INDISPENSABLE BINOMIALS OF TORIC IDEALS

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ABSTRACT. A binomial f belonging to a toric ideal I is indispensable if, for any system $\mathcal{F}$ of binomial generators of I, either f or $-f$ belongs to $\mathcal{F}$. First, it will be proved that a binomial $f \in I$ is indispensable if and only if, for any reduced Gröbner basis $\mathcal{G}$ of I, either f or $-f$ belongs to $\mathcal{G}$. Second, we show that the toric ideal I_G arising from a finite graph G whose complementary graph is weakly chordal is generated by the indispensable binomials if and only if no complete graph of order ≥ 4 is a subgraph of G. Third, we completely classify indispensable binomials of the toric ideal I_G arising from a finite graph G satisfying the odd cycle condition. Finally, the existence of indispensable binomials of I_G will be discussed.

INTRODUCTION

The theory of toric ideals has been developed from viewpoints of combinatorics and computational commutative algebra. One of the recent topics arising in applied mathematics is to study the problem when a toric ideal possesses a unique minimal system of binomial generators. Following [7] a binomial f belonging to a toric ideal I is indispensable if, for any system $\mathcal{F}$ of binomial generators of I, either f or $-f$ belongs to $\mathcal{F}$. When is a toric ideal generated by the indispensable binomials?

The present paper is organized as follows. First of all, in Section 1, we discuss fundamental binomials, indispensable binomials and circuits of toric ideals. It will be proved that every fundamental binomial is indispensable and that every fundamental binomial is a circuit. In addition, we will present (i) a binomial which is both indispensable and a circuit but not fundamental; (ii) a binomial which is indispensable but not a circuit; (iii) a binomial which is a circuit but not indispensable. Second, in Section 2, it will be proved that a binomial f belonging to a toric ideal I is indispensable if and only if, for any reduced Gröbner basis $\mathcal{G}$ of I, either f or $-f$ belongs to $\mathcal{G}$. Third, in Section 3, following [5], we study the toric ideal I_G arising from a finite graph G. Based on the combinatorial classification of indispensable cycles (Theorem 3.2) we will show that the toric ideal I_G of a finite graph G whose complementary graph is weakly chordal is generated by the indispensable binomials if and only if no complete graph of order ≥ 4 is a subgraph of G. See Theorem 3.4. Third, in Section 4, we completely classify indispensable binomials of the toric ideal I_G arising from a finite graph G satisfying the odd cycle condition [2]. See Theorem 4.3. Finally, the existence of indispensable binomials of I_G will be discussed in Section 5. Let G be a finite connected graph with $I_G \neq (0)$ and suppose that G has no vertices of degree 1. (The degree of a vertex i of G is a number of edges of G

with $i \in e$.) Then I_G possesses an indispensable binomial if and only if G is not a complete graph. See Theorem 5.3.

1. Binomials and toric ideals

Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ denote the polynomial ring in d variables over a field K with each $\deg t_i = 1$. Given a finite set $\mathcal{A} = \{f_1, \ldots, f_n\}$ of monomials belonging to $K[\mathbf{t}]$ such that all the f_i's have the same degree, we write $K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}]$ generated by $f_1, \ldots, f_n$ over K. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K with each $\deg x_i = 1$ and $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ the surjective ring homomorphism defined by setting $\pi(x_i) = f_i$ for all $1 \leq i \leq n$. We write $I_{\mathcal{A}}$ for the kernel of π and call $I_{\mathcal{A}}$ the *toric ideal* of $K[\mathcal{A}]$.

A *binomial* of $K[\mathbf{x}]$ is a polynomial f of the form $f = u - v$, where u and v are monomials of $K[\mathbf{x}]$ with $u \neq v$ and with $\deg u = \deg v$. The support of a monomial u of $K[\mathbf{x}]$ is $\mathrm{supp}(u) = \{ x_i : x_i \text{ divides } u \}$ and the support of a binomial $f = u - v$ is $\mathrm{supp}(f) = \mathrm{supp}(u) \cup \mathrm{supp}(v)$. A *binomial ideal* of $K[\mathbf{x}]$ is an ideal of $K[\mathbf{x}]$ generated by binomials. For example, the toric ideal $I_{\mathcal{A}}$ of $K[\mathcal{A}]$ is a binomial ideal.

A binomial $f = u - v$ belonging to $I_{\mathcal{A}}$ is called *primitive* if there is no binomial $g = u' - v' \in I_{\mathcal{A}}$ with $f \neq g$ such that u' divides u and v' divides v. Every primitive binomial is irreducible. The *Graver basis* of $I_{\mathcal{A}}$ is the set of primitive binomials of $I_{\mathcal{A}}$. The Graver basis of $I_{\mathcal{A}}$ is finite and is a system of binomial generators of $I_{\mathcal{A}}$.

We say that an irreducible binomial f belonging to $I_{\mathcal{A}}$ is a *circuit* of $I_{\mathcal{A}}$ if there is no binomial $g \in I_{\mathcal{A}}$ such that $\mathrm{supp}(g) \subset \mathrm{supp}(f)$ and $\mathrm{supp}(g) \neq \mathrm{supp}(f)$. Every circuit is irreducible and primitive. A binomial $f \in I_{\mathcal{A}}$ is a circuit of $I_{\mathcal{A}}$ if and only if $I_{\mathcal{A}} \cap K[\{x_i : x_i \in \mathrm{supp}(f)\}]$ is generated by f.

If $f = u - v$ is a binomial belonging to $I_{\mathcal{A}}$, then we write T_f for the set of those variables t_i such that t_i divides $\pi(u)$ $(= \pi(v))$. Let $K[T_f] = K[\{t_i : t_i \in T_f\}]$ and $\mathcal{A}_f = \mathcal{A} \cap K[T_f]$. The toric ideal $I_{\mathcal{A}_f}$ of $K[\mathcal{A}_f]$ coincides with $I_{\mathcal{A}} \cap K[\{x_i : \pi(x_i) \in K[T_f]\}]$. A binomial $f \in I_{\mathcal{A}}$ is called *fundamental* if $I_{\mathcal{A}_f}$ is generated by f.

Finally, a binomial f belonging to $I_{\mathcal{A}}$ is called *indispensable* if, for any system of binomial generators $\mathcal{F}$ of $I_{\mathcal{A}}$, either f or $-f$ belongs to $\mathcal{F}$.

Theorem 1.1. (a) *Every fundamental binomial is a circuit.*
(b) *Every fundamental binomial is indispensable.*

Proof. (a) Let $f = u - v$ be a binomial belonging to $I_{\mathcal{A}}$. Since $\pi(x_i) \in K[T_f]$ if $x_i \in \mathrm{supp}(f)$, one has $\mathrm{supp}(f) \subset \{x_i : \pi(x_i) \in K[T_f]\}$. Let $g \in I_{\mathcal{A}}$ be a binomial with $\mathrm{supp}(g) \subset \mathrm{supp}(f)$. Then $g \in I_{\mathcal{A}_f}$. If f is fundamental, then $I_{\mathcal{A}_f}$ is generated by f. Since g is a binomial, it follows easily that g is of the form $g = \pm(u'u - v'v)$, where u' and v' are monomials of $K[\mathbf{x}]$ with $\deg u' = \deg v'$. In particular $\mathrm{supp}(f) \subset \mathrm{supp}(g)$. Hence $\mathrm{supp}(f) = \mathrm{supp}(g)$. Thus f is a circuit of $I_{\mathcal{A}}$.

(b) Let $\mathcal{F}$ be any system of binomial generators of $I_{\mathcal{A}}$ and $f = u - v$ a fundamental binomial of $I_{\mathcal{A}}$. If neither f nor $-f$ belongs to $\mathcal{F}$, then one has $0 \neq h = u' - v' \in \mathcal{F}$ such that u' divides u. Hence $\pi(u') \in K[T_f]$. Since $\pi(u') = \pi(v')$, one has

$\pi(v') \in K[T_f]$. Thus $h \in I_{\mathcal{A}_f} = (f)$. This is impossible since $h \neq 0$, $h \neq f$ and $\deg h \leq \deg f$. □

Example 1.2. (a) Let $d = 6$, $n = 8$ and $\mathcal{A} = \{t_1t_2, t_2t_3, t_3t_4, t_4t_5, t_5t_6, t_1t_6, t_2t_6, t_3t_5\}$. The binomial $f = x_1x_3x_5 - x_2x_4x_6$ belonging to $I_\mathcal{A}$ is a circuit and indispensable. However, f is not fundamental.

(b) Let $d = 8$, $n = 10$ and $\mathcal{A} = \{t_1t_2, t_2t_3, t_1t_3, t_3t_4, t_4t_6, t_3t_5, t_5t_6, t_6t_7, t_7t_8, t_6t_8\}$. The binomial $f = x_1x_4x_6x_8x_{10} - x_2x_3x_5x_7x_9$ belonging to $I_\mathcal{A}$ is indispensable. However, f is not a circuit.

(c) Let $d = 4$, $n = 6$ and $\mathcal{A} = \{t_1t_2, t_1t_3, t_1t_4, t_2t_3, t_2t_4, t_3t_4\}$. The binomial $f = x_1x_6 - x_2x_5$ belonging to $I_\mathcal{A}$ is a circuit. However, f is not indispensable.

2. Reduced Gröbner bases

We refer the reader to, e.g., [1] and [6] for fundamental materials on Gröbner bases. Work with the same notation $K[\mathbf{t}] = K[t_1, \ldots, t_d]$, $\mathcal{A} = \{f_1, \ldots, f_n\}$, $K[\mathcal{A}]$, $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ and $I_\mathcal{A}$ as in the previous section.

Given a monomial order $<$ on $K[\mathbf{x}]$, we write $\mathcal{G}_<(I_\mathcal{A})$ for the reduced Gröbner basis of $I_\mathcal{A}$ with respect to $<$. Since $\mathcal{G}_<(I_\mathcal{A})$ is a system of binomial generators of $I_\mathcal{A}$, it follows that if a binomial f belonging to $I_\mathcal{A}$ is indispensable, then either f or $-f$ belongs to $\mathcal{G}_<(I_\mathcal{A})$. More precisely, the indispensable binomial of $I_\mathcal{A}$ can be characterized in terms of the reduced Gröbner bases of $I_\mathcal{A}$. In fact,

Theorem 2.1. *Let f be a binomial belonging to $I_\mathcal{A}$. Then the following conditions are equivalent.*

(i) *f is indispensable*;

(ii) *For any lexicographic order $<_{\text{lex}}$ on $K[\mathbf{x}]$, either f or $-f$ belongs to $\mathcal{G}_{<_{\text{lex}}}(I_\mathcal{A})$*;

(iii) *For any reverse lexicographic order $<_{\text{revlex}}$ on $K[\mathbf{x}]$, either f or $-f$ belongs to $\mathcal{G}_{<_{\text{revlex}}}(I_\mathcal{A})$*;

(iv) *For any monomial order $<$ on $K[\mathbf{x}]$, either f or $-f$ belongs to $\mathcal{G}_<(I_\mathcal{A})$.*

Proof. We already discussed (i) $\Rightarrow$ (iv). Each of (iv) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) is obvious. We will show that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).

((ii) $\Rightarrow$ (i)) Given a binomial $f = u_0 - v_0$ belonging to $I_\mathcal{A}$ which is *not* indispensable, we choose a system of binomial generators $\mathcal{F}$ of $I_\mathcal{A}$ such that neither f nor $-f$ belongs to $\mathcal{F}$. Since $f \in I_\mathcal{A}$, for a monomial w of $K[\mathbf{x}]$ with $w \neq 0$ and for a binomial $g \in \mathcal{F}$, one has $wg = u_0 - v$. Let $h = wg - f = v_0 - v \in I_\mathcal{A}$.

If $h = 0$, then $f = wg$. Since $f \neq g$, one has $w \neq 1$. Thus f is reducible. Hence f can belong to no reduced Gröbner basis of $I_\mathcal{A}$.

Let $h \neq 0$. If h is reducible, then there is an irreducible binomial $h' = u' - v' \in I_\mathcal{A}$ with $u' \neq v_0$ such that u' divides v_0. Let $<_{\text{lex}}$ denote a lexicographic order on $K[\mathbf{x}]$ such that $\text{in}_{<_{\text{lex}}}(h') = u'$. Here $\text{in}_{<_{\text{lex}}}(h')$ is the initial monomial of h' with respect to $<_{\text{lex}}$. Hence neither f nor $-f$ belongs to $\mathcal{G}_{<_{\text{lex}}}(I_\mathcal{A})$.

Similarly, if wg is reducible, then $f \notin \mathcal{G}_{<'_{\text{lex}}}(I_\mathcal{A})$ and $-f \notin \mathcal{G}_{<'_{\text{lex}}}(I_\mathcal{A})$ for some lexicographic order $<'_{\text{lex}}$ on $K[\mathbf{x}]$.

Now, suppose that both binomials wg and h are irreducible. In particular $w = 1$. Let $x_i \in \text{supp}(u_0)$ and $x_j \in \text{supp}(v_0)$. Let $<''_{\text{lex}}$ denote a lexicographic order on $K[\mathbf{x}]$ such that each of x_i and x_j is bigger than x_k for all k with $k \neq i$ and $k \neq j$. Since $x_i \notin \text{supp}(v)$ and $x_j \notin \text{supp}(v)$, one has $\text{in}_{<''_{\text{lex}}}(g) = u_0$ and $\text{in}_{<''_{\text{lex}}}(h) = v_0$. Thus $f \notin \mathcal{G}_{<''_{\text{lex}}}(I_{\mathcal{A}})$ and $-f \notin \mathcal{G}_{<''_{\text{lex}}}(I_{\mathcal{A}})$, as desired.

((iii) $\Rightarrow$ (i)) In the above proof of (ii) $\Rightarrow$ (i), each of the lexicographic orders $<_{\text{lex}}$, $<'_{\text{lex}}$ and $<''_{\text{lex}}$ can be chosen as a reverse lexicographic order. Thus if f is not indispensable, then there is a reverse lexicographic order $<_{\text{revlex}}$ on $K[\mathbf{x}]$ such that $f \notin \mathcal{G}_{<_{\text{revlex}}}(I_{\mathcal{A}})$ and $-f \notin \mathcal{G}_{<_{\text{revlex}}}(I_{\mathcal{A}})$. □

Corollary 2.2. *Let $f_j = t_1^{a_{1j}} \cdots t_d^{a_{dj}}$ and set $\mathbf{a}_j = (a_{1j}, \ldots, a_{dj}) \in \mathbb{Z}^d$. If there exists an indispensable binomial $u - v \in I_{\mathcal{A}}$ such that neither u nor v is squarefree, then there exists no regular unimodular triangulation* [6, Chapter 8] *of the convex hull of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.*

The converse of Corollary 2.2 is false in general. See Example 4.5.

Example 2.3. Let $K[x]$ denote the polynomial ring in one variable over a field K and $I = (x - 1) = (x^2 - x, x^2 - 1)$. Since $\{x - 1\}$ is the only reduced Gröbner basis of I and since $\{x^2 - x, x^2 - 1\}$ is a minimal system of generators of I, Theorem 2.1 cannot be generalized to, e.g., lattice ideals in the obvious way.

3. Weakly chordal graphs

When is a toric ideal generated by the indispensable binomials? Following [5] we study toric ideals arising from finite graphs.

Let G be a finite connected graph on the vertex set $[d] = \{1, \ldots, d\}$ with no loop and no multiple edge and $E(G) = \{e_1, \ldots, e_n\}$ the set of edges of G. Let, as before, $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ denote the polynomial ring in d variables over a field K with each $\deg t_i = 1$. We will associate each edge $e = \{i, j\} \in E(G)$ with the squarefree quadratic monomial $\mathbf{t}^e = t_i t_j$ belonging to $K[\mathbf{t}]$. Let $\mathcal{A}_G = \{\mathbf{t}^{e_1}, \ldots, \mathbf{t}^{e_n}\}$ and $K[G] = K[\mathcal{A}_G]$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K with each $\deg x_i = 1$ and write I_G ($\subset K[\mathbf{x}]$) for the toric ideal of $K[G]$. For notation and terminologies we use in the present section, we follow [5]. In particular if Γ is an even close walk of G, then $f_\Gamma = f_\Gamma^{(+)} - f_\Gamma^{(-)}$ is the binomial [5, p. 512] coming from Γ. One has $f_\Gamma \in I_G$. Moreover, [5, Lamma 1.1] guarantees that I_G is generated by all the binomials f_Γ, where Γ is an even closed walk of G.

The following Example 3.1 is essentially discussed in [5].

Example 3.1. (a) Let G be a finite bipartite graph. For a binomial f belonging to I_G, the following conditions are equivalent: (i) f is indispensable; (ii) f is fundamental; (iii) $f = f_C$ for an even cycle C with no chord.

(b) The toric ideal I_G of a finite bipartite graph G is generated by the fundamental binomials of I_G.

In the study of indispensable binomials of the toric ideal I_G arising from a finite graph G, the most fundamental question is when the binomial f_C coming from an even cycle C is an indispensable binomial of I_G.

Let G be a finite connected graph and $C = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{2q}, v_1\})$ an even cycle of G of length $2q$. A chord $e = \{v_i, v_j\}$ with $1 \leq i < j \leq 2q$ is an *even chord* (resp. *odd chord*) if $j - i$ is odd (resp. even). When $e = \{v_i, v_j\}$ and $e' = \{v_{i'}, v_{j'}\}$ are odd chords of C with $1 \leq i < j \leq 2q$ and $1 \leq i' < j' \leq 2q$, we say that e and e' *cross effectively* in C if either $i < i' < j < j'$ or $i' < i < j' < j$ and if $i' - i$ is odd. (Since each of $j - i$ and $j' - i'$ is even, it follows that each of $j - i'$, $j' - j$ and $j' - i$ is odd.)

Theorem 3.2. *Let G be a finite connected graph and I_G its toric ideal. Given an even cycle C of G of length ≥ 4, the binomial f_C belonging to I_G is indispensable if and only if* (i) *C has no even chord and* (ii) *C has no odd chords e and e' which cross effectively in C.*

Proof. Let C be a cycle of length 4 with the vertex set $\{v_1, v_2, v_3, v_4\}$. It follows easily that the binomial f_C is indispensable if and only if the induced subgraph of G on $\{v_1, v_2, v_3, v_4\}$ is not a complete graph of order 4. In other words, f_C is indispensable if and only if C satisfies the conditions (i) and (ii), as required.

Let C be an even cycle of length ≥ 6. We will show that the binomial f_C is indispensable if and only if C satisfies the conditions (i) C has no even chord and (ii) C has no odd chords e and e' which cross effectively in C.

(First Step) (a) If an even cycle C has an even chord, then (First Step) (a) of [5, p. 519] says that there are even cycles C' and C'' and monomials w and w' with $w \neq 1$ and $w' \neq 1$ such that $f_C = w f_{C'} - w' f_{C''}$. Hence f_C is not indispensable.

(b) Let C be an even cycle of G with no even chord and suppose that C has two odd chords e and e' which cross effectively in C. Let, say, $C = (\{1, 2\}, \{2, 3\}, \ldots, \{2q, 1\})$ and $e = \{1, 2i+1\}$ and $e' = \{2j, 2k\}$ with $1 < 2j < 2i+1 < 2k < 2q$. Let $q \geq 6$. Let C' be the even cycle $(e, \{2i, 2i+1\} \ldots, \{2j, 2j+1\}, e', \{2k, 2k+1\}, \ldots, \{2q, 1\})$ and C'' the even cycle $(e, \{2i+1, 2i+2\}, \ldots, \{2k-1, 2k\}, e', \{2j-1+1, 2j\}, \ldots, \{1, 2\})$. By using the same technique appearing in (First Step) (b) of [5, p. 519], there are monomials w and w' such that $f_C = w f_{C'} - w' f_{C''}$. Hence f_C is not indispensable.

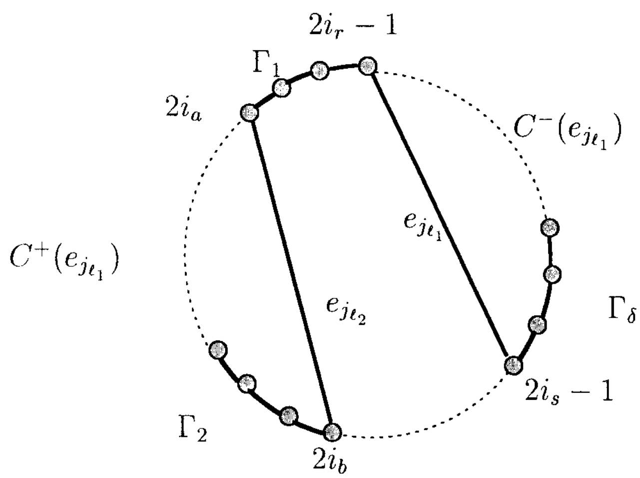
(Second Step) Let C be an even cycle of G of length $2q \geq 6$ with no even chord and with no odd chords e and e' which cross effectively in C, and suppose that the binomial f_C is not indispensable. Let, say, $C = (\{1, 2\}, \{2, 3\}, \ldots, \{2q, 1\})$ and $e_1 = \{1, 2\}, e_2 = \{2, 3\}, \ldots, e_{2q} = \{2q, 1\}$. It follows from [5, Lemma 3.1] that there exists an even closed walk Γ ($\neq C$) of G such that the binomial f_Γ is primitive and $f_\Gamma^{(+)}$ divides $f_C^{(+)}$. Let $f_\Gamma^{(+)} = x_{2i_1-1} x_{2i_2-1} \cdots x_{2i_p-1}$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq q$, and $f_\Gamma^{(-)} = x_{j_1} x_{j_2} \cdots x_{j_p}$. Since $f_\Gamma^{(+)}$ is squarefree and since e_{2i_s+1} and e_{2i_t+1} with $s \neq t$ possess no common vertex, it follows easily from [5, Lemma 3.2] that Γ must be an even cycle. Since $\Gamma \neq C$, one of the edges $e_{j_1}, e_{j_2}, \ldots, e_{j_p}$ must be a chord of C. Moreover, since C possesses no even chord, such a chord must be an odd chord.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_\delta$ denote the "connected components" of $C \cap \Gamma$ and write

$$\Gamma = (e_{j_{\ell_1}}, \Gamma_1, e_{j_{\ell_2}}, \Gamma_2, e_{j_{\ell_3}}, \ldots, e_{j_{\ell_\delta}}, \Gamma_\delta),$$

where $e_{j_{\ell_1}}, e_{j_{\ell_2}}, \ldots, e_{j_{\ell_\delta}}$ are odd chords of C.

Each chord e divides C into $C^+(e)$ and $C^-(e)$ in the obvious way. We will assume that the region surrounded by $C^+(e_{j_{\ell_1}}) \cup e_{j_{\ell_1}}$ contains no chord $e_{j_{\ell_k}}$ with $2 \leq k \leq \delta$. Let, say, $e_{j_{\ell_1}} = \{2i_r - 1, 2i_s - 1\}$ with $r < s$, and $C^+(e_{j_{\ell_1}}) = (e_{2i_r-1}, e_{2i_r}, \ldots, e_{2i_s-2})$. Let $e_{j_{\ell_2}} = \{2i_a, 2i_b\}$. Thus $\Gamma_1 = (e_{2i_r-1}, e_{2i_r}, \ldots, e_{2i_a-1})$. Since the odd chords $e_{j_{\ell_1}}$ and $e_{j_{\ell_2}}$ cannot cross effectively in C, one has $2i_a < 2i_b < 2i_s - 1$. This contradicts our assumption that the region surrounded by $C^+(e_{j_{\ell_1}}) \cup e_{j_{\ell_1}}$ contains no chord $e_{j_{\ell_k}}$ with $2 \leq k \leq \delta$. □



A finite connected graph G is called *weakly chordal* if every cycle of G of length 4 has a chord. We will study the toric ideal of the complementary graph of a weakly chordal graph. It follows easily that the complementary graph of a finite connected graph G is weakly chordal if and only if the following condition is satisfied: $(*)$ If e and e' are edges of G with $e \cap e' = \emptyset$, then there is an edge e'' of G with $e \cap e'' \neq \emptyset$ and with $e' \cap e'' \neq \emptyset$.

Lemma 3.3. *Let G be a finite connected graph whose complementary graph is weakly chordal and suppose that no complete graph of order ≥ 4 is a subgraph of G. Then the toric ideal I_G is generated by the binomials f_C, where C is a cycle of G of even length ≥ 4 satisfying the conditions* (i) *and* (ii) *of Theorem 3.2*

Proof. It follows from (Second Step) (a) and (Third Step) of [5, pp. 520 – 521] that I_G is generated by the binomials f_C, where C is an even cycle. Let $(I_G)_{<q}$ denote the binomial ideal of $K[\mathbf{x}]$ which is generated by the binomials f_C, where C is an even cycle of G of length $< 2q$. Since no complete graph of order ≥ 4 is a subgraph of G, every cycle of G of length 4 satisfies the conditions (i) and (ii) of Theorem 3.2.

If C is an even cycle of length $2q \geq 6$ with an even chord, then (First Step) (a) of the proof of Theorem 3.2 says that $f_C \in (I_G)_{<q}$.

Let C be an even cycle of G with no even chord and suppose that C has two odd chords e and e' which cross effectively in C. Work with the same situation as in (First Step) (b) of the proof of Theorem 3.2. If each of C' and C'' is of length $< 2q$, then $f_C \in (I_G)_{<q}$. Let the length of C' be equal to $2q$. Thus $e = \{1, 2i+1\}$ and $e' = \{2, 2i+2\}$ with $1 < i < q$. Since $q \geq 3$, it follows that either $(2i+1) - 2 \geq 3$ or $(2q+1) - (2i+2) \geq 3$. Let $(2i+1) - 2 \geq 3$. Let $e_1 = \{3, 4\}$ and $e_2 = \{2i+2, 2i+3\}$. Since G satisfies the condition $(*)$, one has an edge e_3 with $e_3 \cap e_1 \neq \emptyset$ and $e_3 \cap e_2 \neq \emptyset$. Since C has no even chord, either $e_3 = \{3, 2i+3\}$ or $e_3 = \{4, 2i+2\}$. In each of the cases, the edge e_3 is an even chord of the cycle C' of length $2q$. Hence the binomial $f_{C'}$ belongs to $(I_G)_{<q}$. Since the cycle C'' is of length 4 and $q \geq 3$, it follows that $f_C = wf_{C'} - w'f_{C''}$ belongs to $(I_G)_{<q}$. □

We are now in the position to discuss the problem when the toric ideal I_G of a finite connected graph G whose complementary graph is weakly chordal is generated by indispensable binomials of I_G.

Theorem 3.4. *Let G be a finite connected graph whose complementary graph is weakly chordal. Then the toric ideal I_G is generated by the indispensable binomials of I_G if and only if no complete graph of order ≥ 4 is a subgraph of G.*

Proof. (**"only if"**) Let G be an *arbitrary* finite connected graph and suppose that a complete graph of order ≥ 4 is a subgraph of G. In particular G possesses a complete subgraph K_4 of order 4. Let $1, 2, 3$ and 4 be the vertices of K_4 and $e_1, \ldots, e_6$ are edges of K_4. Thus K_4 is the induced subgraph of G on $\{1, 2, 3, 4\}$. Let C_1, C_2 and C_3 denote the cycles of length 4 of K_4. Then none of the binomials f_{C_1}, f_{C_2} and f_{C_3} is indispensable. However, since $I_{K_4} = I_G \cap K[x_1, \ldots, x_6]$, any system of binomial generators of I_G must contain at least two of three binomials f_{C_1}, f_{C_2} and f_{C_3}. Hence it is impossible for the toric ideal I_G to be generated by the indispensable binomials.

(**"if"**) Let G be a finite connected graph whose complementary graph is weakly chordal and suppose that no complete graph of order ≥ 4 is a subgraph of G. Lemma 3.3 says that I_G is generated by the binomials f_C, where C is an even cycle of length ≥ 4 satisfying the conditions (i) and (ii) of Theorem 3.2. Theorem 3.2 then guarantees that I_G is generated by the indispensable binomials, as required. □

Remark 3.5. Let G be the graph in [5, Example 2.1]. Then the complementary graph is a cycle of length 5 and hence weakly chordal. Although I_G is generated by indispensable binomials of degree 2, they are not a Gröbner basis with respect to any monomial order since I_G has no quadratic Gröbner basis.

4. The odd cycle condition

Let G be a finite connected graph on the vertex set $[d] = \{1, \ldots, d\}$ with no loop and no multiple edge and $E(G) = \{e_1, \ldots, e_n\}$ the set of edges of G. If C and C' are cycles of G, then a *bridge* between C and C' is an edge $e = \{i, j\}$ of G, where i (resp. j) is a vertex of C (resp. C') and not a vertex of C' (resp. C). We say that G satisfies the *odd cycle condition* [2] if, for any two odd cycles C and C' of G having no common vertex, there exists a bridge between C and C'. We study toric ideal I_G of a finite connected graph G satisfying the odd cycle condition.

A binomial $f \in I_G$ is called *redundant* if f belongs to no minimal system of binomial generators of I_G. In other words, a binomial $f \in I_G$ is redundant if and only if f is contained in the binomial ideal of $K[\mathbf{x}]$ which is generated by the binomials $g \in I_G$ with $\deg g < \deg f$.

When G satisfies the odd cycle condition, which binomials belonging to I_G are indispensable ?

Lemma 4.1. *Let G be a finite connected graph satisfying the odd cycle condition and $\Gamma = (C, C')$ an even closed walk of G, where $C = (\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{2p-1}, i_1\})$ and $C' = (\{j_1, j_2\}, \{j_2, j_3\}, \ldots, \{j_{2q-1}, j_1\})$ are odd cycles of G having exactly one common vertex $i_1 = j_1$.*

(a) If f_Γ be indispensable, then (i) either C or C' is minimal and there is no bridge between C and C'; (ii) if C' is minimal and if $e = \{i_k, i_{k'}\}$ is a chord of C with $1 < k < k' < 2p$, then either $k = 2$ or $k' = 2p-1$, and $k' - k$ is even; (iii) if C' is minimal and if $e = \{i_2, i_k\}$ and $e' = \{i_{k'}, i_{2p-1}\}$ are chords of C with $2 < k < 2p$ and $1 < k' < 2p-1$ such that k is even and k' is odd, then $k' - k = 1$.

(b) Conversely, if C' be minimal and if there is no bridge between C and C', then f_Γ is indispensable if the following conditions are satisfied: (i) if $e = \{i_k, i_{k'}\}$ is a chord of C with $1 < k < k' < 2p$, then either $k = 2$ or $k' = 2p-1$ and $k' - k$ is even; (ii) if $e = \{i_2, i_k\}$ and $e' = \{i_{k'}, i_{2p-1}\}$ are chords of C with $2 < k < 2p$ and $1 < k' < 2p-1$ such that k is even and k' is odd, then $k' - k = 1$.

Proof. (a) If there is a bride $e = \{i_k, j_\ell\}$ between C and C' with $k \neq 1$ and $\ell \neq 1$ or if there is either a chord $e = \{i_1, i_k\}$ of C with $k \neq 1$ or a chord $e' = \{j_1, j_\ell\}$ of C' with $\ell \neq 1$, then (Second Step) of [5, p. 520] guarantees that f_Γ is redundant.

If there is a chord $e = \{i_k, i_{k'}\}$ of C with $1 < k < k' < 2p$ such that $k' - k$ is odd, then f_Γ is redundant.

If there is a chord $e = \{i_k, i_{k'}\}$ of C with $2 < k < k' < 2p-1$ such that $k' - k$ is even, then the odd cycle condition yields a bride $e = \{i_{k''}, j_\ell\}$ between C and C' with $k \leq k'' \leq k'$ and $1 \leq \ell < 2q$. Hence f_Γ is redundant.

If there is a chord $e = \{i_2, i_k\}$ of C with $2 < k < 2p$ such that k is even and if there is a chord $e' = \{j_2, j_\ell\}$ of C' with $2 < \ell < 2q$ such that ℓ is even, then the odd cycle condition yields a bride $e = \{i_{k'}, j_{\ell'}\}$ between C and C' with $2 \leq k' \leq k$ and $2 \leq \ell' \leq \ell$. Hence f_Γ is redundant.

Consequently, it turns out that either C or C' is minimal and there is no bridge between C and C'. Moreover, in case that C' is minimal and $e = \{i_k, i_{k'}\}$ is a chord of C with $1 < k < k' < 2p$, either $k = 2$ or $k' = 2p-1$, and $k' - k$ is even.

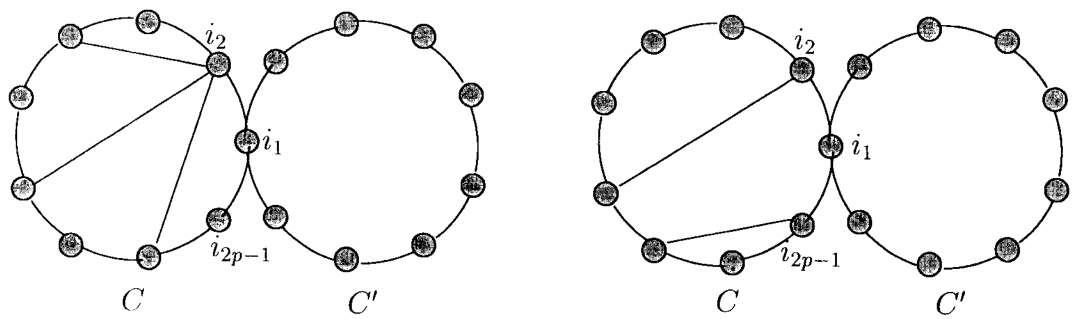
Let $e = \{i_2, i_k\}$ and $e' = \{i_{k'}, i_{2p-1}\}$ be chords of C with $2 < k < 2p$ and $1 < k' < 2p-1$ such that k is even and k' is odd. If $k < k'$ with $k' - k > 1$, then the odd cycle condition yields a chord $e = \{i_a, i_b\}$ of C with $2 \leq a \leq k$ and $k' \leq b \leq 2p-1$. Thus, as we have already discussed, f_Γ is redundant. If $k' < k$, then the same technique as in (First Step) (b) of the proof of Theorem 3.2 enables us to see that f_Γ is not indispensable.

(b) With assuming that the vertices i_1 and j_1 are different, instead of the even closed walk Γ, we may discuss the even cycle

$$(\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{2p-1}, j_1\}, \{j_1, j_2\}, \{j_2, j_3\}, \ldots, \{j_{2q-1}, i_1\}.$$

It is then clear that (Second Step) of the proof of Theorem 3.2 can be applied without modification. $\square$

For example, in each of the figures drawn below, the binomial f_Γ with $\Gamma = (C, C')$ is indispensable.



Lemma 4.2. *Let G be a finite connected graph satisfying the odd cycle condition and $\Gamma = (C, \Gamma_1, C', \Gamma_2)$ an even closed walk of G, where C and C' are odd cycles of G having no common vertex and where Γ_1 and Γ_2 are walks of G both of which combine a vertex i of C with a vertex j of C'. Then Γ is indispensable if and only if each of C and C' is minimal and has exactly one bridge $= \{i, j\}$ and $\Gamma_1 = \Gamma_2 = (\{i, j\})$.*

Proof. If each of C and C' is minimal and has exactly one bridge $= \{i, j\}$ and $\Gamma' = \Gamma'' = (\{i, j\})$, then Γ is an induced subgraph of G. Thus [5, Lemma 3.3] guarantees that f_Γ is fundamental.

Let $\Gamma = (C, \Gamma_1, C', \Gamma_2)$ be an even closed walk of G, where C and C' are odd cycles of G having no common vertex and where Γ_1 and Γ_2 are walks of G both of which combine a vertex i of C with a vertex j of C'. Since G satisfies the odd cycle condition, one has a bridge $e = \{i', j'\}$ between C and C'. Let either $i' \neq i$ or $j' \neq j$. Then (Third Step) of [5, p. 521] guarantees that f_Γ is redundant. Let $e = \{i, j\}$. Since the length of Γ is even, it follows that either (i) both Γ_1 and Γ_2 are of even length or (ii) both Γ_1 and Γ_2 are of odd length.

Let both Γ_1 and Γ_2 be of even length. By using the even closed walks $\Gamma_3 = (C, e, \Gamma_1)$ and $\Gamma_4 = (C', e, \Gamma_2)$, one has $f_\Gamma \in (f_{\Gamma_3}, f_{\Gamma_4})$. Thus f_Γ is redundant.

Let both Γ_1 and Γ_2 are of odd length with $\Gamma_2 \neq (e)$. By using the even closed walk $\Gamma_5 = (C, e, C', \Gamma_1)$ and the even cycle $C'' = (e, \Gamma_2)$, one has $f_\Gamma \in (f_{\Gamma_5}, f_C)$. Thus f_Γ is redundant.

Consequently, it follows that $\Gamma = (C, e, C', e)$ with $e = \{i, j\}$. Finally, if C has a chord $e' = \{a, b\}$, then either (i) there is a bridge $e'' = \{i', j'\}$ between C and C' with either $i' \neq i$ or $j' \neq j$ or (ii) there is an even cycle $C'' = (\Gamma, e'')$ with $\Gamma \subset C$. In each of the cases (i) and (ii) the binomial f_Γ is redundant. □

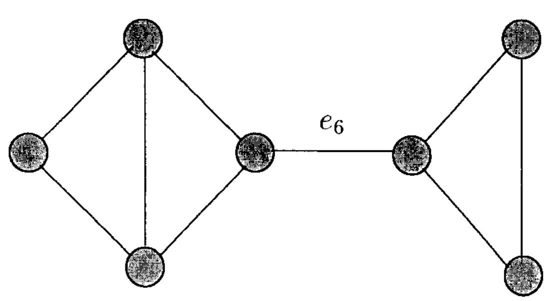
Theorem 3.2 together with Lemmata 4.1 and 4.2 gives the complete classification of indispensable binomials of the toric ideal I_G of a finite connected graph satisfying the odd cycle condition.

Theorem 4.3. *Let G be a finite connected graph satisfying the odd cycle condition. Then the indispensable binomials f_Γ, where Γ is an even closed walk of G, belonging to the toric ideal I_G can be classified as follows:*

- (α) *Γ is an even cycle C of G such that (i) C has no even chord and that (ii) C has no odd chords e and e' which cross effectively in C;*
- (β) *$\Gamma = (C, C')$ is an even closed walk of G, where C and C' are odd cycles of G having exactly one common vertex i and where either C or C' (say, C') is minimal, such that $C = (\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{2p-1}, i_1\})$ with $i = i_1$ possesses the properties (i) if $e = \{i_k, i_{k'}\}$ is a chord of C with $1 < k < k' < 2p$, then either $k = 2$ or $k' = 2p - 1$ and $k' - k$ is even, and (ii) if $e = \{i_2, i_k\}$ and $e' = \{i_{k'}, i_{2p-1}\}$ are chords of C with $2 < k < 2p$ and $1 < k' < 2p - 1$ such that k is even and k' is odd, then $k' - k = 1$;*
- (γ) *$\Gamma = (C, e, C', e)$ is an even closed walk of G, where C and C' are minimal odd cycles of G having no common vertex and exactly one bridge e between C and C'.*

Theorem 4.3 says that if a finite connected graph G satisfies the odd cycle condition, then for each indispensable binomial $f = u - v$ of I_G, either u or v is squarefree. However, the converse is false in general, even though I_G is generated by the indispensable binomials. For example,

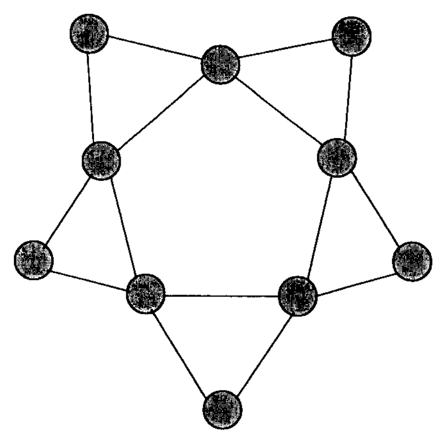
Example 4.4. Let G be the finite graph drawn below. Then G does not satisfy the odd cycle condition. However I_G is generated by the indispensable binomials $f_{\Gamma_1} = x_1x_3 - x_2x_4$ and $f_{\Gamma_2} = x_1x_4x_7x_9 - x_5x_6^2x_8$ where Γ_1 is a cycle of length 4 and $\Gamma_2 = (C, e_6, C', e_6)$ where C and C' are odd cycles of length 3 having no common vertex and exactly one bridge e_6 between C and C'.



We now close the present section with a counter example for the converse of Corollary 2.2.

Example 4.5. Let G be the finite graph drawn below. Then G satisfies the odd cycle condition. Thanks to Theorem 4.3, for each indispensable binomial $f = u - v$ of I_G, either u or v is squarefree. (In this case, I_G is generated by indispensable binomials.) Moreover, in [4], it turns out that there exists no regular unimodular triangulation of the convex hull of the configuration arising from G.

There are 26 indispensable binomials in I_G: 16 even cycles satisfying (α) in Theorem 4.3 (i.e., five even cycles of length 6 with one odd chord, ten even cycles of length 8 with three odd chords, an even cycle of length 10 with five odd chords), 5 even closed walk satisfying (β) in Theorem 4.3 (all of them consist of two odd cycles of length 3 with one common vertex), and 5 even closed walk satisfying (γ) in Theorem 4.3 (all of them consist of two odd cycles of length 3 without common vertex joined by a bridge).



5. The existence of indispensable binomials

We conclude this paper with discussing the existence of indispensable binomials of toric ideals arising from finite graphs. Let, as before, G be a finite connected graph on the vertex set $[d]$ with no loop and no multiple edge and $E(G) = \{e_1, \ldots, e_n\}$ the set of edges of G. Recall that the *degree* of a vertex i of G is a number of edges e of G with $i \in e$.

Proposition 5.1. (a) *Let G be a complete graph. Then I_G has no indispensable binomial. In particular I_G has no fundamental binomials.*

(b) *If G has a vertex i of degree 1 and if G' is the induced subgraph of G on $[d] \setminus \{i\}$, then I_G has an indispensable binomial (resp. a fundamental binomial) if and only if $I_{G'}$ has an indispensable binomial (resp. a fundamental binomial).*

(c) *Let G be a finite connected graph with no vertex of degree 1. Then $I_G = (0)$ if and only if G is an odd cycle.*

Proof. (a) It follows from [6, Theorem 9.1] that I_G is generated by the quadratic binomials. As we have seen, a quadratic binomial coming from a complete graph of order 4 is not indispensable.

(b) We only note that each binomial belonging to a minimal system of binomial generators of I_G does not contain the variable x_j where $i \in e_j$.

(c) This follows from [3, Lemma 1.4] together with Example 3.1. □

Jürgen Herzog asked the authors if $I_G(\neq (0))$ has at least one indispensable binomial when G is not a complete graph. By virtue of Proposition 5.1 (b), when we discuss the existence of indispensable binomials (or fundamental binomials), we may assume that G has no vertex of degree 1.

Lemma 5.2. *If C is an even cycle, then the binomial f_C is fundamental if and only if C has no even chord and has at most one odd chord.*

Proof. Let G' denote the induced subgraph of G on the vertex set of C. When C has an even chord e (resp. two odd chords e' and e''), then there is an even cycle C' of G' with e (resp. e' and e'') its edge. Thus f_C is not fundamental. On the other hand, when C has at most one odd chord and no even chord, the induced subgraph G' can contain neither an even cycle C' with $C' \neq C$ nor two odd cycles with at most one common vertex. Hence G' can contain no primitive even closed walk [5, p. 516]. Thus f_C is fundamental. □

We now come to the main result of this section.

Theorem 5.3. *Let G be a finite connected graph with $I_G \neq (0)$ and suppose that G has no vertex of degree 1. Then the following conditions are equivalent:*

(i) *G is not a complete graph;*

(ii) *I_G possesses a fundamental binomial;*

(iii) *I_G possesses an indispensable binomial.*

Proof. First of all, (ii) $\Rightarrow$ (iii) is obvious. Moreover, (iii) $\Rightarrow$ (i) follows from Proposition 5.1 (a). In order to prove (i) $\Rightarrow$ (ii), suppose that G is not a complete graph and that I_G possesses no fundamental binomial. Example 3.1 says that it can be assumed that G is not bipartite, i.e., G has an odd cycle.

(First Step) It will be proved that, for any even cycle C of G, the induced subgraph of G on the vertex set of C is a complete graph. By Lemma 5.2, it holds in case that the length of C is 4.

Let $\ell \geq 6$ be the length of C and let G' be the induced subgraph of G on the vertex set of C. Since the binomial f_C is not fundamental, Lemma 5.2 guarantees that C has either (i) an even chord or (ii) two odd chords.

When C has two odd chords e and e', then there is an even cycle C' of G' with e and e' its edges whose length is less than that of C. By using the induction on ℓ, the induced subgraph G'' of G' on the vertex set of C' is a complete graph.

It turns out that the cycle C has an even chord $e = \{i, j\}$, which divides C into two even cycles $C^+(e)$ and $C^-(e)$ in the obvious way. Using the induction on ℓ again, the induced subgraph of G' on the vertex set of $C^+(e)$ (resp. $C^-(e)$) is a complete graph. Then each of the vertices i and j is incident to all vertices of C. Let $i'(\neq i,j)$ be a vertex of $C^+(e)$ and let $j'(\neq i,j)$ be a vertex of $C^-(e)$. Since there exists an even cycle $\{i,i',j,j'\}$ of length 4, there exists an edge $\{i',j'\}$ of G. Thus it follows that the induced subgraph of G on the vertex set of C is a complete graph.

(Second Step) The case that G has no odd cycle of length 3 is now discussed. It follows from (First Step) that there exists no even cycle of G and that all odd cycles of G is minimal. Since G is not bipartite, there is an odd cycle C' of G which is minimal. Proposition 5.1 (c) says that there exists another minimal odd cycle C'' of G. Since there exists no even cycle of G, two cycles C' and C'' have at most one common vertex. If C' and C'' have exactly one common vertex, then there exists no bridge between C' and C''. (Otherwise, G has an even cycle.) Thus the graph $\Gamma = C' \cup C''$ is the induced subgraph of G and hence f_Γ is fundamental. A contradiction arises. Hence C' and C'' have no common vertex. Since G is connected, there exists a walk Γ' combining a vertex of C' with a vertex of C''. We assume that the length of Γ' is minimal among all the walks combining a vertex of C' with a vertex of C''. It then follows that the graph $\Gamma = C' \cup \Gamma' \cup C''$ is the induced subgraph of G and f_Γ is fundamental. Again, a contradiction arises.

(Third Step) Finally, the case that there exists an odd cycle C of length 3 is discussed. Let G' be a maximal complete subgraph of G which contains C. Since G is not a complete graph, one has $G \neq G'$. Since G has no vertex of degree 1, one can find a cycle C' of G which is not contained in G'. It follows from (First Step) that if C' is an even cycle, then the induced subgraph of G on the vertex set of C' is a complete graph and, therefore, the induced subgraph on the vertex set of C' has an odd cycle. Thus we may assume that the length of C' is odd and C' is minimal.

Case I: Suppose that G' and C' have at least two common vertices. Then there exists a walk $\Gamma = (\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_{r-1},i_r\})$ such that each of the vertices i_1 and i_r is a vertex of G', each i_k with $k \notin \{1,r\}$ is not a vertex of G', and Γ is a subwalk of C'. If r is even, then $(\Gamma,\{i_r,i_1\})$ is an even cycle and hence the induced subgraph of G on the vertex set of Γ is a complete graph. This contradicts that C' is minimal. Hence r is odd. Note that G' has at least 3 vertices. For each vertex $j(\neq i_1,i_r)$ of G, $(\Gamma,\{i_r,j\},\{j,i_1\})$ is an even cycle of G. Hence the induced subgraph of G on the vertex set of $G' \cup \Gamma$ is a complete graph. This contradicts that G' is a maximal complete subgraph of G.

Case II: Suppose that G' and C' have exactly one common vertex i. Let C'' be an odd cycle of G' of length 3 which contains i. If there exists a bridge e between C' and C'', then there exists an odd cycle of G which is not contained in G' and has two common vertices with G'. This is a contradiction. Hence there exists no bridge between C' and C''. Thus the graph $\Gamma = C' \cup C''$ is the induced subgraph of G and hence f_Γ is fundamental. A contradiction arises.

Case III: Suppose that G' and C' have no common vertex. Since G is connected, there exists a walk Γ' combining a vertex of G' with a vertex of C'. We assume that the pair (Γ', C') satisfies that the length of the walk Γ' is minimal among all the pairs (Γ', C') such that C' is a minimal odd cycle of G containing no vertex of G' and Γ' is a walk combining a vertex of G' with a vertex of C'. Let C'' be an odd cycle of G' of length 3 which contains a vertex of Γ'. It is not difficult to see that the graph $\Gamma = C' \cup \Gamma' \cup C''$ is the induced subgraph of G. Thus f_Γ is fundamental. Again, a contradiction arises. □

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