex. This ring is one of the possible cubical analogues of the Stanley-Reisner ring of a simplicial complex. While in the simplicial case commutative algebra was instrumental in obtaining combinatorial inequalities, this time combinatorics seems to give some commutative algebraic insight.

In the preliminary Section 1 we introduce the basic notions and define a way of associating simplicial complexes to cubical complexes, such that in the case of convex cubical polytopes our definition will

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coincide with the concept of "triangulation via pulling the vertices". We will call our operation the same way.

In Section 2 we show how to reduce greatly the number of relations defining the Stanley ring $K[\mathcal{C}]$ for all cubical complexes $\mathcal{C}$, and we compute the Hilbert-series of $K[\mathcal{C}]$.

In Section 3 we establish a connection between the Stanley ring of a cubical complex and the Stanley-Reisner ring of its triangulations via pulling the vertices: We show that the face ideal of the triangulation via pulling the vertices is the initial ideal with respect to the reverse lexicographic order of the face ideal of our cubical complex. This fact is analogous to Sturmfels' result in [22] on initial ideals of toric ideals.

In Section 4 we take a closer look at shellable cubical complexes and their Stanley rings. Using an idea of Hochster we establish the Cohen-Macaulay property of the rings. For later use in Section 5, we prove that the edge-graph of a shellable cubical complex is bipartite.

Section 5 contains the hardest theorem in this paper. We show that in the case of shellable subcomplexes of the boundary complex of a convex cubical polytope, the Stanley-ring may be defined by homogeneous relations of degree two. By the theorem of Bruggeser and Mani on the shellability of the boundary complex of convex polytopes ([4]), our result applies to the entire boundary complex of a convex cubical polytope.

In Section 6 we introduce the notion of edge-orientable cubical complexes, which turns out to be a cubical analogue of completely balanced simplicial complexes. Not only their Stanley ring contains an explicitly constructible linear system of parameters, but they also have a completely balanced triangulation.

Using almost all previous results of the paper, in Section 7 we construct infinitely many examples verifying a commutative algebraic conjecture of Eisenbud, Mark Green and Harris. According to this conjecture, the h-vector of a polynomial ring in n variables modulo an ideal which has an n-element homogeneous system of parameters of degree two, is the f-vector of a simplicial complex. Taking the face ideal of the boundary complex of any edge-orientable convex cubical polytope, and factoring out by a set of linear forms which is a system of parameters modulo the face ideal we obtain an example verifying the conjecture. The proof of the Eisenbud-Green-Harris conjecture in this very special case uses a theorem of Stanley on the h-vector of completely balanced simplicial complexes, and it does not work, if we drop the condition of edge-orientability. This makes the question interesting, whether convex cubical polytopes with not edge-orientable boundary exist: if yes, (and they probably do,) it may be a challenging task to verify the Eisenbud-Green-Harris conjecture already to this class of polytopes.

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1 Preliminaries

Definition 1 *An (abstract)* simplicial complex $\triangle$ *is a family of sets (called faces) on a vertex set V such that*

(i) $\{v\} \in \triangle$ *holds for every* $v \in V$, *and*

(ii) if $\sigma \in \triangle$ *and* $\tau \subseteq \sigma$ *then* $\tau \in \triangle$.

For every face σ we call $|\sigma| - 1$ the dimension *of σ. The maximal faces are called* facets, *their facets are* subfacets.

Definition 2 *A* cubical complex $\mathcal{C}$ *is a family of finite sets (called faces) on a vertex set V with the following properties.*

(i) For every $\sigma \in \mathcal{C}$ the elements of σ can be represented as the vertices of a finite dimensional cube, where the faces contained in σ are exactly the vertex sets of the faces of this cube. (In particular, we have $\emptyset \in \mathcal{C}$.)

(ii) If $\sigma, \tau \in \mathcal{C}$ *then* $\sigma \cap \tau \in \mathcal{C}$.

For every face σ we call the dimension of the cube associated to σ the dimension *of σ. As before, maximal faces are called* facets, *their facets are* subfacets. *The one-dimensional faces are also called* edges *and two vertices $u, v \in V$ are called* adjacent *if $\{u, v\}$ is an edge. Given a face $\sigma \in \mathcal{C}$ we will denote the subcomplex $\{\tau \in \mathcal{C} : \tau \subseteq \sigma\}$ by $\mathcal{C}|_\sigma$ and call it* the restriction of $\mathcal{C}$ to σ. *Moreover, for every nonempty face σ we will call the complex $\mathcal{C}|_\sigma \setminus \{\sigma\}$ the* boundary *of σ and we will denote it by $\partial(\sigma)$.*

It is a well-known fact that every simplicial complex has a *geometric realization* $\phi : V \rightarrow \mathbf{R}^n$ in a Euclidean space, such that for every $\sigma \in \triangle \setminus \{\emptyset\}$ the set $\text{conv}(\phi(\sigma))$ is a nondegenerate geometric simplex with vertex set $\phi(\sigma)$, and any pair of faces $\sigma, \tau \in \triangle$ we have $\text{conv}(\phi(\sigma)) \cap \text{conv}(\phi(\tau)) = \text{conv}(\phi(\sigma \cap \tau))$. (See for example [17, p.110].) It is not necessarily true, however, that cubical complexes have a similar geometric realization $\phi : V \rightarrow \mathbf{R}^n$. (Now of course we would require $\text{conv}(\phi(\sigma))$ to be a $\dim(\sigma)$-dimensional polytope with vertex set $\phi(\sigma)$ for every $\sigma \in \mathcal{C} \setminus \{\emptyset\}$. An example of a non-representable cubical complex is the complex $\mathcal{C}$ with three squares F_1, F_2, F_3 incident in such a way that they form a Möbius strip. The proof of the fact that this complex has no geometric realization is implicit in the proof of Theorem 7.)

Nevertheless, it is still true even for cubical complexes that their faces may be represented in a standard way.

Definition 3 *The* geometric standard n-cube *is the convex polytope*

$$[0,1]^n = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \;:\; 0 \leq x_i \leq 1\}.$$

We define the (abstract) standard n-cube $\mathcal{C}^n$ *to be the vertex set of* $[0,1]^n$ *together with the inherited face-structure on the vertices. We call any 2^n-element set with an isomorphic face-structure an n-dimensional cube. We call a geometric realization* $\phi : \mathcal{C}^n \longrightarrow \mathbf{R}^n$ standard *if we have* $\phi(V(\mathcal{C}^n)) = \{0,1\}^n$.

Simplicial complexes will occur as *triangulations* of cubical complexes via *pulling the vertices*. We will define these triangulations in an abstract way, i.e., our operation will associate a simplicial complex to even those cubical complexes which have no geometric realizations. For this we use the notion of *cubical span*.

Definition 4 *For a set of vertices $X \subset V$ in a cubical complex $\mathcal{C}$ we define the* cubical span $\text{Cspan}(X)$ *of X to be the smallest face containing the set X. (If there is no such face then we leave* $\text{Cspan}(X)$ *undefined.) For a pair of vertices $\{u,v\}$ and a face $\tau \in \mathcal{C}$ satisfying* $\text{Cspan}(\{u,v\}) = \tau$ *we say that* $\{u,v\}$ *is a* diagonal *of* τ.

Definition 5 *Let $\mathcal{C}$ be a cubical complex on the vertex set V and $<$ a linear order on V. Let us denote the smallest vertex of a face $\sigma \in \mathcal{C} \setminus \{\emptyset\}$ by $\delta_<(\sigma)$. We define the* triangulation of $\mathcal{C}$ via pulling the vertices in order $<$ *to be the family of all sets $\{v_1, \ldots, v_k\}$ such that $k \in \mathbf{N}$, $v_1 > \cdots > v_k$, and for $i = 1, \ldots, k$ we have*

$$v_i = \delta_<(\text{Cspan}(\{v_1, \ldots, v_i\})).$$

Remark Definition 5 may be naturally extended to the face complex of a convex polytope $\mathcal{P}$ by replacing the term $\text{Cspan}(\{v_1, \ldots, v_i\})$ with $\text{Pspan}(\{v_1, \ldots, v_i\})$, where $\text{Pspan}(X)$ stands for the smallest face containing a set of vertices X. (To be read as the *polyhedral span* of the set X.) In [20], Stanley gives an apparently different definition for triangulations $\triangle_<(\mathcal{P})$ of convex polytopes $\mathcal{P}$ via pulling the vertices, but the two definitions may be shown to be equivalent, without much difficulty. It is straightforward from Definition 5 that the restriction of $\triangle_<(\mathcal{C})$ to a face σ is just the triangulation of $\mathcal{C}|_\sigma$ via pulling the vertices with respect to the order induced by $<$ on σ. Hence by [20, Lemma 1.1], whenever a cubical complex $\mathcal{C}$ is geometrically represented, its (abstract) triangulation via pulling the vertices $\triangle_<(\mathcal{C})$ will induce an actual geometric triangulation.

2 Definition and elementary properties of the Stanley ring

R. Stanley suggested investigating the following ring associated to cubical complexes.

Definition 6 *Let $\mathcal{C}$ be a cubical complex, K a field. Associate a variable x_v to each vertex $v \in V$. The* Stanley ring $K[\mathcal{C}]$ *of the complex $\mathcal{C}$ over the field K is the factor ring* $K[x_v : v \in V] \big/ I(\mathcal{C})$, *where the ideal $I(\mathcal{C})$ is generated by the following elements.*

(i) $x_{v_1} \cdot x_{v_2} \cdots x_{v_k}$ *for all* $v_1, \ldots, v_k \in V$ *such that* $\{v_1, \ldots, v_k\}$ *is not contained in any face of* $\mathcal{C}$.

(ii) $x_u \cdot x_v - x_{u'} \cdot x_{v'}$ *for all* $u, u', v, v' \in V$ *such that* $\{u, v\}$ *and* $\{u', v'\}$ *are diagonals of the same face* $\mathrm{Cspan}(\{u, v\}) = \mathrm{Cspan}(\{u', v'\}) \in \mathcal{C}$.

We denote the ideal generated by the elements of type (i), (ii) by $I_1(\mathcal{C})$, $I_2(\mathcal{C})$ respectively. We call $I(\mathcal{C})$ the face ideal *of the cubical complex $\mathcal{C}$.*

In this section we will show that condition (i) can be weakened to requiring the product of at most three variables to be in $I(\mathcal{C})$, whenever the set of their indices is not contained in any face. In Section 5 we will prove that for an important class of cubical complexes (shellable subcomplexes of boundary complexes of convex cubical polytopes), it is even sufficient to set the product of pairs to be zero in $K(\mathcal{C})$ when they are not diagonals of a face. In doing so, the following equivalence relation defined on multisets of vertices will be instrumental. (Entries between brackets "$\lfloor$" and "$\rfloor$" are to be read as a list of elements of a multiset.)

Definition 7 *We call the multisets of vertices* $\lfloor u_1, u_2, \ldots, u_k \rfloor$ *and* $\lfloor v_1, v_2, \ldots, v_l \rfloor$ equivalent, *if $k = l$ and* $\lfloor v_1, v_2, \ldots, v_k \rfloor$ *can be obtained from* $\lfloor u_1, u_2, \ldots, u_k \rfloor$ *by repeated application of the following operation. If* $\mathrm{Cspan}(\{u_1, u_2\})$ *exists, replace* $\lfloor u_1, u_2, u_3, \ldots, u_k \rfloor$ *with* $\lfloor u_1', u_2', u_3, \ldots, u_k \rfloor$, *where* $\lfloor u_1', u_2' \rfloor$ *is any diagonal of* $\mathrm{Cspan}(\{u_1, u_2\})$.

The operation of replacing a diagonal with another one is reversible, and so we defined in deed an equivalence relation. Clearly, if a face $\tau \in \mathcal{C}$ contains $\{u_1, \ldots, u_k\}$ then the same holds for all equivalent multisets $\lfloor v_1, \ldots, v_k \rfloor$. *Hence we can say that a face τ contains or does not contain a given equivalence class of multisets.* In particular, $\mathrm{Cspan}(\lfloor u_1, u_2, \ldots, u_k \rfloor)$ is simultaneously defined or not defined for all multisets of an equivalence class, and its value is constant on an equivalence class, on which it is defined. The definitions yield immediately the following connections between the equivalence classes of multisets and monomials.

Lemma 1 *The monomials of $K[x_v : v \in V]$ have the following properties.*

1. *We have $x_{u_1} \cdots x_{u_k} \in I_1(\mathcal{C})$ if and only if $\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)$ does not exist.*
2. *The differences $x_{\underline{u}} - x_{\underline{v}}$, where $\underline{u}$ and $\underline{v}$ are equivalent multisets of vertices, form a generating system of the K-vector space $I_2(\mathcal{C})$. Consequently, monomials of degree k indexed by equivalent multisets of vertices represent the same element modulo $I_2(\mathcal{C})$.*

The following theorem is the key to understanding the role of the equivalence of multisets of vertices.

Theorem 1 *Monomials not belonging to $I_1(\mathcal{C})$ and associated to multisets from different equivalence classes are linearly independent modulo $I(\mathcal{C})$.*

Proof: Assume that we have a linear combination $\sum_{\underline{v}} \lambda_{\underline{v}} \cdot x_{\underline{v}} \in I(\mathcal{C})$ of monomials $x_{\underline{v}} \notin I_1(\mathcal{C})$, with coefficients $\lambda_{\underline{v}} \in K$, such that all the multisets $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ occurring in this sum belong to different equivalence classes. Let us fix one $x_{\underline{u}} = x_{u_1} \cdots x_{u_k}$, and show that we must have $\lambda_{\underline{u}} = 0$. By $x_{\underline{u}} \notin I_1(\mathcal{C})$ the face $\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)$ must exist. The factor of $K[\mathcal{C}]$ by the ideal $(x_v : v \notin \mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor))$ is the Stanley ring of $\mathcal{C}|_{\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)}$, and we have $x_{\underline{u}} \notin I_1(\mathcal{C}|_{\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)})$. Moreover, if if two multisubsets of $V(\mathcal{C}|_{\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)})$ are not equivalent in $\mathcal{C}$ then they are not equivalent in $\mathcal{C}|_{\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)}$ either. Thus w.l.o.g. we may assume $\mathcal{C} = \mathcal{C}|_{\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)}$, i.e., that $\mathcal{C}$ is a standard n-cube $\mathcal{C}^n$ for some $n \in \mathbf{N}$.

$I_1(\mathcal{C}^n) = 0$ implies $I(\mathcal{C}^n) = I_2(\mathcal{C}^n)$. Let us fix a standard geometric representation ϕ of $\mathcal{C}^n$, and consider the K-algebra homomorphism $\underline{\alpha} : K[x_v : v \in V(\mathcal{C}^n)] \longrightarrow K[y_0, y_1, \ldots, y_n]$ defined by $\underline{\alpha}(x_v) = y_0 \cdot y_1^{\phi_1(v)} \cdots y_n^{\phi_n(v)}$. The kernel of $\underline{\alpha}$ obviously contains all binomials of the form $x_u \cdot x_v - x_{u'} \cdot x_{v'}$, where $\mathrm{Cspan}(\{u, v\}) = \mathrm{Cspan}(\{u', v'\})$. Hence Ker $\underline{\alpha}$ contains $I(\mathcal{C}^n) = I_2(\mathcal{C}^n)$, and monomials associated to equivalent multisets are mapped into the same monomial under $\underline{\alpha}$. Therefore in order to prove $\lambda_{\underline{u}} = 0$, we only need to show that for a multiset $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ not equivalent to $\underline{u}$ we have $\underline{\alpha}(x_{\underline{v}}) \neq \underline{\alpha}(x_{\underline{u}})$, since then the coefficient of the monomial $\underline{\alpha}(x_{\underline{u}})$ in $0 = \sum_{\underline{v}} \lambda_{\underline{v}} \cdot \underline{\alpha}(x_{\underline{v}})$ will be $\lambda_{\underline{u}}$.

Let us denote by $\mathrm{Set}(v)$ the set $\{i : \phi_i(v) = 1\}$. The operation Set is a bijection between $V(C^n)$ and the subsets of $\{1, 2, \ldots, n\}$. A subset X of $V(C^n)$ is a face iff $\{\mathrm{Set}(v) : v \in X\}$ is an interval of the boolean algebra $P(\{1, 2, \ldots, n\})$. Hence we have $\mathrm{Cspan}(\{u, v\}) = \mathrm{Cspan}(\{u', v'\})$ iff for the corresponding subsets $\mathrm{Set}(u) \cap \mathrm{Set}(v) = \mathrm{Set}(u') \cap \mathrm{Set}(v')$ and $\mathrm{Set}(u) \cup \mathrm{Set}(v) = \mathrm{Set}(u') \cup \mathrm{Set}(v')$ hold. For a monomial $x_{\underline{v}} = x_{v_1} \cdots x_{v_l}$ we have $\underline{\alpha}(x_{\underline{v}}) = y_0^{\alpha_0} \cdot y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, where $\alpha_0 = l$, and for $i \geq 1$, α_i is the number of j-s such that $i \in \mathrm{Set}(v_j)$. (We count repeated vertices with their multiplicity.)

Let $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ be an arbitrary multiset of vertices. Replacing any pair (v_i, v_j) with the pair $(\mathrm{Set}^{-1}(\mathrm{Set}(v_i) \cap \mathrm{Set}(v_j)), \mathrm{Set}^{-1}(\mathrm{Set}(v_i) \cup \mathrm{Set}(v_j)))$, we obtain an equivalent multiset of vertices. Using this operation repeatedly, we can reach an equivalent multiset $\underline{v'} = \lfloor v'_1, \ldots, v'_l \rfloor$ such that $\mathrm{Set}(v'_1) \subseteq \cdots \subseteq \mathrm{Set}(v'_l)$ holds. (We can prove this by induction on l.) Now the statement follows from the obvious fact that for this multiset $\lfloor v'_1, \ldots, v'_l \rfloor$ we must have $\mathrm{Set}(v'_j) = \{i \in \{1, 2, \ldots, n\} : \alpha_i \geq l + 1 - j\}$. Therefore $\underline{\alpha}$ assigns different monomials to different equivalence classes of multisets of vertices. □

Corollary 1 *We have $x_{u_1} \cdots x_{u_k} \in I(C)$ if and only if* $\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)$ *does not exist.*

Corollary 2 *Two monomials $x_{u_1} \cdots x_{u_k} \notin I(C)$ and $x_{v_1} \cdots x_{v_l} \notin I(C)$ represent the same class modulo $I(C)$ if and only if $k = l$ and the multisets $\lfloor u_1, \ldots, u_k \rfloor$ and $\lfloor v_1, \ldots, v_k \rfloor$ are equivalent.*

Remark Theorem 1 is also a straight consequence of the proof of Theorem 4. We included an elementary proof, such that we may avoid the use of Gröbner basis theory in this section. Part of the argument presented may also be applied to show the following lemma.

Lemma 2 *Let C be an arbitrary cubical complex and $k \geq 2$. Then any monomial $x_{u_1} \cdot x_{u_2} \cdots x_{u_k}$ such that* $\mathrm{Cspan}(\{u_1, \ldots, u_k\})$ *exists, is equivalent modulo $I_2(C)$ to a monomial $x_{v_1} \cdot x_{v_2} \cdots x_{v_k}$ such that*

$$\mathrm{Cspan}(\{v_1, v_2\}) = \mathrm{Cspan}(\{u_1, \ldots, u_k\}) = \mathrm{Cspan}(\{v_1, \ldots, v_k\})$$

holds.

Proof: Without loss of generality we may assume $C = \mathrm{Cspan}(\{u_1, \ldots, u_k\})$, i.e., that C is a standard n-cube C^n. Let us fix again a geometric realization ϕ and denote by $\mathrm{Set}(v)$ the subset of $\{1, 2, \ldots, n\}$ with characteristic vector $\phi(v)$. We have shown in the proof of Theorem 1 that $\lfloor u_1, \ldots, u_k \rfloor$ is equivalent to a multiset $\lfloor v_1, \ldots, v_k \rfloor$ such that $\mathrm{Set}(v_1) \subseteq \cdots \subseteq \mathrm{Set}(v_k)$ holds. This $\lfloor v_1, \ldots, v_k \rfloor$ will have the required properties. (Observe that in the notation of the proof of Theorem 1, $\{v_1, v_k\}$ will be a diagonal of $\mathrm{Cspan}(\{u_1, \ldots, u_k\})$, whereas in the notation of the statement of this lemma $\{v_1, v_2\}$ is a diagonal. But the difference is only in the numbering of the vertices, which is irrelevant when we investigate multisets of vertices.) □

Using Lemma 2 we can show the following theorem.

Theorem 2 *Let $\mathcal{C}$ be an arbitrary cubical complex. Let $I_1'(\mathcal{C})$ be the ideal of $K[x_v : v \in V]$ generated by all monomials $x_{v_1} \cdots x_{v_k}$ such that $k \leq 3$, and $\{v_1, \ldots, v_k\}$ is not contained in any face of $\mathcal{C}$. Then we have*

$$I(\mathcal{C}) = I_1'(\mathcal{C}) + I_2(\mathcal{C}).$$

Proof: By definition, $I_1'(\mathcal{C})$ is contained in $I_1(\mathcal{C})$. Hence it is sufficient to show that if $\{v_1, \ldots, v_k\}$ is not contained in any face of $\mathcal{C}$ then $x_{v_1} \cdots x_{v_k}$ is congruent modulo $I_2(\mathcal{C})$ to a monomial from $I_1'(\mathcal{C})$. We prove this statement by induction on k. For $k = 2, 3$ we have $x_{v_1} \cdots x_{v_k} \in I_1'(\mathcal{C})$. Assume we know the statement for k, and that we are given $v_1, v_2, \ldots, v_{k+1}$ such that $\{v_1, \ldots, v_{k+1}\}$ is not contained in any face of $\mathcal{C}$. If $\{v_1, \ldots, v_k\}$ is not contained in any face, then we have $x_{v_1} \cdots x_{v_k} \in I_1(\mathcal{C})$, by induction hypothesis we get $x_{v_1} \cdots x_{v_k} \in I_1'(\mathcal{C})$, and so $x_{v_1} \cdots x_{v_k} \cdot x_{v_{k+1}} \in I_1'(\mathcal{C})$. Hence we may assume that $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$ exists. By Lemma 2, the monomial $x_{v_1} \cdots x_{v_k}$ is congruent modulo $I_2(\mathcal{C})$ to a monomial $x_{v_1'} \cdots x_{v_k'}$ such that we have

$$\mathrm{Cspan}(\{v_1', v_2'\}) = \mathrm{Cspan}(\{v_1, \ldots, v_k\}).$$

But then $\mathrm{Cspan}(\{v_1', v_2', v_{k+1}\})$ does not exist and we get

$$x_{v_1'} \cdot x_{v_2'} \cdot x_{v_{k+1}} \in I_1'(\mathcal{C}).$$

This implies

$$x_{v_1'} \cdots x_{v_k'} \cdot x_{v_{k+1}} \in I_1'(\mathcal{C}),$$

and so $x_{v_1} \cdots x_{v_k} \cdot x_{v_{k+1}}$ is congruent modulo $I_2(\mathcal{C})$ to an element of $I_1'(\mathcal{C})$. □

Theorem 1 and its corollaries also allow us to compute the *Hilbert-series* of the Stanley-ring of a cubical complex. Recall, that the Hilbert-series of a finitely generated $\mathbf{N}$-graded K-algebra A is usually defined as

$$\mathcal{H}(A, t) = \sum_{n=0}^{\infty} \dim_K(A_n) \cdot t^n,$$

where A_n is the vector space generated by the homogeneous elements of degree n, and the operator $\dim_K$ stands for taking the vector space dimension. (For details, see e.g. [19, p. 33].)

Theorem 3 *Let $\mathcal{C}$ be a d-dimensional cubical complex and let f_i be the number of i-dimensional faces of $\mathcal{C}$. Then the Hilbert-series $\mathcal{H}(K[\mathcal{C}], t)$ of the graded algebra $K[\mathcal{C}]$ is given by*

$$\mathcal{H}(K[\mathcal{C}], t) = 1 + \sum_{i=0}^{d} f_i \cdot \sum_{k=1}^{\infty} (k-1)^i \cdot t^k. \tag{1}$$

Proof: $K[\mathcal{C}]$ may be written as a direct sum of K-vector spaces as follows.

$$K[\mathcal{C}] = \bigoplus_{\sigma\in\mathcal{C}} \bigoplus_{k=0}^{\infty} \langle x_{u_1}\cdots x_{u_k} \ :\ \mathrm{Cspan}(\lfloor u_1,\ldots,u_k\rfloor) = \sigma\rangle. \tag{2}$$

(Note that this sum includes the vector space generated by the empty product 1 for $\sigma = \emptyset$ and $k = 0$.) It is a consequence of Theorem 1 and its corollaries that for an i-dimensional face $\sigma \in \mathcal{C}$ and a positive integer k, the dimension of $\langle x_{u_1}\cdots x_{u_k} : \mathrm{Cspan}(\lfloor u_1,\ldots,u_k\rfloor) = \sigma\rangle$ is equal to the number of multisets $\lfloor X_1,\ldots,X_k\rfloor$ of subsets of $\{1,2,\ldots,i\}$ such that we have

$$\emptyset = X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k = \{1,2,\ldots,i\}.$$

(For $i = 0$ we write $\emptyset$ instead of $\{1,2,\ldots,i\}$.) The number of such multisets is 1 for $i = 0$, and 0 for $i > 0$, $k = 1$. When $i > 0$ and $k \geq 2$ then for every $j \in \{1,2,\ldots,i\}$ there is a unique $\beta(j) \in \{1,\ldots,k-1\}$ such that $j \notin X_1 \cup X_2 \cup \cdots \cup X_{\beta(j)}$ and $j \in X_{\beta(j)+1} \cap X_{\beta(j)+2} \cap \cdots \cap X_k$. The values $\beta(j)$ may be chosen independently, in $(k-1)^i$ ways. Thus we have

$$\dim\left(\langle x_{u_1}\cdots x_{u_k} \ :\ \mathrm{Cspan}(\lfloor u_1,\ldots,u_k\rfloor) = \sigma\rangle\right) = (k-1)^i,$$

and the theorem follows. □

Introducing

$$\Phi_0(t) \stackrel{\text{def}}{=} \sum_{k\geq 0} t^k = \frac{1}{1-t}, \text{ and } \Phi_r(t) \stackrel{\text{def}}{=} \sum_{k\geq 0} k^r \cdot t^k \text{ for } r \geq 1,$$

we may rewrite equation (1) as

$$\mathcal{H}(K[\mathcal{C}],t) = 1 + \sum_{i=0}^{d} f_i \cdot t \cdot \Phi_i(t). \tag{3}$$

Let D denote the derivation operator of the polynomial ring $\mathbb{Z}[t]$ defined by $D : t \mapsto 1$. Then we have $t \cdot D(\Phi_r(t)) = \Phi_{r+1}(t)$. It is well-known that D satisfies the operator identity

$$(t\cdot D)^n = \sum_{k=0}^{n} S(n,k)\cdot t^k \cdot D^k,$$

where the letters $S(n,k)$ denote the Stirling numbers of the second kind. (See e.g. [15, p. 218, Section 6.6, formula (34)].) Using this formula for D allows for us to obtain

$$\Phi_i(t) = \sum_{j=0}^{i} S(i,j)\cdot t^j \cdot D^j\left(\frac{1}{1-t}\right) = \sum_{j=0}^{i} S(i,j)\cdot t^j \cdot \frac{j!}{(1-t)^{j+1}}.$$

Assuming $S(0,0) = 1$, this formula holds even for $i = 0$. Thus (3) is equivalent to

$$\mathcal{H}(K[\mathcal{C}],t) = 1 + \sum_{i=0}^{d} f_i \cdot t \cdot \sum_{j=0}^{i} S(i,j) \cdot t^j \cdot \frac{j!}{(1-t)^{j+1}}. \tag{4}$$

Introducing

$$f_j^{\triangle} \stackrel{\text{def}}{=} \begin{cases} \sum_{i=j}^{d} f_i \cdot S(i,j) \cdot j! & \text{when } 0 \leq j \leq d \\ 1 & \text{when } j = -1 \end{cases}, \tag{5}$$

we may transform (4) into the following equivalent form.

$$\mathcal{H}(K[\mathcal{C}],t) = \frac{\sum_{i=-1}^{d} f_i^{\triangle} \cdot t^{i+1} \cdot (1-t)^{d-i-1}}{(1-t)^d}. \tag{6}$$

3 Initial ideals and triangulations

In this section we describe the connection between the Stanley-Reisner ring of a triangulation of a cubical complex $\mathcal{C}$ via pulling the vertices, and the Stanley ring of this cubical complex, using the language and standard facts of Gröbner basis theory.

Let us first recall the definition of the *Stanley-Reisner ring of a simplicial complex* $\triangle$. (See e.g. [19].)

Definition 8 *Given a simplicial complex $\triangle$ with vertex set V, we define the* Stanley-Reisner ring *$K[\triangle]$ of $\triangle$ to be the factor ring $K[x_v : v \in V] \big/ I(\triangle)$, where the ideal $I(\triangle)$ is generated by the set $\{\mathbf{x}_{v_1} \cdots x_{v_k} : k \in \mathbb{N}, \{v_1, \ldots, v_k\} \notin \triangle\}$. We call $I(\triangle)$ the* face ideal *of $\triangle$.*

Note that both $K[\mathcal{C}]$ and $K[\triangle_<(\mathcal{C})]$ are the factors of the same polynomial ring $K[x_v : v \in V]$. To express the connection between the face ideals $I(\triangle_<(\mathcal{C}))$ and $I(\mathcal{C})$, we need the following concepts of Gröbner basis theory.

Definition 9 *Consider an arbitrary polynomial ring $K[X]$ over a field K. A* monomial order *on the set of monomials of $K[X]$ is a linear order $<$ on the semigroup of monomials such that if m_1, m_2 and n are monomials then*

$$m_1 > m_2 \textit{ implies } n \cdot m_1 > n \cdot m_2.$$

Given a monomial order $<$, for every polynomial $p \in K[X]$ we define the initial term $\mathrm{init}_<(p)$ *of p to be the largest term with respect to the term order $<$. Given an ideal I of $K[X]$ we denote by*

$\mathrm{init}_<(I)$ *the ideal generated by the initial terms of elements of* I*. A generating system* $\{p_1, \ldots, p_k\}$ *of* I *is called a* Gröbner basis *with respect to the term order* $<$*, if* $\mathrm{init}_<(I)$ *is generated by the set* $\{\mathrm{init}_<(p_1), \ldots, \mathrm{init}_<(p_k)\}$.

In particular, we will use *reverse lexicographic term orders*, which are defined as follows.

Definition 10 *Let* $K[X]$ *be a polynomial ring and* $<$ *a linear order on the set of variables* X*. We define the* reverse lexicographic order $<_{\mathrm{rlex}}$ induced by $<$ *as follows. Given two monomials* m *and* n*, we write both of them in the form* $m = x_1^{a_1} \cdots x_k^{a_k}, n = x_1^{b_1} \cdots x_k^{b_k}$ *where* $x_1 > \cdots > x_k$*. We set* $m <_{\mathrm{rlex}} n$ *iff* $\deg(m) < \deg(n)$ *holds or we have* $\deg(m) = \deg(n)$ *and* $a_i > b_i$ *for the last index* i *with* $a_i \neq b_i$.

Using the above definitions, the relation between $I(\mathcal{C})$ and $I(\triangle_<(\mathcal{C}))$ may be stated as follows.

Theorem 4 *Let* $\mathcal{C}$ *be a cubical complex on the vertex set* V *and* $<$ *any linear order on the vertices. Then we have the following identity.*

$$\mathrm{init}_{<_{\mathrm{rlex}}}(I(\mathcal{C})) = I(\triangle_<(\mathcal{C})).$$

In words, the initial ideal of the face ideal of $\mathcal{C}$ *with respect to the reverse lexicographic order induced by* $<$ *is the face ideal of the triangulation of* $\mathcal{C}$ *via pulling the vertices with respect to the order* $<$.

Proof: By Definition 6 and Corollary 1, the face ideal $I(\mathcal{C})$ is a *binomial ideal*, i.e., it has a generating system consisting only of binomials. (A binomial is a linear combination of at most two monomials.) Moreover, all monomials of $I(\mathcal{C})$ belong to $I_1(\mathcal{C})$. It is well known in the theory of binomial ideals, that every binomial ideal with respect to any term order has a reduced Gröbner basis consisting only of binomials. (See e.g. [8, Proposition 1].) This is true, because if we start Buchberger's algorithm (described e.g. in [3, Section 5.5]) to compute a Gröbner basis on a set of binomials, every newly added polynomial will be a binomial. In our case, we have a reduced Gröbner basis $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_1$ is the set of monomials minimally generating $I_1(\mathcal{C})$, and $\mathcal{G}_2$ is the reduced Gröbner basis with respect to $<_{\mathrm{rlex}}$ of $I_2(\mathcal{C})$. It is easy to verify that the elements of $\mathcal{G}_2$ will be differences of monomials of the form $x_{u_1} \cdots x_{u_k} - x_{v_1} \cdots x_{v_k}$, where $\mathrm{Cspan}(\{u_1, \ldots, u_k\})$ exists and the multisets $\lfloor u_1, \ldots, u_k \rfloor$ and $\lfloor v_1, \ldots, v_k \rfloor$ are equivalent. Moreover, a monomial $x_{u_1} \cdots x_{u_k} \notin I_1(\mathcal{C})$ is not the leading term of a binomial $x_{u_1} \cdots x_{u_k} - x_{v_1} \cdots x_{v_k} \in I_2(\mathcal{C})$ if and only if the set $\{u_1, \ldots, u_k\}$ is a face of $\triangle_<(\mathcal{C})$. In fact $x_{u_1} \cdots x_{u_k} \notin I_1(\mathcal{C})$ holds iff $\{u_1, \ldots, u_k\}$ is contained in some face $\sigma \in \mathcal{C}$. Assuming $u_1 \geq \cdots \geq u_k$, there is a multiset $\lfloor v_1, \ldots, v_k \rfloor$ equivalent to $\lfloor u_1, \ldots, u_k \rfloor$ with $x_{v_1} \cdots x_{v_k} <_{\mathrm{rlex}} x_{u_1} \cdots x_{u_k}$ iff for some $i \in \{2, \ldots, k\}$ we have $u_i \neq \delta_<(\mathrm{Cspan}(u_1, \cdots, u_{i-1})$. (This is a consequence of Lemma 2.) Thus $\mathrm{init}_{<_{\mathrm{rlex}}}(I(\mathcal{C}))$ is generated by square-free monomials, and these square-free monomials are the same as those generating $I(\triangle_<(\mathcal{C}))$. □

Corollary 3 *For any cubical complex $\mathcal{C}$, and any of its triangulations via pulling the vertices $\triangle_<(\mathcal{C})$ we have*

$$\mathcal{H}(K[\mathcal{C}],t) = \mathcal{H}(K[\triangle_<(\mathcal{C})],t). \tag{7}$$

Proof: It is well-known in the theory of Gröbner bases that for every polynomial ring $K[X]$, every ideal I of this polynomial ring, and every term order $\prec$ the Hilbert-series of $K[X] / I$ is equal to the Hilbert-series of $K[X] / \mathrm{init}_\prec(I)$. (See e.g. [3, Lemma 9.26 and Proposition 6.52].) Thus we have

$$\mathcal{H}(K[\mathcal{C}],t) = \mathcal{H}\left(K[x_v \,:\, v \in V] \Big/ \mathrm{init}_{<_{\mathrm{rlex}}}(I(\mathcal{C})), t\right),$$

and the statement follows from Theorem 4. □

Corollary 4 *For any d-dimensional cubical complex $\mathcal{C}$, and any order $<$ on its vertices, the vector $(f_{-1}^{\triangle}, \ldots, f_d^{\triangle})$ given by (5) is the f-vector of the simplicial complex $\triangle_<(\mathcal{C})$.*

Proof: This is a straight consequence of (6), Corollary 3, and [19, Ch. II., §1, 1.4 Theorem]. □

Remarks

1. The fact that a square-free initial ideal is the face ideal of a simplicial complex is widely used, see eg.[22, Section 6], or [10, Introduction].
2. The above proof was inspired by (and is much simpler than) the analogous result of B. Sturmfels for initial ideals of toric ideals in [22]. The special case when the cubical complex is a standard n-cube is implicit in [22, Corollary 5.2.].
3. As a special case of Corollary 4, we obtain that every triangulation via pulling the vertices of a standard n-cube has the same f-vector. This could also be deduced from [20, Corollary 2.7], because using [20, Theorem 2.3] it is easy to show that the standard n-cube is a compressed polytope.

4 Shellable cubical complexes

In the definition of shellable cubical complexes we will need the notions of *ball* and *sphere*. Remember that an n-cube $\mathcal{C}^n$ has a *standard geometric realization* $\phi : V(\mathcal{C}^n) \longrightarrow \mathbf{R}^n$, where ϕ is a bijection between $V(\mathcal{C}^n)$ and $\{0,1\}^n$, such that vertices connected by an edge go into to the vertices of $[0,1]^n$ connected by an edge. (See Definition 3.)

Definition 11 *A collection* $\{F_1, F_2, \ldots, F_k\}$ *of facets of the boundary of an n-cube is called an* $(n-1)$-dimensional ball *or* $(n-1)$-dimensional sphere *respectively, if the set* $\bigcup_{i=1}^{k} \operatorname{conv}(\phi(F_i))$ *is homeomorphic to an* $(n-1)$*-dimensional ball or sphere respectively.*

As in [16], we encode the nonempty faces of C^n with vectors $(u_1, u_2, \ldots, u_n) \in \{0, 1, *\}^n$ in the following way. Consider a standard geometric realization $\phi : C^n \longrightarrow \mathbf{R}^n$ For a nonempty face $\sigma \in C^n$ and $i \in \{1, 2, \ldots, n\}$ set $u_i = 0$ or 1 respectively if the i-th coordinate of every element of $\phi(\sigma)$ is 0 or 1 respectively. Otherwise we set $u_i = *$. Using this coding, the facets of C^n will correspond to the vectors $(u_1, \ldots, u_n)$ for which exactly one of the u_i-s is not a $*$-sign.

Definition 12 *Let* A_i^0 *resp.* A_i^1 *stand for the facet* $(u_1, u_2, \ldots, u_n)$ *with* $u_i = 0$ *resp.* $u_i = 1$ *and* $u_j = *$ *for* $j \neq i$. *Let* $\{F_1, \ldots, F_k\}$ *be a collection of facets of* $\partial(C^n)$. *Let r be the number of i-s such that exactly one of* A_i^0 *and* A_i^1 *belong to* $\{F_1, \ldots, F_k\}$, *and let s be the number of i-s such that both* A_i^0 *and* A_i^1 *belong to* $\{F_1, \ldots, F_k\}$. *We call* (r, s) *the* type *of* $\{F_1, \ldots, F_k\}$.

Note that when the type of $\{F_1, \ldots, F_k\}$ is (r, s) then there are exactly $n - r - s$ coordinates i such that neither A_i^0 nor A_i^1 belong to $\{F_1, \ldots, F_k\}$.

The following observation, originally due to Ron Adin [1], gives a full description of those collections of facets $\{F_1, \ldots, F_k\}$ which are an $(n-1)$-dimensional ball or sphere.

Lemma 3 *The collection of facets* $\{F_1, \ldots, F_k\}$ *of the boundary of an n-cube is an* $(n-1)$*-sphere if and only if it has type* $(0, n)$ *and it is an* $(n-1)$*-ball if and only if its type* (r, s) *satisfies* $r > 0$.

Remark Lemma 3 allows us to define $(n-1)$-balls or $(n-1)$-spheres combinatorially, by prescribing their types.

Definition 13 *A cubical complex* C *is* pure *if all facets of* C *have the same dimension. We define* shellable cubical complexes *as follows.*

1. *The empty set is a ((−1)-dimensional) shellable cubical complex.*
2. *A point is a (0-dimensional) shellable complex.*
3. *A d-dimensional pure complex* C *is shellable if its facets can be listed in a linear order* $F_0, F_1, \ldots, F_n$ *such that for each* $k \in \{1, 2, \ldots, n\}$ *the subcomplex* $C|_{F_k} \cap (C|_{F_1} \cup \cdots \cup C|_{F_{k-1}})$ *is a pure complex of dimension* $(d-1)$ *such that its maximal dimensional faces form a* $(d-1)$*-dimensional ball or sphere.*

By abuse of notation we will say that the attachment of $\mathcal{C}|_{F_k}$ to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ in a shelling $F_0, F_1, \ldots, F_k$ has type (r, s) if set of facets of $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$ considered as a collection of facets of $\mathcal{C}|_{F_k}$ has type (r, s).

The definition of shellability often allows us to prove properties of shellable cubical complexes by *induction on shelling*, i.e. by induction on their dimension and the number of their facets. One of the most important results obtainable this way is the following theorem.

Theorem 5 *If $\mathcal{C}$ is a shellable cubical complex, then the Stanley-ring $K[\mathcal{C}]$ is a Cohen-Macaulay ring.*

Proof (Sketch): The basic idea of the proof is due to Hochster. We observe that if I_1 and I_2 are perfect ideals of the same dimension, k, and $I_1 + I_2$ is a perfect ideal of dimension $k-1$, then $I_1 \cap I_2$ is a perfect ideal. (This is item (ii) in the proof of [12, Theorem 2°], essentially equivalent to [13, Proposition 18].) Hence, like in the proof of [12, Theorem 2°], we can prove the perfectness of the face ideal $I(\mathcal{C})$ of a a shellable cubical complex $\mathcal{C}$ by induction on the number of facets. We leave the details to the reader. (Note the well-known fact that the perfectness of an ideal in a polynomial ring implies the Cohen-Macaulay property of the factor by this ideal–see, e.g. [2, Theorem 3.5.8].) □

Another example to the use of induction on shelling is the following elementary lemma, which we will use later.

Lemma 4 *The edge-graph of a shellable cubical complex of dimension at least 2 is bipartite.*

Proof: We use induction on the number of facets. Let $F_1, \ldots, F_k$ be a shelling of $\mathcal{C}$. By induction hypothesis, the complex $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ has a bipartite edge-graph. Clearly $\mathcal{C}|_{F_k}$ has a bipartite edge-graph: when we represent its vertices, as vertices of the standard d-cube $[0,1]^d$, an appropriate coloring with 2 colors is to color the vertices according with the parity of the sum of their coordinates. It is easy to check that the edge-graph of $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$ is a connected graph. Thus the induction step follows from the fact that essentially there is only one way to color a connected bipartite graph. □

5 A homogeneous generating system of degree 2 for $I(\mathcal{C})$

Theorem 2 inspires the following question. Let $I_1''(\mathcal{C})$ be the ideal generated by the monomials $x_u \cdot x_v$ such that the pair $\{u, v\} \subseteq V$ is not contained in any face. When do we have $I(\mathcal{C}) = I_1''(\mathcal{C}) + I_2(\mathcal{C})$? For such complexes $I(\mathcal{C})$ is generated by homogeneous forms of degree 2.

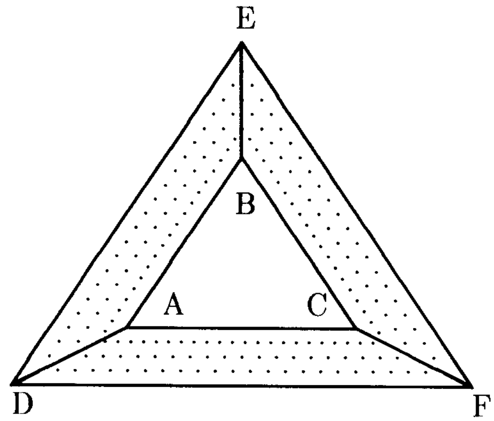

Figure 1: A not well behaved cubical complex

Definition 14 *We call a cubical complex* well behaved *when it satisfies* $I(C) = I_1''(C) + I_2(C)$.

The following lemma is a straightforward consequence of Theorem 2 and of the trivial inclusion $I_1''(C) \subseteq I_1'(C)$.

Lemma 5 *A cubical complex is well behaved iff for every triple* $\lfloor u_1, u_2, u_3 \rfloor$ *either* $\{u_1, u_2, u_3\}$ *is contained in a face of* C *or there is a* $\lfloor v_1, v_2, v_3 \rfloor$ *equivalent to* $\lfloor u_1, u_2, u_3 \rfloor$ *such that* $\{v_1, v_2\}$ *is not contained in any face of* C.

We will use the statement of Lemma 5 as an equivalent definition of well behaved cubical complexes.

Example Figure 1 represents a not well behaved cubical complex. The facets of the complex are $ABED, BCEF$ and $ACDF$.

It is easy to verify that for the triple $\lfloor A, C, E \rfloor$ and for any equivalent triple, any two elements of the triple are contained in a face, but there is no face containing all three of them.

Conjecture 1 *Every shellable cubical complex is well behaved.*

The following lemmas are statements about the properties of an eventual minimal counterexample to Conjecture 1. (Minimality will always mean minimality of the number of facets.) At the end we will not get a proof of the conjecture, but the properties to be shown will allow us to exclude all shellable subcomplexes of a boundary complex of a convex cubical polytope from the class of shellable not well behaved cubical complexes.

Definition 15 *Let C be a not well behaved shellable cubical complex. We call the triple $\lfloor u_1, u_2, u_3 \rfloor$ a* counter-evidence *if any two of u_1, u_2 and u_3 are contained in some face of C, but no face contains the set $\{u_1, u_2, u_3\}$, and the same holds for all equivalent triples in C.*

Clearly C is a counterexample to Conjecture 1 exactly when C contains a counterevidence. The following lemma tells us, how a counterevidence must lie in a minimal counterexample.

Lemma 6 *Let C be a minimal not well behaved shellable cubical complex and $F_1, F_2, \ldots, F_k$ a shelling of C. Then every counter-evidence $\lfloor u_1, u_2, u_3 \rfloor$ has exactly one element outside F_k, and two elements in F_k.*

Proof: The triple $\lfloor u_1, u_2, u_3 \rfloor$ cannot be a counter-evidence, if all three u_i-s lie in F_k. Thus at least one of them must lay outside F_k. If at least two of u_1, u_2, u_3 are not in F_k, then any face containing at least two of them is also contained in $C|_{F_1} \cup \cdots \cup C|_{F_{k-1}}$. In particular, we must have $\{u_1, u_2, u_3\} \subset F_1 \cup \cdots \cup F_{k-1}$. By minimality, $\lfloor u_1, u_2, u_3 \rfloor$ is not a counter-evidence in the shellable complex $C|_{F_1} \cup \cdots \cup C|_{F_{k-1}}$. Thus there is an equivalent $\lfloor v_1, v_2, v_3 \rfloor$ such that $\{v_1, v_2, v_3\}$ is contained in some face of $C|_{F_1} \cup \cdots \cup C|_{F_{k-1}}$. But then the same holds in C and so $\lfloor u_1, u_2, u_3 \rfloor$ is not a counter-evidence. Hence the only way for $\lfloor u_1, u_2, u_3 \rfloor$ to be a counter-evidence is to contain exactly one element outside F_k. □

Corollary 5 *Let C be a minimal not well behaved shellable cubical complex and $F_1, F_2, \ldots, F_k$ a shelling of C and $\lfloor u_1, u_2, u_3 \rfloor$ a counter-evidence such that $u_1, u_2 \in F_k$ and $u_3 \notin F_k$ Then for every $u \in \mathrm{Cspan}(\{u_1, u_2\})$, the face $\mathrm{Cspan}(\{u_3, u\})$ has exactly half of its vertices in F_k.*

Proof: If necessary, we can replace the diagonal $\lfloor u_1, u_2 \rfloor$ by another diagonal of $\mathrm{Cspan}(\{u_1, u_2\})$ such that $u = u_1$ holds, and so $\mathrm{Cspan}(\{u_3, u\})$ exists, and we may assume $u_1 = u$. By $u_3 \notin F_k$, at most half of $\mathrm{Cspan}(\{u_3, u\})$ belongs to F_k. If less then half is contained in F_k then the diagonal $\lfloor u_1, u_3 \rfloor$ may be replaced by a diagonal $\lfloor u_1', u_3' \rfloor$ such that both u_1' and u_3' are outside F_k. Thus the triple $\lfloor u_1', u_2, u_3' \rfloor$ (which is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$) will not be counter-evidence by Lemma 6. □

Proposition 1 *Assume that for the shelling $F_1, \ldots, F_k$ of C, the attachment of $C|_{F_k}$ to $\bigcup_{i=1}^{k-1} C|_{F_i}$ has type $(r, 0)$. Then C can not be a minimal not well behaved shellable complex.*

Proof: Assume the contrary. When the type is $(r, 0)$ then $F_k \setminus (F_1 \cup \cdots \cup F_{k-1})$ is not empty: there is at least one vertex which was added when we added F_k. Let $\lfloor u_1, u_2, u_3 \rfloor$ be a counterexample with

$u_1, u_2 \in F_k$, $u_3 \notin F_k$. If $\mathrm{Cspan}(\{u_1, u_2\})$ contains a newly added vertex, then –after replacing eventually u_1 and u_2 with another diagonal of $\mathrm{Cspan}(\{u_1, u_2\})$– we may assume $u_2 \in F_k \setminus (F_1 \cup \cdots \cup F_{k-1})$. But then $\{u_2, u_3\}$ is not contained in any face of $\mathcal{C}$, and we get a contradiction. Thus we must have $\mathrm{Cspan}(\{u_1, u_2\}) \subset F_1 \cup \cdots \cup F_{k-1}$. In this case, however, any face containing at least two of u_1, u_2 and u_3 is contained in $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ and so $\lfloor u_1, u_2, u_3 \rfloor$ is a counter-evidence in this smaller complex already. □

Lemma 7 *Let $\mathcal{C}$ be a minimal not well behaved shellable cubical complex and $F_1, F_2, \ldots, F_k$ a shelling of $\mathcal{C}$. Assume that there is a pair H, H' of subfacets which are opposite facets of $\partial(\mathcal{C}|_{F_k})$ and both belong to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$. Let $\lfloor u_1, u_2, u_3 \rfloor$ be a counter-evidence such that $u_1, u_2 \in F_k, u_3 \notin F_k$. Then there is a face of $\mathcal{C}$ containing* $\mathrm{Cspan}(\{u_1, u_2\}) \cap H$ *and u_3.*

Proof: If $\mathrm{Cspan}(\{u_1, u_2\}) \cap H = \emptyset$ then we have $\mathrm{Cspan}(\{u_1, u_2\}) \subseteq H' \in \mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$, meaning that already $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ was a counterexample. Similar contradiction with minimality arises when we assume $\mathrm{Cspan}(\{u_1, u_2\}) \cap H' = \emptyset$. Thus we may suppose $u_1 \notin H$ and $u_2 \in H$, and we may renumber all equivalent triples $\lfloor v_1, v_2, v_3 \rfloor$ such that $v_3 \notin F_k$, $v_1 \in H'$ and $v_2 \in H$ holds. Let u_1' be the projection of u_1 onto H. Then we have $\mathrm{Cspan}(\{u_1', u_2\}) = \mathrm{Cspan}(\{u_1, u_2\}) \cap H$. If the set $\{u_1', u_2, u_3\}$ is contained in a face, then we are done. Otherwise, given the fact that the cubical span of any two of u_1', u_2 and u_3 is contained in $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$, a well behaved shellable cubical complex, we obtain that the triple $\lfloor u_1', u_2, u_3 \rfloor$ is equivalent to a triple $\lfloor z_1, z_2, z_3 \rfloor$ such that there is no face of $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ containing $\{z_1, z_2\}$. Consider a sequence of replacing diagonals, which demonstrates the equivalence of $\lfloor u_1', u_2, u_3 \rfloor$ and $\lfloor z_1, z_2, z_3 \rfloor$. Assume that $\lfloor u_1, u_2, u_3 \rfloor$ was chosen from its equivalence class such that this derivation of equivalence is the shortest possible.

If the first step is replacing the triple $\lfloor u_1', u_2, u_3 \rfloor$ with $\lfloor w_1', w_2, u_3 \rfloor$ where $\{w_1', w_2\}$ is a diagonal of $\mathrm{Cspan}(\{u_1', u_2\})$ then we get a contradiction with the minimality of the derivation. In fact, let w_1 be the projection of w_1' onto H'. It is easy to check that $\{w_1, w_2\}$ is a diagonal of $\mathrm{Cspan}(\{u_1, u_2\})$, so $\lfloor w_1, w_2, u_3 \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$, and there is a shorter derivation of equivalence between $\lfloor w_1', w_2, u_3 \rfloor$ and $\lfloor z_1, z_2, z_3 \rfloor$. We also get a contradiction when we assume that our first step was to replace $\lfloor u_1', u_2, u_3 \rfloor$ with $\lfloor u_1', w_2, w_3 \rfloor$, where $\{w_2, w_3\}$ is a diagonal of $\mathrm{Cspan}(\{u_2, u_3\})$. In this case, the very same replacement can be performed on $\lfloor u_1, u_2, u_3 \rfloor$ and we obtain the equivalent triple $\lfloor u_1, w_2, w_3 \rfloor$, from which we have a shorter derivation.

Hence we are left with the case when the first step of the derivation involves replacing $\lfloor u_1', u_2, u_3 \rfloor$ with $\lfloor w_1', u_2, w_3 \rfloor$, where $\{w_1', w_3\}$ is a diagonal of $\mathrm{Cspan}(\{u_1', u_3\})$ such that $w_1' \in H$ and $w_3 \notin F_k$. Let q_1 be the projection of u_2 onto H'. Then $\lfloor q_1, w_1', w_3 \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$, so the first step is again unnecessary.

Therefore we may assume that at least two of u'_1, u_2 and u_3 are not contained in a common face. This pair cannot be $\{u'_1, u_2\} \subseteq F_k$ and it cannot be $\{u_2, u_3\}$ because then $\lfloor u_1, u_2, u_3 \rfloor$ is not a counterexample. Finally, if $\{u'_1, u_3\}$ is not contained in any face, then we obtain a contradiction since $\lfloor q_1, u'_1, u_3 \rfloor$ is not a counterevidence but is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$. □

Lemma 8 *Assume $F_1, \ldots, F_k$ is a shelling of a minimal not well behaved complex $\mathcal{C}$, and $\lfloor u_1, u_2, u_3 \rfloor$ is a counter-evidence with $u_3 \notin F_k$. Assume H and H' are opposite facets of $\partial(\mathcal{C}|_{F_k})$, such that they both belong to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$. Then there is no edge $\{v, w\} \in$ Cspan$(\{u_1, u_2\})$ for which $v \in H, w \in H'$ would hold and* Cspan$(\{u_3, v, w\})$ *would exist.*

Proof: Assume the contrary. Since $\{v, w\}$ is an edge, either $\{u_3, v\}$ or $\{u_3, w\}$ is a diagonal of Cspan$(\{u_3, v, w\})$. W.l.o.g. we may assume that $\{u_3, v\}$ is a diagonal. We also may assume that the triple $\lfloor u_1, u_2, u_3 \rfloor$ was chosen in such a way that $u_1 = v$ holds. (If not, we can replace the pair $\lfloor u_1, u_2 \rfloor$ with another pair containing $v \in$ Cspan$(\{u_1, u_2\})$.) Let u'_3 be the vertex diagonally opposite to w in Cspan$(\{u_3, v, w\})$. Then $\lfloor u_1, u_2, u_3 \rfloor$ is equivalent to $\lfloor w, u_2, u'_3 \rfloor$ and here we have $u'_3 \notin F_k$ and $\{w, u_2\} \subseteq H' \in \mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$, contradicting the assumption about the minimality of $\mathcal{C}$. □

Proposition 2 *If $\mathcal{C}$ has a shelling $F_1, \ldots, F_k$ such that the attachment of $\mathcal{C}|_{F_k}$ to $\bigcup_{i=1}^{k-1} \mathcal{C}|_{F_i}$ has type (r, s) with $s \geq 2$, then $\mathcal{C}$ is not a minimal not well behaved shellable complex.*

Proof: Assume the contrary. Let H_1, H'_1 and H_2, H'_2 be pairs of subfacets which are opposite in F_k, all belonging to $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$. Assume furthermore that $\lfloor u_1, u_2, u_3 \rfloor$ is a counterevidence satisfying $u_1, u_2 \in F_k$, $u_3 \notin F_k$. Then Cspan$(\{u_1, u_2\}) \cap H_i$ and Cspan$(\{u_1, u_2\}) \cap H'_i$ are non-empty by the minimality of $\mathcal{C}$. By Lemma 7, Cspan$(\{u_1, u_2\}) \cap H_1$ is contained in a face with u_3. But then we can find $v, w \in$ Cspan$(\{u_1, u_2\}) \cap H_1$ such that $\{v, w\}$ is an edge and we have $v \in H_2, w \in H'_2$, contradicting Lemma 8. □

Lemma 9 *Let $\mathcal{C}$ be a minimal not well behaved shellable complex. Let H and H' be opposite facets of $\partial(\mathcal{C}|_{F_k})$ such that H also belongs to $(\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$ but H' does not. Assume, there is a counter-evidence $\lfloor u_1, u_2, u_3 \rfloor$, such that $u_1, u_2 \in F_k, u_3 \notin F_k$, and* Cspan$(\{u_1, u_2\}) \cap H \neq \emptyset$ *hold. Then for any $u \in$* Cspan$(\{u_1, u_2\}) \cap H'$ *we have*

$$\text{Cspan}(\{u_3, u\}) \cap F_k \subseteq H'.$$

Proof: Note first that $\mathrm{Cspan}(\{u_1, u_2\}) \cap H' \neq \emptyset$ otherwise $\lfloor u_1, u_2, u_3 \rfloor$ would also be a counter-evidence in $(\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$. W.l.o.g. we may assume $u = u_1$ and so $u_2 \in H$. By Corollary 5 the face $\mathrm{Cspan}(\{u_3, u_1\})$ has exactly half of its vertices in F_k. In particular, u_3 is connected by an edge to a unique vertex $v \in F_k$ and we have $\mathrm{Cspan}(\{u_3, u_1\}) \cap F_k = \mathrm{Cspan}(\{u_1, v\})$. Thus we only need to show $v \in H'$. If not, then we can replace the diagonal $\lfloor u_3, u_1 \rfloor$ with a diagonal $\lfloor u_3', v \rfloor$ and obtain an equivalent triple $\lfloor v, u_2, u_3' \rfloor$ with $v, u_2 \in H$, contradicting the assumption of minimality of $\mathcal{C}$. □

Proposition 3 *Let $\mathcal{C}$ be shellable d-dimensional minimal not well behaved cubical complex, with shelling $F_1, F_2, \ldots, F_k$. Then the type of the attachment of $\mathcal{C}|_{F_k}$ to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ can not be $(r, d-r)$.*

Proof: Assume the contrary. By Proposition 1 and Proposition 2 we may assume that the type of the attachment of $\mathcal{C}|_{F_k}$ to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ is $(d-1, 1)$. Thus, taking a standard geometric realization ϕ of $\mathcal{C}|_{F_k}$, we may assume that exactly the following facets of $\partial(\mathcal{C}|_{F_k})$ belong to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$: $A_1^0, A_2^0, \ldots, A_{d-1}^0, A_d^0$ and A_d^1.

Let $\lfloor u_1, u_2, u_3 \rfloor$ be a counter-evidence such that $u_1, u_2 \in F_k, u_3 \notin F_k$, and $\dim \mathrm{Cspan}(\{u_1, u_2\})$ is maximal under these conditions. Then, similarly to Corollary 5, we can show that for every $u \in \mathrm{Cspan}(\{u_1, u_2\})$ the face $\mathrm{Cspan}(\{u_3, u\})$ exists and has exactly half of its vertices in $\mathrm{Cspan}(\{u_1, u_2\})$. In fact, w.l.o.g. we may assume $u_1 = u$ and so $\mathrm{Cspan}(\{u_3, u\})$ exist. At most half of $\mathrm{Cspan}(\{u_3, u_1\})$ may belong to $\mathrm{Cspan}(\{u_1, u_2\})$, because otherwise F_k would also contain more than half of the vertices of $\mathrm{Cspan}(\{u_3, u_1\})$, in contradiction with Corollary 5. If less than half of the vertices of $\mathrm{Cspan}(\{u_3, u_1\})$ belong to $\mathrm{Cspan}(\{u_1, u_2\})$ then there is a $u_1' \in \mathrm{Cspan}(\{u_3, u_1\})$ such that u_1' is connected to u_1 by an edge, and $u_1' \in F_k \backslash \mathrm{Cspan}(\{u_1, u_2\})$. Let u_3' be the vertex diagonally opposite to u_1' in $\mathrm{Cspan}(\{u_3, u_1\})$. The triple $\lfloor u_1', u_2, u_3' \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$, it satisfies, $u_1', u_2 \in F_k$ (hence we must have $u_3' \notin F_k$), and $\mathrm{Cspan}(\{u_1', u_2\})$ properly contains $\mathrm{Cspan}(\{u_1, u_2\})$, contradicting the maximality of $\dim \mathrm{Cspan}(\{u_1, u_2\})$.

By the minimality of $\mathcal{C}$, for any $i \in \{1, 2, \ldots, d-1\}$ the vertices u_1 and u_2 cannot be both contained in A_i^0, otherwise the triple $\lfloor u_1, u_2, u_3 \rfloor$ is already a counter-evidence in $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$. Similarly, the last coordinate of u_1 and u_2 can not agree. These considerations show that the vertices $u, v \in F_k$ defined by $\phi(u) \stackrel{\mathrm{def}}{=} (1, 1, \ldots, 1, 0)$ and $\phi(v) \stackrel{\mathrm{def}}{=} (1, 1, \ldots, 1, 1)$ both belong to $\mathrm{Cspan}(\{u_1, u_2\})$. We claim that $\mathrm{Cspan}(\{u_3, u\})$ and $\mathrm{Cspan}(\{u_3, v\})$ are edges. In fact, as noted above, half of $\mathrm{Cspan}(\{u_3, u\})$ is contained in $\mathrm{Cspan}(\{u_1, u_2\})$. It is sufficient to show therefore that $\mathrm{Cspan}(\{u_3, u\}) \cap \mathrm{Cspan}(\{u_1, u_2\})$ is zerodimensional. If not, then it contains a vertex u' for which $\phi(u')$ differs from $\phi(u)$ from exactly one coordinate, say the j-th one. When $j = d$ then we get a contradiction by Lemma 8, when $j \leq d-1$ we get a contradiction by Lemma 9. Hence $\{u_3, u\}$ is an edge and similarly $\{u_3, v\}$ is an edge. But then u_3, u and v form a triangle in the edge-graph of $\mathcal{C}$, which cannot be bipartite therefore, contradicting

Lemma 4. □

Propositions 1, 2 and 3 imply the following theorem.

Theorem 6 *Every shellable cubical complex of dimension* 2 *is well behaved.*

Proof: Take a minimal counterexample $\mathcal{C}$ with shelling $F_1, \ldots, F_k$. By Lemma 3, the possible types of attachments of $\mathcal{C}|_{F_k}$ to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ are the following: $(1,0), (2,0), (1,1)$ and $(0,2)$. The types $(1,0)$ and $(2,0)$ are excluded by Proposition 1, type $(0,2)$ is forbidden by Proposition 2, and finally type $(1,1)$ is disallowed by Proposition 3. □

Finally, we prove the main theorem of this section.

Theorem 7 *Let $\mathcal{C}$ be a $(d-1)$-dimensional shellable subcomplex of the boundary complex of a d-dimensional convex cubical polytope P. Then $\mathcal{C}$ is well behaved.*

Proof: Let $\mathcal{C}$ be a minimal counterexample, and $F_1, F_2, \ldots, F_k$ a shelling of $\mathcal{C}$. By the previous results, we may assume $d \geq 4$, and that the type of attachment of $\mathcal{C}|_{F_k}$ to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$ is $(r, 1)$ with $0 < r \leq d-3$. Let H_1 and H_2 be the only pair of opposite facets of $\partial(\mathcal{C}|_{F_k})$ such that they both belong to $\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}}$.

Let us take a counter-evidence $\lfloor u_1, u_2, u_3 \rfloor$ such that $u_1, u_2 \in F_k$, $u_3 \notin F_k$ hold, and the dimension of $\mathrm{Cspan}(\{u_1, u_2\})$ be maximal under these conditions. Let us denote $\mathrm{Cspan}(\{u_1, u_2\})$ by τ_3. By Lemma 7, the faces $\tau_i \stackrel{\mathrm{def}}{=} \mathrm{Cspan}(\{u_3\} \cup (\tau_3 \cap H_i))$ exist for $i = 1, 2$. As in the proof of Theorem 6, the maximality of $\dim \mathrm{Cspan}(\{u_1, u_2\})$ implies that exactly half of the vertices of τ_1 or τ_2 belong to τ_3. Thus we have

$$\frac{|\tau_1|}{2} = |\tau_1 \cap \tau_3| = |\tau_3 \cap H_1| = \frac{|\tau_3|}{2} = |\tau_3 \cap H_2| = |\tau_2 \cap \tau_3| = \frac{|\tau_1|}{2},$$

and so τ_1, τ_2 and τ_3 have the same dimension. Let us denote this dimension by δ.

Let S be the affine hull of u_3 and τ_3. It is a $(\delta+1)$-dimensional plane, and it intersects the polytope P in a $(\delta+1)$-dimensional polytope P'. Clearly, S contains both τ_1 and τ_2, because half of these faces is a $(\delta-1)$-face of τ_3 (and so belongs to S) and the affine span of u_3 and $\tau_i \cap \tau_3$ contains τ_i. $(i = 1, 2.)$

Consider the "pyramid" $Q \stackrel{\mathrm{def}}{=} \mathrm{conv}(u_3, \tau_3)$. We may assume that $\mathrm{relint}(Q) \subset \mathrm{int}\,(P)$ otherwise Q is contained in a face of $\partial(P)$, and so there is a face of $\partial(P)$ containing τ_1, τ_2 and τ_3. It is easy to convince ourselves, however, that no cube can contain three equidimensional faces with the intersection

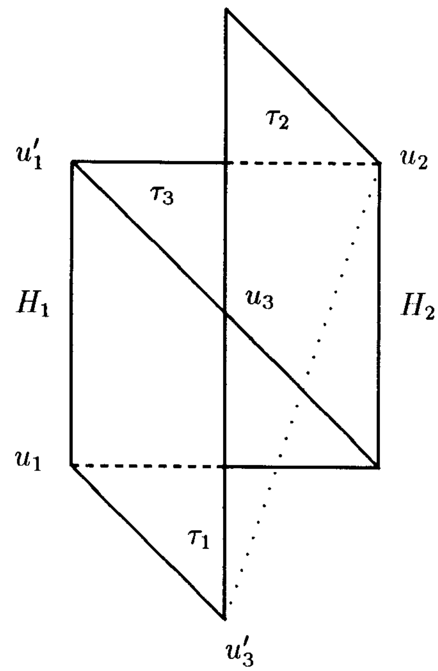

Figure 2: Illustration to the proof of Theorem 7

properties of τ_1, τ_2 and τ_3: if half of the vertices of τ_1 and τ_2 intersect τ_3 in opposite halves of τ_3 then $\tau_1 \cap \tau_2$ would be empty, and we need $u_3 \in \tau_1 \cap \tau_2$. The affine hull of Q is S.

W.l.o.g. we may assume that u_1 is diagonally opposite to u_3 in τ_1. (If not, we may replace $\lfloor u_1, u_2 \rfloor$ by another diagonal of τ_3.) Let u_3' be the vertex of $\tau_1 \setminus \tau_3$ which is connected to u_1 by an edge. (In other words, let u_3' be the vertex diagonally opposite to u_3 in the face $\tau_1 \setminus \tau_3$.) Let u_1' be the vertex diagonally opposite to u_1 in $\tau_3 \cap \tau_1$. Then u_1' is diagonally opposite to u_3' in τ_1 and so $\lfloor u_1', u_2, u_3' \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$, hence there is a face containing u_3' and u_2. In particular, the line segment connecting u_3' and u_2 belongs to $\partial(P)$ and thus it cannot have any common point with relint(Q).

Consider now those supporting hyperplanes of Q in S which intersect Q in a facet of Q. These hyperplanes are δ-dimensional, and with the exception of the affine hull of τ_3, they all arise as the affine hull of u_3 and of a $(\delta - 1)$-face of τ_3. For each such hyperplane K, let us call that half-space of K in S which contains Q, the *positive half* of K. If K contains u_1', u_2 and u_3, then u_3' is in the strict positive half of K. In fact, K then intersects τ_3 in a $(\delta - 1)$-face of τ_3 and so it intersects $\tau_1 \cap \tau_3$ in a $(\delta - 2)$-face. $K \cap \tau_1$ contains this $(\delta - 2)$-face and u_3, and so $K \cap \tau_1$ contains a $(\delta - 1)$-face of τ_1. Thus $K \cap \tau_1$ is equal to this $(\delta - 1)$-face, because otherwise K would contain the whole affine hull of τ_1 which does not contain u_2. The $(\delta - 1)$-face $K \cap \tau_1$ of τ_1 contains u_1' and so it cannot contain u_3' which is diagonally opposite to $u_1' \in K$ in τ_1.

The only hyperplanes of facets of Q through u_2 which don't contain both u_3 and u_1' are aff (τ_3) and aff (τ_2). The vertex u_3' cannot be in the strict positive half of both of them, because otherwise the line segment connecting u_3' with u_2 would contain a point of relint(Q) close to u_2. Therefore either $u_3' \in$ aff (τ_3) or $u_3' \in$ aff (τ_2) must hold. Equivalently, we have either $u_3' \in \tau_3$ or $u_3' \in \tau_2$.

Recall that $u_3' \in \tau_1 \setminus \tau_3$, so $u_3' \notin \tau_3$. Therefore u_3' must belong to τ_2 and so $\tau_1 \cap \tau_2$ contains the $(\delta - 1)$-face Cspan$(\{u_3, u_3'\})$, hence $\tau_1 \cap \tau_2$ must also be $(\delta - 1)$-dimensional. We claim that in this case u_2 is connected to u_3 by an edge. In fact by what was said above, the line segment connecting u_3' and u_2 intersects relint(τ_2) and so u_3' and u_2 are diagonally opposite in τ_2, and u_3 is connected to u_2 by an edge. Therefore u_1', u_2 and u_3 form a triangle, contradicting Lemma 4. □

Corollary 6 *The boundary complex of a convex cubical polytope is well behaved.*

Proof: As a special case of the results shown in [4], the boundary complex of a convex cubical polytope is shellable. Hence we may apply Theorem 7. □

6 Edge-orientable cubical complexes

Definition 16 *We call two edges* $\{u, v\}$ *and* $\{u', v'\}$ *of a cubical complex* C parallel *if there is a facet* $F \in C$ *containing* $\{u, u', v, v'\}$, *and a subfacet* $H \subset F$ *such that* $|\{u, v\} \cap H| = |\{u', v'\} \cap H| = 1$.

We can turn the edge-graph of C *into a directed graph by defining a function*

$$\pi : V \times V \longrightarrow \{-1, 0, 1\},$$

satisfying the following properties

(i) $\pi(u, v) = -\pi(v, u)$ *holds for all* $u \neq v$,

(ii) $\pi(u, v) = 0$ *if and only if* $\{u, v\}$ *is not an edge of* C.

(We say when $\pi(u, v) = 1$ *that "the edge points from* u *towards* v*".)*

We call π *an* orientation of the edge-graph of C *or edge-orientation on* C *if it satisfies the following condition: given two parallel edges* $\{u, v\}$ *and* $\{u', v'\}$, *a facet* F *containing these edges and a subfacet* $H \subset F$ *such that* $\{u, v\} \cap H = u$, $\{u', v'\} \cap H = u'$, *we have*

$$\pi(u, v) = \pi(u', v')$$

We call C edge-orientable, if its edge-graph has an orientation π.

In plain English, edge-orientability means that we can direct the edges of C such that "parallel edges point in the same direction." As a consequence of Jordan's theorem, the boundary complex of a 3-dimensional cubical polytope is edge-orientable. Another often studied class of cubical polytopes with edge-orientable boundary, is the class of cubical zonotopes. For general cubical polytopes in higher dimensions, edge-orientability means that every $(d-2)$-dimensional manifold connecting the midpoints of parallel edges is orientably embedded into the boundary of the polytope. It seems to be intuitively clear to the author that there are convex 4-dimensional cubical polytopes, of which the boundary complex contains a "Möbius strip" of 3-dimensional cubes, but we leave the verification of this conjecture to the reader.

Conjecture 2 *There exist convex cubical polytopes, of which boundary complex is not edge-orientable.*

The following lemma shows the existence of a labeling for a shellable and edge-orientable complex C which will have important applications.

Lemma 10 *Let C be a shellable and edge-orientable complex of dimension at least 2 and π an orientation of the edge-graph of C. Then there is a labeling*

$$\theta : V \longrightarrow \mathbf{Z}$$

such that for every edge $\{u,v\}$ we have

$$\theta(v) - \theta(u) = \pi(u,v). \tag{8}$$

Proof: The proof is analogous to the proof of Lemma 4. Assume C is a counterexample with a minimal number of facets. Let $F_1, F_2, \ldots, F_k$ be a shelling of C. The complex $C|_{F_1} \cup \cdots \cup C|_{F_{k-1}}$ is shellable, and the restriction of π provides an edge-orientation on it. Hence, by the minimality of C, there is a labeling θ' on it which satisfies equation (8) for every pair of vertices of $C|_{F_1} \cup \cdots \cup C|_{F_{k-1}}$. On the other hand it is easy to see that there is a labeling θ'' on the cube $C|_{F_k}$: we can take a standard geometric realization $\phi : C|_{F_k} \rightarrow [0,1]^{\dim(C)}$, such that the only vertex with no incoming edges in $C|_{F_k}$ goes into $(0,0,\ldots,0)$, and the only vertex with no outgoing edges in $C|_{F_k}$ goes into $(1,1,\ldots,1)$. Then we can set $\theta''(v)$ to be the sum of the coordinates of $\phi(v)$ for every $v \in F_k$. It is easy to check that this labeling will also satisfy (8) for every pair of vertices of F_k.

Clearly, if a labeling θ satisfies (8) in a complex then the same holds for $\theta + c$, where c is an arbitrary constant. Thus we may assume that we have a $v_0 \in F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ such that $\theta'(v_0) = \theta''(v_0)$ holds. But then, as we have observed in the proof of Lemma 4, the edge-graph of the complex $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$ is connected. It is easy to see that if θ' and θ'' are labelings satisfying (8) in a directed graph G, which has a connected graph as underlying undirected graph, then their difference is constant. Thus, by $\theta'(v_0) = \theta''(v_0)$, the restriction of θ' to $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$ is equal to the restriction of θ'' to $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$. Therefore we can define

$$\theta(v) \stackrel{\text{def}}{=} \begin{cases} \theta'(v) & \text{when } v \in F_1 \cup \cdots \cup F_{k-1} \\ \theta''(v) & \text{when } v \in F_k \end{cases}$$

and obtain a labeling for $\mathcal{C}$ that satisfies (8), contradicting our assumption. □

Lemma 11 *Let $\mathcal{C}$ be a shellable, edge-orientable cubical complex, and π be an edge-orientation of $\mathcal{C}$. Then the transitive closure $<_\pi$ of the relation*

$$u \stackrel{\text{def}}{<_\pi} v \quad \textit{whenever } \pi(u,v) = 1$$

is a partial order on the vertex set V of $\mathcal{C}$.

Proof: We only need to show that there is no sequence of vertices $v_1, v_2, \ldots, v_k$ such that

$$\pi(v_1, v_2) = \pi(v_2, v_3) = \cdots = \pi(v_{k-1}, v_k) = \pi(v_k, v_1) = 1$$

would hold. If we had such a sequence then for a labeling θ satisfying (8) we would have $\theta(v_{i+1}) = \theta(v_i) + 1$ for $i = 1, 2, \ldots, k-1$, and $\theta(v_k) + 1 = \theta(v_1)$. But this would imply

$$\theta(v_1) = \theta(v_1) + k,$$

a contradiction. □

Definition 17 *Let $\mathcal{C}$ a shellable, edge-orientable cubical complex and π an edge-orientation of $\mathcal{C}$. We call the partial order described in Lemma 11 the* partial order induced by π, *and we denote it by $<_\pi$.*

We define the triangulation $\triangle_\pi(\mathcal{C})$ of $\mathcal{C}$ induced by π *as follows.*

1. *We set $V(\triangle_\pi(\mathcal{C})) \stackrel{\text{def}}{=} V(\mathcal{C})$.*

2. *A set $\{v_1, \ldots, v_k\} \subseteq V(\mathcal{C})$ is a face of $\triangle_\pi(\mathcal{C})$ if and only if* $\text{Cspan}(\{v_1, v_2, \ldots, v_k\})$ *exists and $\{v_1, v_2, \ldots, v_k\}$ is a chain in the partially ordered set $(V, <_\pi)$.*

Lemma 12 *Given a shellable and edge-orientable cubical complex $\mathcal{C}$ and an edge-orientation π of $\mathcal{C}$, we have $\triangle_\pi(\mathcal{C}) = \triangle_<(\mathcal{C})$ for any linear extension $<$ of the partial order $<_\pi$.*

Proof: Take an arbitrary subset $\{v_1, \ldots, v_k\}$ of the vertex set V. Without loss of generality we may assume $v_1 > \cdots > v_k$.

If $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$ does not exist then $\{v_1, \ldots, v_k\}$ does not belong to any of $\triangle_\pi(\mathcal{C}), \triangle_<(\mathcal{C})$. Thus we may assume that $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$ exist, and w.l.o.g. we may even assume that there is no vertex outside $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$, i.e., $\mathcal{C}$ is a standard n-cube $\mathcal{C}^n = \mathrm{Cspan}(\{v_1, \ldots, v_k\})$ for some n.

In this special case the statement can be easily shown by induction on k. In fact, $\{v_1, \ldots, v_k\} \in \triangle_\pi(\mathcal{C})$ holds iff we have $\{v_1, \ldots, v_{k-1}\} \in \triangle_\pi(\mathrm{Cspan}(\{v_1, \ldots, v_{k-1}\}))$ and v_k is the unique minimal element of $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$ with respect to the partial order $<_\pi$. Similarly, we have $\{v_1, \ldots, v_k\} \in \triangle_<(\mathcal{C})$ iff $\{v_1, \ldots, v_{k-1}\} \in \triangle_<(\mathrm{Cspan}(\{v_1, \ldots, v_{k-1}\}))$, and v_k is the unique minimal element of $\mathrm{Cspan}(\{v_1, \ldots, v_k\}$ with respect to the linear order $<$. Observe finally that $<$, being an extension of $<_\pi$, the unique minimal element of $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$ is the same with respect to both orders. □

Recall that a d-dimensional pure simplicial complex $\triangle$ is *completely balanced* if the vertex set of $\triangle$ may be colored with $d+1$ colors such that no two vertices of the same color belong to a common face.

Lemma 13 *Let $\mathcal{C}$ be a d-dimensional shellable edge-orientable complex and π an edge-orientation of $\mathcal{C}$. Then $\triangle_\pi(\mathcal{C})$ is a completely balanced simplicial complex.*

Proof: As shown in Lemma 10, there is a labeling θ of the vertices of $\mathcal{C}$ satisfying (8). Color the vertex v with the modulo $(d+1)$ equivalence class of $\theta(v)$. We claim that $\triangle_\pi(\mathcal{C})$ becomes a completely balanced complex, with this coloring. In fact, let us take a a face $\{v_1, v_2, \ldots, v_k\} \in \triangle_\pi(\mathcal{C})$. By the definition of the triangulation, there is a facet $F \in \mathcal{C}$ containing $\{v_1, v_2, \ldots, v_k\}$, and w.l.o.g. we may assume $v_1 <_\pi v_2 <_\pi \cdots <_\pi v_k$. It is an easy consequence of (8) that then we have

$$\theta(v_1) < \theta(v_2) < \cdots < \theta(v_k).$$

The values of θ on F are $d+1$ consecutive integers, hence no two of the above $\theta(v_i)$-s can be congruent modulo $(d+1)$, and so $\triangle_\pi(\mathcal{C})$ is a balanced complex. □

Remark The coloring described in the proof of Lemma 13 allows as to give an explicit system of linear parameters for $K[\mathcal{C}]$. In fact, $\{\sum_{\mathrm{color}(v)=i} x_v \,:\, i = 1, 2, \ldots, d+1\}$ is such a system. We leave the proof to the reader.

7 The Eisenbud-Green-Harris conjecture

Using the Stanley ring of the boundary complex $\mathcal{C}$ of an edge-orientable convex cubical polytope we may construct an interesting example to a conjecture of D. Eisenbud, M. Green and J. Harris. Before stating the conjecture, let us recall the definition of the *h-vector of a graded algebra.* It is a well known fact that the Hilbert-series of a Noetherian $\mathbb{N}$-graded algebra A may be written in the following form.

$$\mathcal{H}(A,t) = \frac{\sum_{i=0}^{l} h_i \cdot t^i}{\prod_{i=1}^{s}(1-t^{e_i})}, \tag{9}$$

where $d \stackrel{\text{def}}{=} \sum_{i=1}^{s} e_i$ is the Krull-dimension of A, i.e. the maximum length of an increasing chain of prime ideals. (See e.g. [21].)

Definition 18 *We call the vector $(h_0, \ldots, h_l)$ in (9) the* h-vector *of the graded Noetherian algebra A.*

In particular, for a simplicial complex $\triangle$ or a cubical complex $\mathcal{C}$ we define the h-vector of the simplicial or cubical complex to be the h-vector of their Stanley rings.

Now we may formulate the Eisenbud-Green-Harris conjecture as follows. (See [7, Conjecture ($\mathrm{V_m}$)].)

Conjecture 3 *Let I be an ideal of a polynomial ring $K[x_1, \ldots, x_r]$ which contains a regular sequence of length r in degree 2. Then the h-vector of the graded algebra $K[x_1, \ldots, x_r]/I$ is the f-vector of some simplicial complex.*

Example Let $\mathcal{C}$ be the boundary complex of a $(d+1)$-dimensional convex cubical polytope, and assume that $\mathcal{C}$ is edge-orientable with an edge-orientation π. Assume furthermore that K is an infinite field. Then the Stanley ring $K(\mathcal{C})$ is a d-dimensional Cohen-Macaulay ring, and it contains a linear system of parameters $l_1, \ldots, l_d$. We claim that the polynomial ring $K[x_v : v \in V]/(l_1, \ldots, l_d)$ and the natural image of the face ideal $I(\mathcal{C})$ in this ring provide an example for Conjecture 3.

In fact, by Theorem 7 the face ideal $I(\mathcal{C})$ is generated by homogeneous elements of degree 2, and so the same holds for the image $\overline{I(\mathcal{C})}$ of $I(\mathcal{C})$ in $K[x_v : v \in V]/(l_1, \ldots, l_d)$. Thus $\overline{I(\mathcal{C})}$ contains a maximal regular system of parameters in degree 2. The factor of $K[x_v : v \in V]/(l_1, \ldots, l_d)$ by $\overline{I(\mathcal{C})}$ is Artinian, isomorphic to $K[\mathcal{C}]/(l_1, \ldots, l_d)$, and its h-vector is equal to the h-vector of the cubical complex $\mathcal{C}$ by the Cohen-Macaulay property of $K[\mathcal{C}]$. By Corollary 3 this h-vector is the also the h-vector of any triangulation via pulling the vertices $\triangle_<(\mathcal{C})$ of $\mathcal{C}$. By Lemma 12, whenever we take a linear extension $<$ of the partial order $<_\pi$, the simplicial complex $\triangle_<(\mathcal{C})$ is equal to the simplicial complex $\triangle_\pi(\mathcal{C})$. Thus we are left to show that the h-vector of $\triangle_\pi(\mathcal{C})$ is the f-vector of some other simplicial complex. But by Lemma 13 the simplicial complex $\triangle_\pi(\mathcal{C})$ is completely balanced, and –being a triangulation of a sphere– it is a Cohen-Macaulay simplicial complex by [19, Corollary 4.4]. Therefore its h-vector is the f-vector of another simplicial complex by [18, 4.5 Corollary].

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