JOURNAL OF COMBINATORIAL THEORY, Series B 68, 277–294 (1996)
ARTICLE NO. 0069

1417801

A Pseudoconfiguration of Points without Adjoint

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Received October 17, 1995

We give an example of a simple oriented matroid $\mathcal{G}$ that admits an oriented adjoint. Furthermore, any adjoint of the underlying matroid D, does not itself admit an adjoint. D arises from the well-known non-Desargues matroid by a coextension by a coparallel element and, hence, has rank 4. The orientability of D and some of its adjoints follows from an apparently new oriented matroid construction given in the paper that is a very special case of an amalgam of two copies of one oriented matroid. © 1996 Academic Press, Inc.

1. INTRODUCTION

One source of motivation for matroid theory is the study of the combinatorial structure of point configurations in linear space. While ordinary matroid theory may be viewed as a quite general abstraction of projective geometry over arbitrary fields, the study of oriented matroids was motivated by combinatorial questions arising from geometry in real linear space. The Topological Representation Theorem [5, Theorem 5.2.1.] states that, indeed, the axioms of oriented matroids are strong enough to guarantee a geometric situation in Euclidean space. By that theorem any oriented matroid can be represented by an *arrangement of pseudohyperplanes in real projective space*, where "pseudohyperplane" means linear hyperplane with "some local deformations" allowed. This is called a Type I representation of an oriented matroid.

Any matroid represented by a point configuration in linear space may via polarity as well be represented by a hyperplane arrangement and vice versa; this is the well-known point-hyperplane duality from projective geometry. While any oriented matroid admits a representation of the latter kind with slightly deformed "hyperplanes" (pseudohyperplanes) the situation is more difficult with "pseudoconfigurations of points" (cf. [5, Chap. 5.3]).

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0095-8956/96 $18.00

A standard approach to define the dependencies in matroids in a geometric representation is to give a list of points, lines, planes, etc. They need not necessarily represent a linear configuration (see [12]). The information about the orientation of a hyperplane in oriented matroid theory is encoded in the partition it induces for the points which it does not contain. Thus, hyperplanes should partition space and, hence, be somewhat closer to real hyperplanes. So, a natural extension of the geometric representation by points in real linear space is to embed points into a configuration of pseudohyperplanes. Such a model is called a Type II representation of an oriented matroid. Unfortunately, such a representation does not always exist.

Given a Type II representation of an oriented matroid $\mathcal{M}$ by the Topological Representation Theorem, the underlying pseudohyperplane arrangement represents an oriented matroid which we call the adjoint $\mathcal{M}^{\Delta}$ of $\mathcal{M}$. The elements (points) of $\mathcal{M}$ correspond to some hyperplanes of $\mathcal{M}^{\Delta}$ and the elements of $\mathcal{M}^{\Delta}$ are in one-to-one correspondence with the hyperplanes of $\mathcal{M}$. Thus, in the unoriented case an adjoint corresponds to an embedding of the dual of the geometric lattice of a matroid into a geometric lattice of the same rank such that the map is one-to-one from copoints to points.

It was observed already in the first paper about adjoints [8] that they may not exist. One reason for that is the following. Starting with an (oriented) matroid $\mathcal{M}$ it is immediate that $\mathcal{M}$ is a submatroid of $(\mathcal{M}^{\Delta})^{\Delta}$. Thus, iterating the process of taking adjoints of adjoints yields an increasing chain of submatroids the union of which has to have a modular lattice. Hence, for nonlinear matroids of rank at least four at some point the process has to get stuck. To get a better understanding of "reasons" for nonlinearity, G. Ziegler (see [5, Exercise 7.15]) asked for an example of an oriented matroid which has a Type II representation but no double adjoint.

We will prove that a certain coextension D of the non-Desargues matroid has a Type II representation but a double adjoint fails to exist already in the unoriented case. This coextension may also serve for a matroid theory proof of Desargues' Theorem as suggested by T. Brylawski (see [18, Exercise 7.53]). To construct the Type II representation of D we give an apparently new construction method for oriented matroids which we call the *squint of an oriented matroid*, which is a very special case of an amalgam of oriented matroids.

The paper is organized as follows. In the next section we discuss some background on adjoints of matroids and provide some technical results used in Section 3. In Section 3 we introduce our example, shortly discuss the existence of an unoriented adjoint, and prove the nonexistence of a double adjoint. Section 4 reviews basic facts on topological representations of oriented matroids focusing on the rank 3 case. In the final section we present our construction method and as corollary derive an oriented adjoint of an orientation of our example.

We assume some familiarity with matroid and oriented matroid theory; standard references are [18, 14, 17, and 5]. We will frequently refer to the latter of these. The notation used is standard; however, we remark that we denote by $L(M)$ the geometric lattice of flats of a matroid M. Conversely, for a geometric lattice L we let $M(L)$ denote the simple matroid on the atoms of L that has L for its lattice of flats.

2. BASICS ON ADJOINTS

There is a standard notion of duality in lattice theory. If $L=(S, \wedge, \vee)$ is a lattice, $L^{op}:=(S, \vee, \wedge)$ is a lattice as well —the *opposite* (or, *order dual*) obtained from L by "turning L upside down." For a geometric lattice L the opposite lattice L^{op} will in general fail to be semimodular and hence fail to be geometric. There is a noteworthy exception: opposite lattices of modular geometric lattices are modular geometric and, indeed, a special case of this is the well-known point–hyperplane duality from projective geometry.

For geometric lattices L in general, however, the best one can hope for with regard to this kind of duality is that the opposite lattice L^{op} may be embedded into a geometric lattice L^{Δ} of the same rank as L. More precisely, one defines

DEFINITION 1. Let L be a geometric lattice of rank r. A geometric lattice L^{Δ} is called an *adjoint* of L if L and L^{Δ} have the same rank and there exists an order-reversing injective map $\Phi\colon L \to L^{\Delta}$, taking the coatoms of L bijectively onto the atoms of L^{Δ}.

It is easy to see that the property of having an adjoint is stable under taking minors. It follows that coordinatizable matroids admit adjoints, since coordinatizing a matroid is the same thing as representing the matroid as a restriction minor of a coordinatizable projective space.

In general a matroid does not admit an adjoint. Already Cheung, who first defined the notion of adjoint, showed in [8] that the notorious Vamos matroid does not admit an adjoint. In [2] it was shown that the dual of a matroid that admits an adjoint does not in general admit an adjoint itself; the example given in that paper is the non-Desargues matroid shown in Fig. 1 which also plays a prominent role in the present paper. In the next section we give an example of a rank 4 (connected) matroid that admits an adjoint, but any adjoint of which does not itself admit an adjoint. In [13] Mason gave an example for a matroid of this kind. His example has the

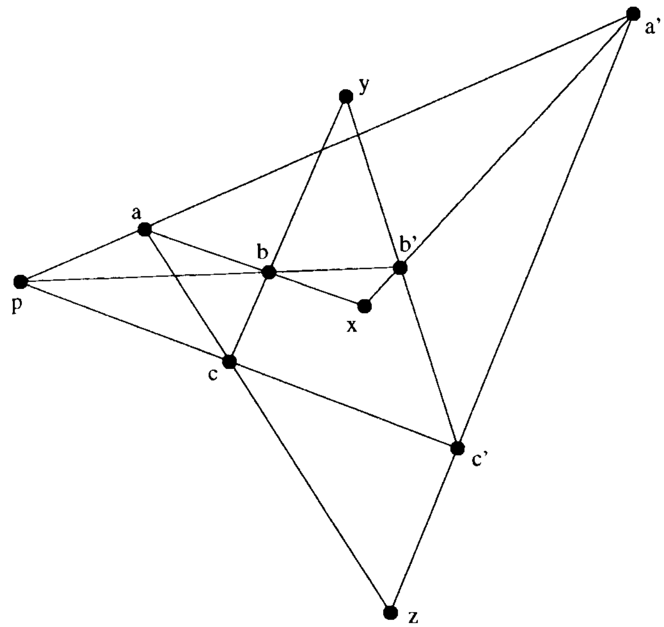

FIG. 1. The non-Desargues matroids.

Fano matroid as a minor and is therefore nonorientable; our example is apparently the first orientable matroid of that kind that admits an oriented adjoint (cf. [5, Ex. 7.15]). The example is a simple coextension of the non-Desargues matroid. In hindsight, some matroid "around" the non-Desargues matroid was, indeed, to be expected to provide an example of this phenomenon.

We need some preparatory observations. Lemma 1 provides an easy to use criterion for detecting some independent sets in an adjoint of a matroid. The condition is not necessary. Proposition 1 states that the images of any pair of flats of the original lattice L under Φ are a modular pair in the adjoint L^{Δ}. As already alluded to in the Introduction, this proposition implies that iterated taking of adjoints, if possible, "converges" to a modular geometric lattice and, hence, by the theorem of Veblen-Young (see [4]) to a projective geometry if the matroid $M(L)$ started with is connected.

Let L denote a geometric lattice. Let $\mathscr{H}$ denote the set of copoints (hyperplanes) of L and $\mathscr{P}$ denote the set of points of L. Recall that a geometric lattice is atomic as well as coatomic; i.e.,

$$\forall a \in L: a = \bigvee \{p \in \mathscr{P} \mid p \leqslant a\}$$

and

$$\forall a \in L: a = \bigwedge \{h \in \mathscr{H} \mid a \leqslant h\}.$$

LEMMA 1. *Let L be a geometric lattice of rank r and L^{Δ} an adjoint of L with corresponding embedding Φ: $L \to L^{\Delta}$. If $h_1, ..., h_m$ are copoints of L such that*

$$1 > h_1 > h_1 \wedge h_2 > \cdots > h_1 \wedge \cdots \wedge h_m$$

then $\Phi(h_1), ..., \Phi(h_m)$ is an independent set of $M(L^{\Delta})$.

Proof. From the fact that L is coatomic it easily follows that any set of hyperplanes with the assumed property may be augmented to a set $\{h_1, ..., h_r\} \subseteq \mathscr{H}$ (renumbered, if necessary) such that

$$1 \to h_1 \to h_1 \wedge h_2 \to \cdots \to h_1 \wedge \cdots \wedge h_r,$$

where $\to$ denotes the covering relation on L. Then

$$\left(\bigwedge\{h_i \mid 1 \leqslant i \leqslant j-1\}\right)_{1 \leqslant j \leqslant r+1}$$

is a maximal chain in L and, as the rank of L^{Δ} is r and Φ is order-reversing and injective it follows that this maximal chain in L gets mapped to a maximal chain in L^{Δ} by Φ. It follows that $\Phi(h_1), ..., \Phi(h_r)$ span $M(L^{\Delta})$ and hence are a basis. ∎

Note that in particular the restriction of the rank function of L^{Δ} to the image of Φ is seen to be the corank function of L. Furthermore, it follows that Φ is cover preserving.

The following fact is implicitly contained in [5, Exercise 7.17]. We omit its straightforward proof.

PROPOSITION 1. *Let L be a geometric lattice and L^{Δ} an adjoint of L with corresponding order-reversing injective function Φ: $L \to L^{\Delta}$ mapping the copoints of L onto the points of L^{Δ}. Then $\Phi(x)$, $\Phi(y)$ is a modular pair of L^{Δ} for any $x, y \in L$; i.e.,*

$$r_{L^{\Delta}}(\Phi(x)) + r_{L^{\Delta}}(\Phi(y)) = r_{L^{\Delta}}(\Phi(x) \vee \Phi(y)) + r_{L^{\Delta}}(\Phi(x) \wedge \Phi(y)).$$

It follows from the above that for nonlinear connected matroids of rank $\geqslant 4$ an infinite sequence of iterated adjoints cannot exist and the process has to stop after finitely many steps with a matroid that does not admit an adjoint. The next section gives an example of a matroid that has an adjoint, any adjoint of which, however, does not admit an adjoint.

3. THE UNORIENTED EXAMPLE

Our example is a principal coextension of the non-Desargues matroid. For a discussion of coextensions in general see Chapter 7 on matroid constructions in [18] or [14]. We do not need to get involved into the details of coextensions, as our example is easily enough understood without knowing about coextensions.

Consider the non-Desargues matroid N on the set $\{p, a, b, c, a', b', c', x, y, z\}$, "affinely" represented in Fig. 1, the labeling of the points being the one from the figure. Coextend N by a point w coparallel to p; i.e., consider the matroid $D := (N^* +_{\mathscr{F}^*} w)^*$, where $\mathscr{F}^* = \{x \in L(M^*) \mid p \leqslant x\}$ denotes the principal filter of $L(M^*)$ generated by p. An "affine" picture of D is given in Fig. 2—solid lines are in the "original non-Desargues plane," the dotted line is in "space." Since we coextend by a point w coparallel to p the point p slides up into space along the line from p to w. We have renamed p to p' and indicated by an unfilled circle that the point p of the original non-Desargues plane in D no longer exists.

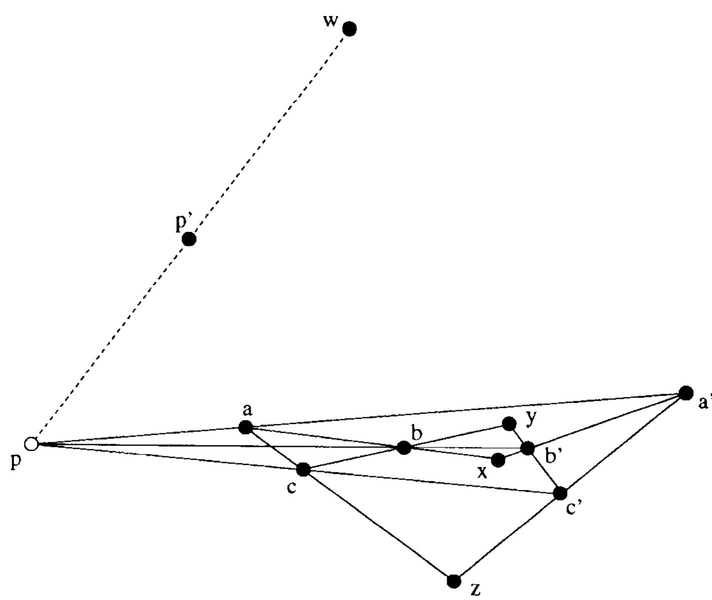

FIG. 2. Coextending the non-Desargues matroid by a point coparallel to p.

Not much is known in way of sufficient conditions for the existence of an adjoint. In [1], however, it has been observed that for matroids of rank 4 to admit an adjoint it is sufficient that the geometric lattice of the matroid is pseudomodular. For a definition of this notion and first properties see [6]. Although D is not pseudomodular it is quite easily seen that D "extended by p," i.e., extended by a point with respect to the modular cut of D generated by the line $\overline{wp'}$ and the plane $\overline{abc}$, is pseudomodular of rank 4 and, therefore, has an adjoint. As the class of matroids with adjoint is minor closed, we get that D itself also admits an adjoint. This, however, also follows from the next section of the present paper, where it is shown that D is orientable and admits an oriented adjoint.

We now show that any adjoint of our matroid D cannot itself have an adjoint.

Suppose we had an adjoint $(D^{\Delta})^{\Delta}$ of an adjoint of D. The arguments we are going to give may be followed in Fig. 3, where the final situation leading to a contradiction is pictured. We argue "in $(D^{\Delta})^{\Delta}$" now, using the fact that flats of D when considered as flats of $(D^{\Delta})^{\Delta}$ intersect modularly. This follows from Proposition 1. Hence, the coplanar lines $\overline{aw}$, $\overline{a'p'}$ intersect in

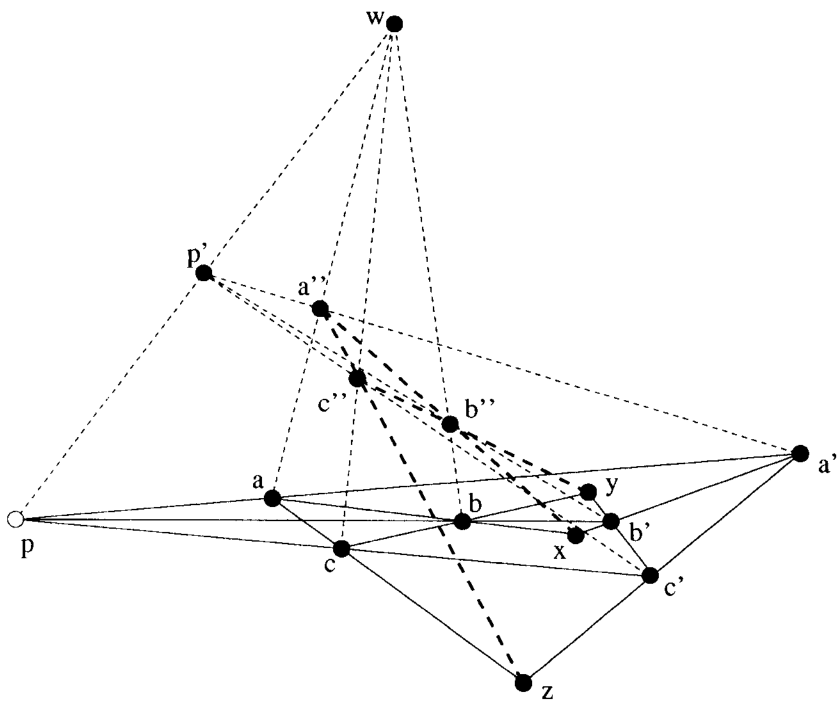

FIG. 3. No matroid.

a point a'' and, similarly, we get points $b'' = \overline{bw} \cap \overline{b'p'}$ and $c'' = \overline{cw} \cap \overline{c'p'}$. The intersection of the plane $\overline{p'a''b''}$ and the "original non-Desargues plane" $H := \overline{abc}$ is the line $\overline{a'b'}$, and the intersection of $\overline{wa''b''}$ and H is the line $\overline{ab}$. As both of these lines contain x it follows that the line $\overline{a''b''}$ meets H in x. Similarly, $\overline{a''c''} \cap H = z$ and $\overline{b''c''} \cap H = y$. Now, the planes H and $\overline{a''b''c''}$ are different, and, thus, can intersect in at most a line, yet we have $\overline{a''b''c''} \cap \overline{abc} \supseteq \overline{xyz} = H$, a contradiction.

Note that the argument we have given is a proof of the theorem of Desargues for coordinatizable projective geometries of rank $\geqslant 3$ "by matroid theory" (cf. Brylawski's exercise 7.53 in [18]). In fact, we have shown that in any projective geometry of rank $\geqslant 4$ the theorem of Desargues holds, as is well known.

4. ORIENTED MATROIDS AND PSEUDOPOINT CONFIGURATIONS

Oriented matroids were introduced in the late 1970s in [7] and [9]. From the various axiom systems for oriented matroids we will consider the circuit axioms (cf. [5, 3.2.1]) here which may give the easiest access for readers familiar with matroid and graph theory. In order to simplify notation in the proof of Theorem 2 our notation is slightly different from the one used in [5].

DEFINITION 2. Let E be a set. For our purposes it will be convenient to consider two copies of E namely E^+, E^- with different signs. The *support* $\operatorname{supp}(X) \subseteq E$ of a subset $X \subseteq E^+ \cup E^-$ of the signed sets is the set of elements of the underlying groundset. A *signed subset* $X \subseteq E^+ \cup E^-$ of E is a set with $\operatorname{supp}(X \cap E^+) \cap \operatorname{supp}(X \cap E^-) = \emptyset$. For short, we write $X^+ := X \cap E^+$ and $X^- := X \cap E^-$. The *separator* of two signed subsets X, Y is defined as

$$\operatorname{sep}(X, Y) := \operatorname{supp}(X \cap -Y).$$

A collection $\mathscr{C}$ of signed subsets of a set E is the set of *signed circuits* of an *oriented matroid* on E if it satisfies the following axioms:

(C0) $\emptyset \notin \mathscr{C}$,

(C1) $\mathscr{C} = -\mathscr{C}$, (symmetry)

(C2) $\forall X, Y \in \mathscr{C}\colon \operatorname{supp}(X) \subseteq \operatorname{supp}(Y) \Rightarrow X \in \{Y, -Y\}$, (incomparability)

(C3) $\forall X \neq -Y \in \mathscr{C} \quad \forall e \in \operatorname{sep}(X, Y) \quad \exists Z \in \mathscr{C}\colon Z \subseteq X \cup Y \setminus \{e^+, e^-\}$ (weak elimination).

For our application oriented matroids of rank 3 are of importance. Here their topological Type I representation is equivalent to the well-known arrangements of pseudolines as introduced by Levi [11] (see also [10] and [5, 6.2.3]).

Choosing a positive and a negative side of each pseudoline, one derives signed coordinates for the vertices of the arrangement. To be more precise, let v be a vertex of the arrangement and let C be the set of lines such that v is positive with respect to the orientation of each line in C (i.e., it is on the positive side). Let C' denote the set of lines which have v on their negative side. Then v gives rise to the signed cocircuits $C^+ \cup C'^-$ and $C^- \cup C'^+$. It is not difficult to see that this collection of signed sets satisfies the signed circuit axioms of oriented matroid theory. Since the underlying matroid has corank 3, for our purposes it is more appropriate to think of the vertices in terms of cocircuits of the oriented matroid. The pseudoline arrangement is called a Type I representation of this oriented matroid. In just the same way a signed set is assigned to the vertices of the arrangement, any point of the plane may be assigned a signed set. The collection of all the signed sets obtained this way are the *covectors* of the oriented matroid. With pseudohyperplanes and arrangements of these appropriately defined (see [5, Chap. 5]) the above notions carry over to higher dimensions and the Topological Representation Theorem states that arrangements of pseudohyperplanes and oriented matroids are equivalent.

As mentioned in the Introduction, there is another topological representation of oriented matroids in terms of pseudoconfigurations of points. To cut down on the topology we have to introduce, we retranslate this notion into a combinatorial definition. For the general topological definition of a Type II representation we refer the reader to [5, 5.3]. That the collection of signed sets defined in Definition 3 is, indeed, the collection of cocircuits of an oriented matroid is proved in [5, Prop. 5.3.2].

DEFINITION 3. Let $\mathcal{M}^A$ be an oriented matroid of rank r on the groundset E and P a subset of the cocircuits of $\mathcal{M}^A$ with $|P| \geqslant r$ such that

(i) the union of the supports of any $r-1$ elements of P is a proper subset of E,

(ii) the collection of subsets

$$\{\{C \in P \mid e \notin \operatorname{supp}(C)\} \mid e \in E\}$$

is an antichain with respect to inclusion.

For $e \in E$ let

$$P_1(e) := \{C \in P \mid e^+ \in C\} \qquad \text{and} \qquad P_2(e) := \{C \in P \mid e^- \in C\}.$$

Then the collection $\mathscr{C}$ of signed sets

$$\{P_1(e)^+ \cup P_2(e)^-, P_2(e)^+ \cup P_1(e)^- \mid e \in E\}$$

is the collection cocircuits of an oriented matroid $\mathscr{M}$ on P. The pair $(\mathscr{M}^\Delta, P)$ is called a *Type II representation* of $\mathscr{M}$.

As suggested by the notation, $\mathscr{M}^\Delta$ then is an oriented adjoint of $\mathscr{M}$. We note that the matroid underlying $\mathscr{M}^\Delta$ is an adjoint of the matroid underlying $\mathscr{M}$.

DEFINITION 4 [3]. Let $\mathscr{M}$ be an oriented matroid of rank r on a set E. An *oriented adjoint* of $\mathscr{M}$ is an oriented matroid $\mathscr{M}^\Delta$ of the same rank r on a subset $E^\Delta \subseteq \mathscr{C}^*$ of signed cocircuits of $\mathscr{M}$ with $\mathscr{C}^* = E^\Delta \dot{\cup} -E^\Delta$ such that the signed sets

$$Z[e] = \{Y^+ \in E^{\Delta +} \mid Y \in E^\Delta \quad \text{and} \quad e^+ \in Y\}$$
$$\cup \{Y^- \in E^{\Delta -} \mid Y \in E^\Delta \quad \text{and} \quad e^- \in Y\}$$

are cocircuits of $\mathscr{M}^\Delta$.

From a combinatorial point of view Type II representations and oriented adjoints are equivalent. The nice thing about Type II representations is that one model represents two oriented matroids; the Type II represented oriented matroid $\mathscr{M}$ and the Type I represented oriented matroid $\mathscr{M}^\Delta$.

THEOREM 1 [5, 5.3.6]. *A simple oriented matroid has a representation by a pseudoconfiguration of points (a Type II representation) if and only if it has an adjoint.*

As for unoriented matroids of rank 3, we also have that oriented matroids of rank 3 admit an adjoint. Crucial in the oriented case is Levi's Enlargement Lemma, which we present next. For a proof see [10].

LEMMA 2 (Levi's Enlargement Lemma). *Let x and y be two vertices in a pseudoline arrangement $\mathscr{A}$ which do not both lie on any of the pseudolines from $\mathscr{A}$. Then there exists a proper enlargement of $\mathscr{A}$ by a pseudoline which contains x and y. Furthermore, this pseudoline can be chosen such that it does not contain any other vertex of $\mathscr{A}$.*

Now, given a rank 3 oriented matroid $\mathscr{M}$ and a topological representation as an (oriented) arrangement of pseudolines E with vertices E^Δ we enlarge the configuration to an arrangement $(E^\Delta)^\Delta$ such that any two vertices in E^Δ are contained in a pseudoline of $(E^\Delta)^\Delta$.

Then the vertices E^{Δ} together with the oriented matroid $(\mathcal{M}^{\Delta})^{\Delta}$ (Type I) represented by $(E^{\Delta})^{\Delta}$ provide a Type II representation of an oriented matroid $\mathcal{M}^{\Delta}$ which is an adjoint of $\mathcal{M}$. By what we have already said $(\mathcal{M}^{\Delta})^{\Delta}$ is also an oriented adjoint of $\mathcal{M}^{\Delta}$, or an oriented *double adjoint* of $\mathcal{M}$.

Figure 4 shows a Type II representation of an oriented non-Desargues matroid. The "uninteresting" 2-point-lines are dotted. We have given a pseudoline arrangement as claimed in Levi's Enlargement Lemma; i.e., apart from the points p, a, b, c, a', b', c', x, y, z any point of the plane lies on at most 2 of the (pseudo)-lines. The cocircuits of a Type II represented oriented matroid may be read off the arrangement as follows:

Any pseudoline from the arrangement partitions the given points not on the line into two sets C, C' according to which side of the line they are. Then $C^+ \cup C'^-$ and $C'^+ \cup C^-$ are the two signed sets determined by that line. The cocircuits of the Type II represented matroid are all the signed sets that arise in this fashion. Note that these cocircuits do not depend on an orientation of the pseudolines. The four cocircuits of the Type II represented non-Desargues matroid corresponding to be two bold (pseudo)-lines are U with $U^+ = \{p^+, a^+, b^+, c^+, x^+\}$ and $U^- = \{a'^-, b'^-, c'^-\}$, its negative

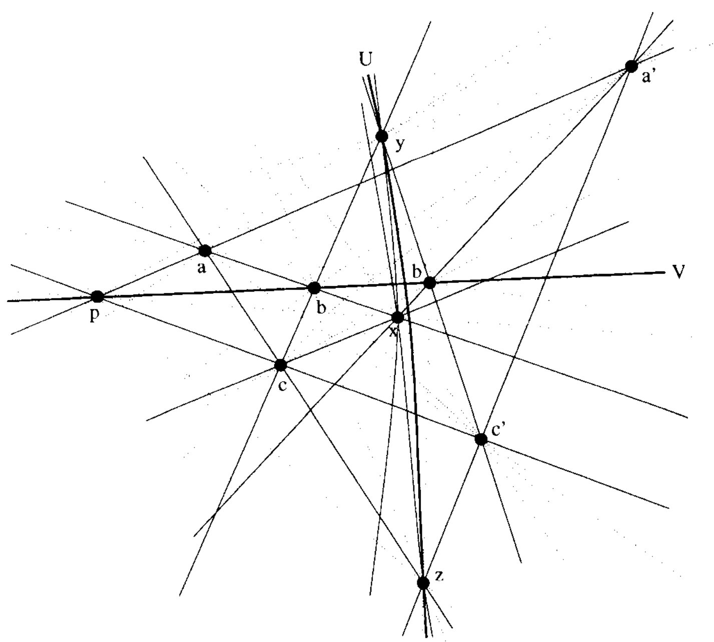

FIG. 4. A Type II representation of an oriented non-Desargues matroid. Arbitrarily orienting each pseudoline defines an oriented adjoint.

$-U$, V with $V^+ = \{a^+, a'^+, y^+\}$ and $V^- = \{c^-, c'^-, x^-, z^-\}$ and its negative $-V$.

The reader may have noticed that in the arrangement in Fig. 4, as are the lines supported by x, y, z, the pseudoline through b and z is not straight, nor is the one through c and a'. The reason is that, e.g., the straight line through c and b', the one through a and c', and the one through b and z (seem to) intersect in one point and we perturbed these lines in order to present an arrangement as claimed by Levi's Enlargement Lemma. However, even if it is in an "unnecessary" special position, the arrangement with all dotted lines straight is still a Type II representation of the same oriented non-Desargues matroid and hence gives rise to an oriented adjoint. This shows that adjoints of matroids are far from being unique.

The reader may recall from the discussion at the beginning of this section how to derive the signed cocircuits of the pseudoline arrangement in Fig. 4 interpreted as Type I representation of an oriented matroid. A complete list of these cocircuits fell victim to a garbage collection. Nevertheless, that oriented matroid is an oriented adjoint of the oriented non-Desargues matroid.

5. THE ADJOINT OF D IS A SQUINT

The following construction is a very special case of an amalgam of oriented matroids. For us it serves as a tool to construct an oriented adjoint of our example D from an oriented adjoint of the non-Desargues matroid as given in Fig. 4. Before we give the formal definition we sketch the construction in the realizable case.

Assume we are given a hyperplane arrangement $\mathscr{H}$ in $\mathbb{R}^n$ which has a vertex p such that for any other vertex v of $\mathscr{H}$ the line through p and v is a line of the arrangement. From this we construct a hyperplane arrangement in $\mathbb{R}^{n+1}$ as follows. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ and choose two points p', w outside of $\mathbb{R}^n \times \{0\}$ such that the line through p' and w contains p. Now, for each hyperplane $h \in \mathscr{H}$ we derive two hyperplanes $\tilde{h}_1, \tilde{h}_2$ of $\mathbb{R}^{n+1}$ by considering the span of h and p' resp. h and w. These two hyperplanes coincide if and only if $p \in h$. Our arrangement now consists of all these hyperplanes and $\mathbb{R}^n \times \{0\}$. What are the vertices of this arrangement?

(i) The old vertices of $\mathscr{H}$ remain vertices.

(ii) Let v_1 and v_2 be vertices of $\mathscr{H}$ such that the line through them is a line of $\mathscr{H}$ and meets p. Then the lines $\overline{v_1 p'}$ and $\overline{v_2 w}$ will be coplanar lines of $\tilde{\mathscr{H}}$. Hence their intersection is a new vertex of $\tilde{\mathscr{H}}$.

(iii) Finally, p' and w are vertices of $\tilde{\mathscr{H}}$.

On the other hand, a closer inspection shows that the properties of p guarantee that there will be no other vertices. Figure 5 depicts this construction in the rank 2 case.

In terms of cocircuits of oriented matroids this gives rise to the following definition.

DEFINITION 5. Let $\mathcal{M}$ be a simple oriented matroid on a finite set E, given by the list of its cocircuits $\mathcal{C}$ and P be a modular cocircuit of $\mathcal{M}$ with the property that $\forall C^1, C^2 \in \mathcal{C}: r(\mathcal{M}) - r(E \backslash (\text{supp}(P) \cup \text{supp}(C^1) \cup \text{supp}(C^2))) \leqslant 3$. Let $S = \text{supp}(P)$ and S' be a set disjoint from E such that $\tilde{\phi}: S \to S'$ is a bijection and h an element different from $E \cup S'$. Extend $\tilde{\phi}$ to a map $\phi: \mathcal{C} \to 2^{\pm S'}$ by first intersecting the signed set corresponding to a cocircuit C with $S^+ \cup S^-$ and then mapping it to $S'^+ \cup S'^-$ the obvious way. Consider the following set of signed sets which we call the *squint of $\mathcal{C}$ through P*:

(i) $\forall C \in \mathcal{C}: C \cup \phi(C)$;

(ii) $\forall C^1, C^2$ such that C^1, C^2 and P are on a coline and C^1 arises from an elimination between P and C^2: $C^2 \cup \phi(C^1) \cup h^+$;

(iii) $h^+ \cup \phi(P)$ and $h^+ \cup -P$;

(iv) the negative of the above.

We remark that r in the definition refers to the rankfunction of the underlying matroid. The stated rankcondition is trivially satisfied for

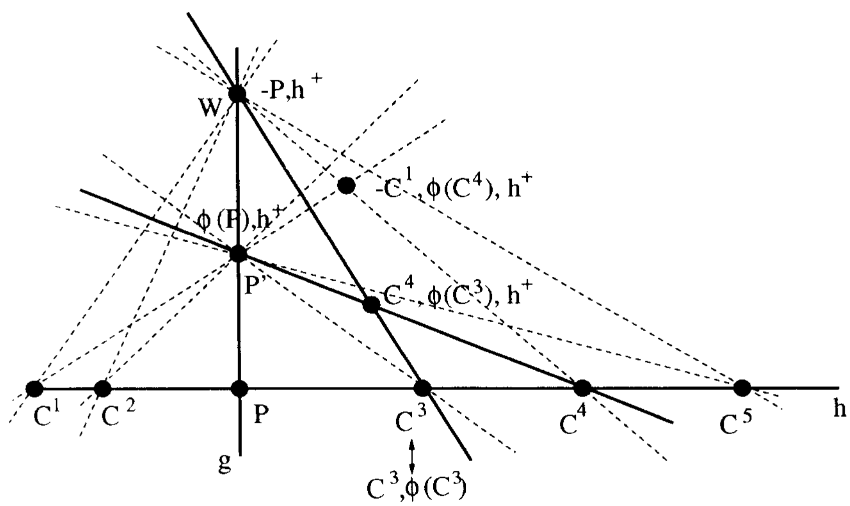

FIG. 5. The rank 2 case.

oriented matroids of rank 3 and guarantees that any two circuits together with P span a pseudoline arrangement in $\mathcal{M}$.

Note that the construction is almost symmetric in S and S'. We say a signed set is of type 1 if it arises from the first case in the definition. Analogously, we define type 2 and type 3 signed sets.

If $\mathcal{C}$ is of rank 2 and, hence, realizable, this construction yields the signed cocircuits of a line arrangement which is constructed as follows. Start with points on a line h as defined by $\mathcal{C}$ and add a line g through P. Choose two points W, P' in one component of $g \backslash h$ such that P' is closer to P and add the lines through all points of h and W resp. P' (see Fig. 5). Note that the (pseudo)-lines in this arrangement may be taken to be straight, i.e., that the squint of a rank 2 oriented matroid is realizable.

THEOREM 2. *The squint of $\mathcal{C}$ through P satisfies the circuit axioms for oriented matroids.*

Proof. By construction the system is an antichain and symmetric. We have to verify the circuit elimination axiom. Due to the condition

$$r(\mathcal{M}) - r(E \backslash (\operatorname{supp}(P) \cup \operatorname{supp}(C^1) \cup \operatorname{supp}(C^2))) \leqslant 3$$

we may, in each of the cases to consider, restrict our attention to the pseudoline arrangement which is spanned by C^1, C^2 and P in the topological representation of $\mathcal{M}$.

Clearly, an elimination exists if both circuits are of type 1.

(i) Let $C \cup \phi(C)$ be a type 1 signed set, $C^2 \cup \phi(C^1) \cup h^+$ be of type 2 and e be an element in $\operatorname{sep}(C, C^i)$, for an $i \in \{1, 2\}$.

If C is on the coline through C^1, C^2 then we can find an elimination in the line arrangement corresponding to the squint of the rank 2 minor induced by that coline.

So assume C is not on the coline through C^1, C^2. This situation is pictured in Fig. 6. Consider the pseudoline arrangement in the plane spanned by P, C and C^i. By Levi's Enlargement Lemma we may choose an extension of that rank 3 oriented matroid by two elements i and j which are lines through C and C^i resp. C and C^j. Let $\tilde{C}$ denote the elimination between C and C^i on e in the extended arrangement. If $\tilde{C} \backslash \{i, j\}$ is not a cocircuit of $\mathcal{C}$ it is an edge of the pseudoline arrangement and both its vertices are eliminations of C and C^i on e. Let one of them be C^{i+2} and denote by g the pseudoline through P and C^{i+2}. This pseudoline intersects j and again we find $C^{j+2} \in \mathcal{C}$, which is an elimination of C and C^j on g. Now, if $C^3 = C^4$, the signed set $C^3 \cup \phi(C^3)$ is an elimination that belongs to our set, and, if $C^3 \neq C^4$, it is obvious that C^3 is an elimination of C^4 and P, so that the signed set $C^4 \cup \phi(C^3) \cup h^+$ is as desired.

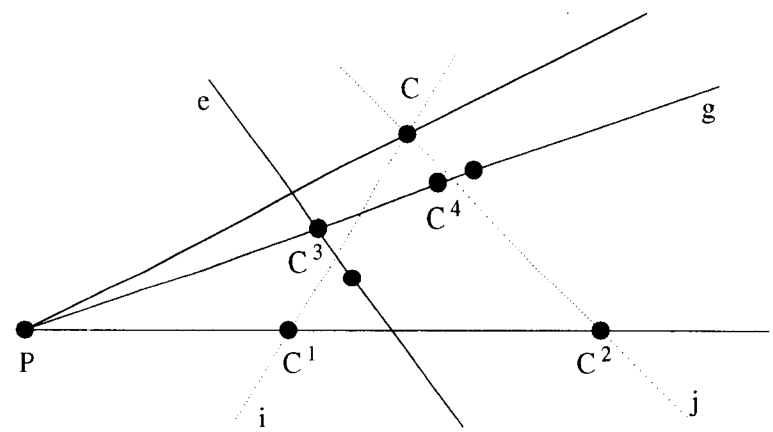

FIG. 6. Elimination between type 1 and type 2.

(ii) If one of the two cocircuits is of type 3 and the other is of any type, an elimination exists, since then again all cocircuits involved live in the squint of a rank 2 minor of $\mathscr{C}$ and the situation is as shown in Fig. 5.

(iii) We are left with the case where both signed sets are of type 2.

As a first subcase we show how to eliminate h from signed sets $C^2 \cup \phi(C^1) \cup h^+$ and $C^3 \cup \phi(C^4) \cup h^-$ (see Fig. 7, left-hand side). Again we augment the corresponding pseudoline arrangement by pseudolines through C^1 and C^4 resp. C^2 and C^3. The intersection of these pseudolines is a point in some cell of the original arrangement the vertices of which are contained in $C^1 \cup C^4$ as well as in $C^2 \cup C^3$ due to compatibility of the covector of a cell and the covectors of cells on its boundary. Thus, any such vertex $\tilde{C}$ defines a signed set $\tilde{C} \cup \phi(\tilde{C})$ that is an elimination on h.

Assume now wlog, that we are to eliminate an element $e \in \mathrm{sep}(C^1, C^4)$ (see Fig. 7 right case). Again, first we consider the augmented arrangement

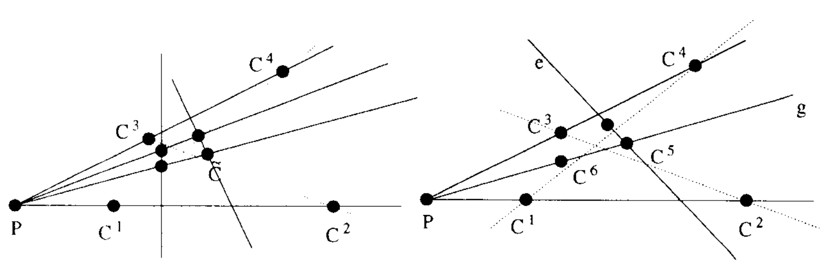

FIG. 7. Eliminations between type 2 and type 2.

and the intersection of e and the line through C^1 and C^4. "Nearby" we find a cocircuit C^5 of the original arrangement. Let g denote the coline through C^5 and P, and let C^6 be a vertex at the cell corresponding to the intersection of g and the (possibly added) line through C^2 and C^3. Now either $C^6 \cup \phi(C^5) \cup h^+$ or $C^6 \cup \phi(C^5) \cup h^-$ is in our set, depending on the relative position of C^5 and C^6 with respect to P, or $C^5 = C^6$ and $C^5 \cup \phi(C^5)$ is as desired.

The case where we have to eliminate an element between two type 2 signed sets which both are positive on h is similar.

(iv) The remaining cases can be reduced to the above by taking the inverse all over.

To complete our project we have to give an orientation of the matroid from Section 3 and show that the above construction can be utilized to find an oriented adjoint of this orientation. To describe signed cocircuits of a coextension by a coparallel for an oriented matroid is easy. Let $e \in E$ be an element of the groundset of an oriented matroid $\mathcal{M}$ and $f \notin E$ and $\mathscr{C}$ be the set of its cocircuits. Then $\{C \mid C \in \mathscr{C}\} \cup \{C \cup f^{-i} \backslash e^i \mid e^i \in C \in \mathscr{C}\} \cup \{e^+, f^+\}$ is the set of cocircuits of an oriented matroid.

Coextension by coparallel elements is a well known tool in oriented matroid theory and usually serves to link realizability results from oriented matroid theory to questions about the face lattices of polyhedra (see, e.g., [16; 5, Section 9.3; 15]). The authors of the latter article named the coextension by a single coparallel element the *Lawrence extension.*

The squint construction, however, yields that a single Lawrence extension of a pseudoline arrangement does not destroy polarity; i.e., it implies Theorem 3. We will not give a detailed proof for that theorem but will rather make it plausible by giving the arguments for the matroid D considered in Section 3.

THEOREM 3. *A coextension of an oriented matroid of rank* 3 *by a coparallel element* (*in the sense of* [5, 4.1.10]) *admits an oriented adjoint.*

Applied to our example the method is as follows. Start with an oriented adjoint $\mathcal{N}$ for the non-Desargues matroid, e.g., the oriented matroid Type 1 represented in Fig. 4. Denote the cocircuits of the adjoint that correspond to the labelled vertices in Fig. 4 by the corresponding capital letters. Enlarge the pseudoline arrangement by a set $\mathscr{L}$ of lines through P such that for any cocircuit C of $\mathcal{N}$ there is a pseudoline connecting P and C. Then P becomes a modular cocircuit in the enlarged arrangement. Let $\tilde{\mathscr{S}}$ denote the squint of that arrangement through P. Consider the restriction minor $\mathscr{S} = \tilde{\mathscr{S}} \backslash \mathscr{L}$ and let $\mathscr{C}$ denote its set of cocircuits. Now, $\mathscr{S}$ is seen to be an oriented adjoint of an orientation $\mathscr{D}$ of D, if we can identify the points of D with a subset of $\mathscr{C}$ that satisfies the conditions in Definition 3 such that the matroid underlying that Type II represented oriented matroid is D. We identify the cocircuits, leaving the (easy) verification that they together with $\mathscr{S}$ satisfy the requirements of Definition 3 to the reader. The points a, b, c, a', b', c', x, y, z on the non-Desargues plane correspond to $A \cup \phi(A)$, $B \cup \phi(B), \ldots$ with entries corresponding to $\mathscr{L}$ deleted. The point w corresponds $P \cup h^+$ and the point p' corresponds to $\phi(P) \cup h^+$ ($\mathscr{S}$-entries deleted).

By the results of Section 3 the existence of an adjoint of any adjoint of $\mathscr{D}$ fails already in the unoriented case. This answers a question of G. Ziegler (cf. [5, Exercise 7.17]):

COROLLARY 1. *The oriented matroid $\mathscr{D}$ has an adjoint, but none of its adjoints has one.*

REFERENCES

1. M. Alfter and W. Hochstättler, On pseudomodular matroids and adjoints, *Discrete Appl. Math.* **60** (1995), 3–11.
2. M. Alfter, W. Kern, and A. Wanka, On adjoints and dual matroids, *J. Combin. Theory Ser. B* **60** (1990), 208–213.
3. A. Bachem and W. Kern, Adjoints of oriented matroids, *Combinatorica* **6** (1986), 299–308.
4. G. Birkhoff, "Lattice Theory," 3rd ed., Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, 1967.
5. A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler, "Oriented Matroids," Cambridge Univ. Press, Cambridge, 1993.
6. A. Björner and L. Lovász, Pseudomodular lattices and continuous matroids, *Acta Sci. Math.* **51** (1987), 295–308.
7. R. G. Bland and M. Las Vergnas, Orientability of matroids, *J. Combin. Theory Ser. B* **25** (1978), 94–123.
8. A. L. C. Cheung, Adjoints of a geometry, *Canad. Math. Bull.* **17**, No. 3 (1974), 363–365.
9. J. Folkman and J. Lawrence, Oriented matroids, *J. Combin. Theory Ser. B* **25** (1978), 199–236.
10. B. Grünbaum, "Arrangements and Spreads," CBMS Regional Conference Series in Math. Vol. 10, Amer. Math. Soc., Providence, 1972.
11. F. Levi, Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade, *Ber. Math.-Phys. Kl. Sächs. Akad. Wiss.* **78** (1926), 256–267.
12. J. H. Mason, Geometrical realization of combinatorial geometries, *Proc. Amer. Math. Soc.* **30** (1971), 15–21.
13. J. H. Mason, Glueing matroids together: A study of Dilworth truncations and matroid analogues of exterior and symmetric powers, *in* "Algebraic Methods in Graph Theory" (L. Lovász and V. T. Sós, Eds.), Amsterdam, 1981.
14. J. G. Oxley, "Matroid Theory," Oxford Univ. Press, Oxford/New York/Tokyo, 1992.

15. J. Richter-Gebert and G. M. Ziegler, Realization spaces of 4-polytopes are universal, *Bull. Amer. Math. Soc.* **32** (1995), 403–412.
16. B. Sturmfels, Boundary complexes of convex polytopes cannot be characterized locally, *Bull. London Math. Soc.* **35** (1987), 314–326.
17. D. J. A. Welsh, "Matroid Theory," Academic Press, New York/London/San Francisco, 1976.
18. N. White (Ed.), "Theory of Matroids," Cambridge Univ. Press, Cambridge, 1986.