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STAR-SHAPED COMPLEXES AND EHRHART POLYNOMIALS

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ABSTRACT. We study Ehrhart polynomials of star-shaped triangulations of balls by means of Cohen-Macaulay rings and canonical modules.

A polyhedral complex Γ in $\mathbb{R}^N$ is a finite set of convex polytopes in $\mathbb{R}^N$ such that

(1.1) if $\mathscr{P} \in \Gamma$ and $\mathscr{F}$ is a face of $\mathscr{P}$, then $\mathscr{F} \in \Gamma$, and

(1.2) if $\mathscr{P}, \mathscr{Q} \in \Gamma$, then $\mathscr{P} \cap \mathscr{Q}$ is a face of $\mathscr{P}$ and of $\mathscr{Q}$.

We are concerned with a polyhedral complex Γ in $\mathbb{R}^N$ which satisfies the following conditions:

(2.1) every vertex α of $\mathscr{P} \in \Gamma$ has integer coordinates, i.e., $\alpha \in \mathbb{Z}^N$, and

(2.2) the underlying space $X := \bigcup_{\mathscr{P} \in \Gamma} \mathscr{P} (\subset \mathbb{R}^N)$ of Γ is homeomorphic to the d-ball.

Let ∂X denote the boundary of X; thus ∂X is homeomorphic to the $(d-1)$-sphere. Given an integer $n > 0$, write nX for $\{n\alpha; \alpha \in X\}$ and define $i(X, n)$ to be $\#(nX \cap \mathbb{Z}^N)$, the cardinality of $nX \cap \mathbb{Z}^N$. In other words, $i(X, n)$ is equal to the number of rational points $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in X$ with each $n\alpha_i \in \mathbb{Z}$. It is known that

(3.1) $i(X, n)$ is a polynomial in n of degree d, called the *Ehrhart polynomial* of X,

(3.2) $i(X, 0) = 1$, and

(3.3) $(-1)^d i(X, -n) = \#[n(X - \partial X) \cap \mathbb{Z}^N]$ for every $1 \leq n \in \mathbb{Z}$.

Define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1 + \sum_{n=1}^{\infty} i(X, n)\lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i.$$

Then

(4.1) $\delta_0 = 1$ and $\delta_1 = \#(X \cap \mathbb{Z}^N) - (d+1)$,

(4.2) $\delta_i = 0$ for each $i > d$, and

(4.3) $\delta_d = \#[(X - \partial X) \cap \mathbb{Z}^N]$.

We say that $\delta(X) = (\delta_0, \delta_1, \ldots, \delta_d)$ is the *δ-vector* of X. We refer the reader to, e.g., [6, Chapter IX], for geometric proofs of the above fundamental results

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due to Ehrhart. Note that, even though X is not necessarily convex, the proofs in [6] are valid without modification since X is homeomorphic to the d-ball.

Some algebraic technique [1] is indispensable for the study of combinatorics on δ-vectors. Fix a field k, and let $\xi_1, \xi_2, \dots, \xi_N$, t be (commutative) indeterminates over k. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in nX \cap \mathbb{Z}^N$, then we set $\xi^\alpha t^n = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_N^{\alpha_N}t^n$. We write $[A_k(\Gamma)]_n$ for the vector space spanned by all monomials $\xi^\alpha t^n$ with $\alpha \in nX \cap \mathbb{Z}^N$. Thus, in particular, $\dim_k[A_k(\Gamma)]_n = i(X, n)$. Let $A_k(\Gamma)$ denote $\bigoplus_{n\geq 0}[A_k(\Gamma)]_n$ with $[A_k(\Gamma)]_0 = k$, and define multiplication $(\xi^\alpha t^n)(\xi^\beta t^m)$ of monomials $\xi^\alpha t^n$ and $\xi^\beta t^m$ in $A_k(\Gamma)$ as follows: $(\xi^\alpha t^n)(\xi^\beta t^m) = \xi^{\alpha+\beta}t^{n+m}$ if there exists $\mathscr{P} \in \Gamma$ with $\alpha \in n\mathscr{P}$ and $\beta \in m\mathscr{P}$; $(\xi^\alpha t^n)(\xi^\beta t^m) = 0$ otherwise. Then $A_k(\Gamma)$ is a noetherian (i.e., finitely generated) graded algebra over k and the Hilbert series $F(A_k(\Gamma), \lambda) := \sum_{n=0}^{\infty}\dim_k[A_k(\Gamma)]_n\lambda^n$ is $(\delta_0+\delta_1\lambda+\delta_2\lambda^2+\cdots+\delta_d\lambda^d)/(1-\lambda)^{d+1}$. Let $\Omega(A_k(\Gamma)) = \bigoplus_{n\geq 1}[\Omega(A_k(\Gamma))]_n$ be the graded ideal of $A_k(\Gamma)$ which is generated by those monomials $\xi^\alpha t^n$ such that $0 < n \in \mathbb{Z}$ and $\alpha \in n(X - \partial X) \cap \mathbb{Z}^N$. Since X is homeomorphic to the d-ball, $A_k(\Gamma)$ is Cohen-Macaulay [10, Lemma 4.6]. Thus, a well-known technique of commutative algebra enables us to obtain $\delta(X) \geq 0$, i.e., each $\delta_i \geq 0$ (cf. Stanley [8]). On the other hand, the same technique as in the proof of [2, Theorem (5.6.1)] enables us to show that $\Omega(A_k(\Gamma))$ is the canonical module of $A_k(\Gamma)$.

We say that X is "star-shaped" with respect to a point $\alpha \in X - \partial X$ if $t\alpha + (1-t)\beta \in X - \partial X$ for every point $\beta \in X$ and for each real number $0 < t < 1$.

Theorem. *We employ the same notation as used above. Suppose that the set $(X - \partial X) \cap \mathbb{Z}^N$ is nonempty and that the underlying space X is star-shaped with respect to some $v_1 \in (X - \partial X) \cap \mathbb{Z}^N$. Then the δ-vector $\delta(X) = (\delta_0, \delta_1, \dots, \delta_d)$ of X satisfies the linear inequalities as follows:*

(5.1) $\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_d + \delta_{d-1} + \cdots + \delta_{d-i}, \qquad 0 \leq i \leq [d/2];$

(5.2) $\delta_1 \leq \delta_i, \qquad 2 \leq i < d.$

Sketch of proof. First, recall that a simplicial complex in $\mathbb{R}^N$ is a polyhedral complex Δ in $\mathbb{R}^N$ such that every convex polytope belonging to Δ is a simplex in $\mathbb{R}^N$. Fix an arbitrary simplicial complex $\Delta(0)$ in $\mathbb{R}^N$ with the vertex set $\partial X \cap \mathbb{Z}^N$ whose underlying space is the boundary ∂X of X. Since X is star-shaped with respect to $v_1 \in (X - \partial X) \cap \mathbb{Z}^N$, we can define the cone $\Delta(1)$ over $\Delta(0)$ with apex v_1, i.e., $\Delta(1)$ is the simplicial complex in $\mathbb{R}^N$ which consists of those simplices σ such that either $\sigma \in \Delta(0)$ or σ is the convex hull of $\tau \cup \{v_1\}$ in $\mathbb{R}^N$ for some $\tau \in \Delta(0)$. The vertex set of $\Delta(1)$ is $(\partial X \cap \mathbb{Z}^N) \cup \{v_1\}$ and the underlying space of $\Delta(1)$ is X. Let $(X - \partial X) \cap \mathbb{Z}^N = \{v_1, v_2, \dots, v_\ell\}$ and, for each $2 \leq j \leq \ell$, construct a simplicial complex $\Delta(j)$ with the vertex set $(\partial X \cap \mathbb{Z}^N) \cup \{v_1, v_2, \dots, v_j\}$ and with the underlying space X by the same way as in [7]. We write Δ for $\Delta(\ell)$. Then the element $\theta = \xi^{v_1}t + \xi^{v_2}t + \cdots + \xi^{v_\ell}t$ of $[\Omega(A_k(\Delta))]_1$ is a nonzero divisor on $A_k(\Delta)$. Hence, it follows from a standard technique of commutative algebra [11] (see also [4]) that $\sum_{0\leq j\leq i}\delta_j \leq \sum_{0\leq j\leq i}\delta_{d-j}$ for every $0 \leq i \leq [d/2]$. On the other hand, let $h(\Delta) = (h_0, h_1, \dots, h_d, 0)$ be the h-vector (e.g., [9]) of the simplicial complex Δ. Then $h_1 \leq h_i$ for each $2 \leq i < d$ (cf. [7]). Also, $h_1 = \delta_1$. Since $h_i \leq \delta_i$, $0 \leq i \leq d$, by [1], we have $\delta_1 \leq \delta_i$ for each $2 \leq i < d$ as desired. Q.E.D.

[1] We refer to, e.g., [6, Chapter IV] for "Commutative Algebra for Combinatorialists".

Remark. (a) In the above sketch of proof, let $A_k(\Delta)^*$ denote the graded subalgebra of $A_k(\Delta)$ generated by $[A_k(\Delta)]_1$ over k. Then $A_k(\Delta)^*$ coincides with the Stanley-Reisner ring [9] of the simplicial complex Δ. Thus $A_k(\Delta)^*$ is Cohen-Macaulay with the Hilbert series

$$F(A_k(\Delta)^*, \lambda) = (h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_d\lambda^d)/(1-\lambda)^{d+1}.$$

Moreover, $A_k(\Delta)$ is finitely generated as a module over $A_k(\Delta)^*$.

(b) By the similar method as in [3, Theorem (1.3)], without the hypothesis that $(X - \partial X) \cap \mathbb{Z}^N$ is nonempty and X is star-shaped, we can prove that the δ-vector $\delta(X) = (\delta_0, \delta_1, \dots, \delta_d)$ of X satisfies the linear inequality

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_0 + \delta_1 + \cdots + \delta_i + \delta_{i+1}$$

for every $0 \leq i \leq [(d-1)/2]$.

Example. Let $N = d = 3$ and $X = \mathscr{P} \cup \mathscr{Q}$, where $\mathscr{P} \subset \mathbb{R}^3$ (resp. $\mathscr{Q} \subset \mathbb{R}^3$) is the tetrahedron with the vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, -1, -1)$ (resp. $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$). Then $(X - \partial X) \cap \mathbb{Z}^3 = \{(0, 0, 0)\}$ and X is not star-shaped with respect to $(0, 0, 0)$. However, X is star-shaped with respect to, e.g., $(1/3, 1/3, 1/3)$. We have $\delta(X) = (1, 2, 1, 1)$ which fails to satisfy (5.1) for $i = 1$ and (5.2) for $i = 2$.

Corollary [3, 7, 11]. *Let $\mathscr{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d, and suppose that $(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N$ is nonempty. Then the δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of $\mathscr{P}$ satisfies the following linear inequalities:*

(6.1) $\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_d + \delta_{d-1} + \cdots + \delta_{d-i}, \qquad 0 \leq i \leq [d/2];$

(6.2) $\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_0 + \delta_1 + \cdots + \delta_i + \delta_{i+1}, \qquad 0 \leq i \leq [(d-1)/2];$

(6.3) $\delta_1 \leq \delta_i, \qquad 2 \leq i < d.$

We conclude the paper with a remark about the question when $A_k(\Gamma)$ is Gorenstein. For a while, we assume that $N = d$ and the origin of $\mathbb{R}^d$ is contained in the interior of X. We say that $\delta(X) = (\delta_0, \delta_1, \dots, \delta_d)$ is symmetric if $\delta_i = \delta_{d-i}$ for every $0 \leq i \leq d$. It follows from, e.g., [5] that X is star-shaped with respect to the origin if $\delta(X)$ is symmetric. On the other hand, $\delta(X)$ is symmetric if and only if there exists a polyhedral complex Γ in $\mathbb{R}^d$ with the underlying space X such that $A_k(\Gamma)$ is Gorenstein, i.e., the canonical module $\Omega(A_k(\Gamma))$ of $A_k(\Gamma)$ is generated by a single element of $A_k(\Gamma)$.

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CASTELNUOVO REGULARITY AND GRADED RINGS ASSOCIATED TO AN IDEAL

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Abstract. We compare the Castelnuovo regularity defined with respect to different homogeneous ideals in a graded ring and use the result we obtain to prove a generalized Goto-Shimoda theorem for ideals of positive height in a Cohen-Macaulay local ring.

1. Introduction

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $I \subseteq R$ an ideal. A number of papers in the past ten years or so have studied the transfer of the Cohen-Macaulay property of R to various graded rings associated to I, with particular attention being paid to $\mathscr{G} = \mathscr{G}(I)$ and $\mathscr{R} = \mathscr{R}(I)$ — the associated graded ring and the Rees ring of R with respect to I. In [H], Huneke showed that $\mathscr{G}$ is Cohen-Macaulay whenever $\mathscr{R}$ is Cohen-Macaulay (and the ideal has positive height) and pointed out that the converse need not hold. Since then, numerous authors have studied additional conditions required for $\mathscr{R}$ to be Cohen-Macaulay when $\mathscr{G}$ is Cohen-Macaulay. One of the most important theorems to emerge from these endeavors is the so-called Goto-Shimoda theorem ([GS, Theorem 3.1]) which we now state.

Theorem (Goto-Shimoda). *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Then $\mathscr{R}$ is Cohen-Macaulay if and only if $\mathscr{G}$ is Cohen-Macaulay and $JI^{d-1} = I^d$ for every minimal reduction J of I.*

It is fair to say that [GS] has provided the impetus for a large amount of research. Notable among subsequent endeavors is [GHO], where the Goto-Shimoda theorem was extended to equimultiple ideals (i.e., ideals whose height equals their analytic spread). The theorem in [GHO] reads exactly the same as the one above, only the assumption that I is $\mathfrak{m}$-primary is replaced by the (more general) assumption that I is equimultiple and $d = \dim(R)$ is replaced by $s = s(I)$, the analytic spread of I. Little progress was made on extending the

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