

A Lower Bound Theorem for Ehrhart Polynomials of Convex Polytopes

TAKAYUKI HIRI

Department of Mathematics, Faculty of Science,
Hokkaido University, Kita-ku, Sapporo 060, Japan

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope; i.e., a convex polytope any of whose vertices has integer coordinates of dimension d , and let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer n we write $i(\mathcal{P}, n)$ for the number of those rational points $(\alpha_1, \alpha_2, \dots, \alpha_N)$ in $\mathcal{P}$ such that each $n\alpha_i$ is an integer. In other words,

$$i(\mathcal{P}, n) = \#(n\mathcal{P} \cap \mathbb{Z}^N).$$

Here $n\mathcal{P} := \{n\alpha; \alpha \in \mathcal{P}\}$ and $\#(X)$ is the cardinality of a finite set X . The systematic study of $i(\mathcal{P}, n)$ originated in the work of Ehrhart (cf. [Ehr]), who established that the function $i(\mathcal{P}, n)$ possesses the following fundamental properties:

(0.1) $i(\mathcal{P}, n)$ is a polynomial in n of degree d . (Thus $i(\mathcal{P}, n)$ can be defined for every integer n .)

$$(0.2) \quad i(\mathcal{P}, 0) = 1.$$

(0.3) ("loi de réciprocité") $(-1)^d i(\mathcal{P}, -n) = \#(n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N)$ for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of $\mathcal{P}$. See, e.g., [Sta₃, pp. 235-241; and H₅] for an introduction to Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \dots$ of integers by the formula

$$(1 - \lambda)^{d+1} \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n) \lambda^n \right] = \sum_{i=0}^{\infty} \delta_i \lambda^i. \quad (1)$$

Then, the basic facts (0.1) and (0.2) on $i(\mathcal{P}, n)$ together with a fundamental result on generating functions, e.g., [Sta₃, Corollary 4.3.1], guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence $\delta(\mathcal{P}) := (\delta_0, \delta_1, \dots, \delta_d)$ which appears in Eq. (1) is the δ -vector of $\mathcal{P}$. Thus $\delta_0 = 1$ and $\delta_1 = \#(\mathcal{P} \cap \mathbb{Z}^N) - (d+1)$. On the other hand, it follows easily from (0.3) that $\delta_d = \#((\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N)$. Moreover, each δ_i is non-negative [Sta₁].

Now, our result in this paper is

(1.1) THEOREM. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ and suppose that $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N$ is non-empty; i.e., $\delta_d \neq 0$. Then we have the inequality $\delta_1 \leq \delta_i$ for every $1 \leq i < d$.

The proof of Theorem 1.1 relies on the following two well-known facts. See, e.g., [Sta₂] for fundamental definitions and results concerning f -vectors and h -vectors of triangulations of balls and spheres.

(1.2) LEMMA (cf. [B-M]). Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$. Also, let Δ be a triangulation of $\mathcal{P}$ with the vertex set $\mathcal{P} \cap \mathbb{Z}^N$ whose h -vector is $h(\Delta) = (h_0, h_1, \dots, h_d, h_{d+1})$. (Thus, $h_1 = \delta_1, h_d = \delta_d$ [Sta₂, pp. 80-81] and $h_{d+1} = 0$ [Sta₂, p. 67].) Then $h_i \leq \delta_i$ for every $0 \leq i \leq d$.

(1.3) LEMMA ([Bar₁, Bar₂]). Let $h(\Delta) = (h_0, h_1, \dots, h_d)$ be the h -vector of a triangulation of the boundary $\partial\mathcal{P}$ of a convex polytope $\mathcal{P}$ of dimension d . (Thus, in particular, $h_i = h_{d-i}$ for each $0 \leq i \leq d$ [Sta₂, p. 77].) Then $h_i \leq h_i$ for every $1 \leq i < d$.

We are now in the position to give a proof of Theorem 1.1. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and suppose that $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N$ is non-empty, say $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N = \{v_1, v_2, \dots, v_l\}$. First, we take any triangulation $\Delta(0)$ of the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ with the vertex set $\partial\mathcal{P} \cap \mathbb{Z}^N$, and then we construct a triangulation $\Delta(j)$ of $\mathcal{P}$ with the vertex set $(\partial\mathcal{P} \cap \mathbb{Z}^N) \cup \{v_1, \dots, v_j\}$ for each $1 \leq j \leq l$ in the following way:

- (i) Define $\Delta(1)$ to be the triangulation of $\mathcal{P}$ which consists of those simplices $\sigma \subset \mathcal{P}$ such that σ is the convex hull of $\tau \cup \{v_1\}$ in $\mathbb{R}^N$ for some $\tau \in \Delta(0)$; i.e., $\Delta(1)$ is the cone over $\Delta(0)$ with apex v_1 .
- (ii) If $\Delta(j)$ is constructed and $1 \leq j < l$, then let $\tau(j) \in \Delta(j)$ be the smallest face which contains v_{j+1} and write τ' for the subdivision of $\tau(j)$ which is the cone over the boundary $\partial\tau(j)$ of $\tau(j)$ with apex v_{j+1} . Also, let $\text{link}_{\Delta(j)}(\tau(j))$ (resp., $\text{star}_{\Delta(j)}(\tau(j))$) be the link [Sta₂, p. 70] (resp., star [Sta₂, p. 72]) of $\tau(j)$ in $\Delta(j)$. We then define $\Delta(j+1)$ to be the triangulation of $\mathcal{P}$ which consists of those simplices $\sigma \subset \mathcal{P}$ such that σ is either (a) the convex hull of $\zeta \cup \xi$ in $\mathbb{R}^N$ for some $\zeta \in \tau'$ and $\xi \in \text{link}_{\Delta(j)}(\tau(j))$ or (b) $\sigma \in \Delta(j) - \text{star}_{\Delta(j)}(\tau(j))$.

We now investigate the relation between the h -vector of $\Delta(j)$ and that of $\Delta(j+1)$ for each $0 \leq j < l$. We write $(h_0^{(j)}, h_1^{(j)}, h_2^{(j)}, \dots)$ for the h -vector $h(\Delta(j))$ of $\Delta(j)$. Since $\Delta(1)$ is the cone over $\Delta(0)$ with apex v_1 , we know that $h_1^{(0)} = h_1^{(1)}$ for each $0 \leq i \leq d$. Thus, by virtue of Lemma 1.3, we have

$h_i^{(1)} \leq h_i^{(j)}$ for every $1 \leq i < d$. Now, let $j \geq 1$ and suppose that $h_i^{(j)} \leq h_i^{(j+1)}$ for every $1 \leq i < d$. Let v be a vertex of $\tau(j)$ and set $\Delta^* = \text{star}_{\Delta(j+1)}(\{v_{j+1}\})$. Then we easily see

$$f_i(\Delta(j+1)) = f_i(\Delta(j)) + f_{i-1}(\text{link}_{\Delta^*}(\{v\})) \quad (2)$$

for every $1 \leq i \leq d$. Here, e.g., $(f_0(\Delta(j)), f_1(\Delta(j)), f_2(\Delta(j)), \dots)$ is the f -vector of Δ_j ; i.e., $f_i(\Delta(j))$ is the number of i -dimensional simplices of $\Delta(j)$. On the other hand, we set

$$\rho := \text{link}_{\Delta(j+1)}(\{v\}) \cap \text{link}_{\Delta(j+1)}(\{v_{j+1}\}).$$

Then ρ is a triangulation of a $(d-2)$ -sphere and $\text{link}_{\Delta^*}(\{v\})$ is the cone over ρ with apex v_{j+1} . Let $h(\rho) = (h_0, h_1, \dots, h_{d-1})$ be the h -vector of ρ . It follows from Eq. (2) that

$$h_i^{(j+1)} = h_i^{(j)} + h_{i-1}$$

for each $1 \leq i \leq d$. Hence $h_i^{(j+1)} \leq h_i^{(j+1)}$ for every $1 \leq i < d$ because of each $h_i \geq h_0$ ($=1$).

Thanks to Lemma 1.2, the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of $\mathcal{P}$ satisfies $h_i(l) \leq \delta_i$ for every $0 \leq i \leq d$. On the other hand, we know $h_i^{(j)} \geq h_1^{(j)} (= \delta_1)$ for each $1 \leq i < d$, thus we have the inequality $\delta_1 \leq \delta_i$ for every $1 \leq i < d$ as required. Q.E.D.

(1.4) Remark. (a) When $\tau(j) \in \Delta(j)$ is a facet (maximal face) of $\Delta(j)$, then $h(\rho) = (1, 1, \dots, 1)$. This fact immediately shows that if $(h_0, h_1, \dots, h_d, 0)$ is the h -vector of a triangulation of a d -ball, then $(h_0, h_1 + 1, h_2 + 1, \dots, h_d + 1, 0)$ is also the h -vector of a triangulation of a d -ball. Thus, in particular, given positive integers d and n , there exists a triangulation Δ of a d -ball with the h -vector $h(\Delta) = (1, n, 1, \dots, 1, 0) \in \mathbb{Z}^{d+2}$.

(b) Let Δ be a triangulation of a d -ball and suppose that each facet possesses a vertex contained in the interior $\Delta - \partial\Delta$ of Δ . Then the h -vector $h(\Delta) = (h_0, h_1, \dots, h_d, 0)$ of Δ satisfies the linear inequality

$$h_0 + h_1 + \dots + h_i \leq h_d + h_{d-1} + \dots + h_{d-i}$$

for every $0 \leq i \leq [d/2]$.

(c) A technique similar to what was done in the proof of Theorem 1.1 enables us to obtain the linear inequalities

$$\delta_{d-1} + \delta_{d-2} + \dots + \delta_{d-i} \leq \delta_2 + \delta_3 + \dots + \delta_i + \delta_{i+1},$$

$0 \leq i \leq [(d-1)/2]$, for the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of an arbitrary integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d . In particular, $\delta_{d-1} \leq \delta_2$.

however, [Sta₅, Ex. 3.4] shows that unfortunately we cannot expect $\delta_{d+1-i} \leq \delta_i$ when $2 < i$.

It would, of course, be of great interest to find a combinatorial characterization of the δ -vectors of integral convex polytopes.

We refer the reader to [Sta₄, (3.4); Sta₄, (4.1); Sta₅, (3.3)] for further information about δ -vectors ($=h^*$ -vectors). Also, see [H₁, H₂, H₃, and H₄].

ACKNOWLEDGMENTS

The author is grateful to Professor Louis J. Billera for an e-mail discussion on triangulations of convex polytopes and to Professor Richard P. Stanley for sending the papers [Sta₄] and [Sta₅].

REFERENCES

- [Bar₁] D. BARNETTE, The minimum number of vertices of a simple polytope, *Israel J. Math.* 10 (1971), 121–125.
- [Bar₂] D. BARNETTE, A proof of the lower bound conjecture for convex polytopes, *Pacific J. Math.* 46 (1973), 349–354.
- [B-M] U. BETKE AND P. McMULLEN, Lattice points in lattice polytopes, *Mh. Math.* 99 (1985), 253–265.
- [Ehr] E. EHRHART, "Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire," Birkhäuser, Basel/Stuttgart, 1977.
- [H₁] T. HIBI, Some results on Ehrhart polynomials of convex polytopes, *Discrete Math.* 83 (1990), 119–121.
- [H₂] T. HIBI, Dual polytopes of rational convex polytopes, *Combinatorica* 12 (1992), 237–240.
- [H₃] T. HIBI, A combinatorial self-reciprocity theorem for Ehrhart quasi-polynomials of rational convex polytopes, *European J. Combin.*, in press.
- [H₄] T. HIBI, Ehrhart polynomials of convex polytopes, h -vectors of simplicial complexes, and nonsingular projective toric varieties; in "DIMACS Series in Discrete Mathematics and Theoretical Computer Science," Vol. 6, pp. 165–177, Amer. Math. Soc., Providence, RI, 1991.
- [H₅] T. HIBI, "Algebraic Combinatorics on Convex Polytopes," Carlaw, Sydney, 1992.
- [Sta₁] R. STANLEY, Decompositions of rational convex polytopes, *Ann. Discrete Math.* 6 (1980), 333–342.
- [Sta₂] R. STANLEY, "Combinatorics and Commutative Algebra," Birkhäuser, Boston/Basel/Stuttgart, 1983.
- [Sta₃] R. STANLEY, "Enumerative Combinatorics," Vol. 1, Wadsworth & Brooks/Cole, Monterey, CA, 1986.
- [Sta₄] R. STANLEY, On the Hilbert function of a graded Cohen-Macaulay domain, *J. Pure Appl. Algebra* 73 (1991), 307–314.
- [Sta₅] R. STANLEY, A monotonicity property of h -vectors and h^* -vectors, *European J. Combin.* 14 (1993), 251–258.