Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes

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0. Introduction

Let G be a reductive linear algebraic group acting rationally on an n-dimensional vector space V over a field K ([13], Chapter 1, § 1). The author has conjectured that the ring of invariants R^G of the induced representation of G on the symmetric algebra $R = K[x_1, \cdots, x_n]$ of V is Cohen-Macaulay. We shall prove this result here in the case where G is a torus, $GL(1, K)^m$. In fact, we shall show that if M is a semigroup of monomials $x_1^{h_1} \cdots x_n^{h_n}$ in the variables $x_1, \cdots, x_n$ such that $K[M] \subset K[x_1, \cdots, x_n]$ is normal (this condition is independent of the field K), then $R[M] \subset R[x_1, \cdots, x_n]$ is Cohen-Macaulay for every Cohen-Macaulay ring R. This implies the desired result. The proof depends in an essential way on a recent result concerning the shellability of (real) polytopes, Proposition 2 and its Corollary in [2]. Throughout, "polytope" means real or rational convex polytope.

We make some remarks on Cohen-Macaulay rings for the convenience of the reader unfamiliar with them. Let R be a Noetherian ring. We say that $r_1, \cdots, r_n \in R$ form an *R-sequence* if $(r_1, \cdots, r_n)R \neq R$ and for each i, $0 \leqq i \leqq n-1$, r_{i+1} is not a zerodivisor modulo $(r_1, \cdots, r_i)R$. We define the *grade* of a proper ideal I of R to be the length of any maximal R-sequence contained in I. A Noetherian ring R is *Cohen-Macaulay* if one of the following equivalent conditions holds:

(1) For every prime P of R, grade P = rank P (where rank P = Krull dim R_P).

(2) Any ideal generated by an R-sequence has no embedded primes.

(3) For any prime P, every system of parameters in R_P is an R_P-sequence.

(4) For every maximal ideal M, some system of parameters in R_M is an R_M-sequence.

If R is a finitely generated graded algebra over a field K, it turns out that R is Cohen-Macaulay if and only if R is a finitely generated graded *free* module over the polynomial subring generated by some (equivalently,

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every) homogeneous system of parameters. In this situation, if we represent R as S/I, where S is a polynomial ring over K and I is a homogeneous ideal, then it is a theorem that R is Cohen-Macaulay if and only if $dh_S S/I = \text{grade } I$. In this case, I is unmixed and grade I is the same as the rank of any minimal prime of I. See § 3 of [12] for further information.

1. Reformulation of the problem

We note that several special cases of the conjecture mentioned above were established in [12] (announced in [11]), [10] and even [4], [3], [5], [6], [16], and [17], although the point of view taken in the latter papers was different. Chow's motivation in [4] was that the result that a Segre product of polynomial rings is Cohen-Macaulay was needed for application to counting problems in the study of zeta functions. Barshay [1] needed a result of this type for the same reasons; he derived it from Chow's results [4]. The rings Barshay considered arose naturally as the rings generated over a field by those monomials whose exponents satisfy certain homogeneous linear equations [1], p. 451. The connection with our work becomes clear if we reformulate our problem:

The representation of $G = GL(1, K)^m$ can be diagonalized, and we can assume that $a = (a_1, \cdots, a_m) \in G$ acts by taking $x_j \in K[x_1, \cdots, x_n]$ to

$$a_1^{t_{1j}} \cdots a_m^{t_{mj}} x_j ,$$

$1 \leqq j \leqq n$, where the t_{ij} are the mn integers which determine the diagonal representation. R^G is therefore spanned, as a K-vector space, by those monomials $x_1^{h_1} \cdots x_n^{h_n}$ such that

$$(*) \qquad t_{i1}h_1 + \cdots + t_{in}h_n = 0 , \qquad 1 \leqq i \leqq m .$$

Conversely, if (*) is a given system of m linear homogeneous equations in n unknowns over the rationals Q we can, after clearing denominators, assume that the coefficients t_{ij} are integers, and it is evident how to write down a representation of $GL(1, K)^m$ on the 1-forms of $R = K[x_1, \cdots, x_n]$ such that the ring of invariants, R^G, will be spanned as a K-vector space by those monomials $x_1^{h_1} \cdots x_n^{h_n}$ such that $(h_1, \cdots, h_n)$ is a nonnegative integral solution for (*).

By a *monomial* in the variables $x_1, \cdots, x_n$ we shall always mean a term of the form $x_1^{h_1} \cdots x_n^{h_n}$, where $h_1, \cdots, h_n$ are nonnegative integers. We can regard, somewhat imprecisely, a given set M of monomials as a subset of any polynomial ring $R[x_1, \cdots, x_n]$ in the variables $x_1, \cdots, x_n$. Throughout, "semigroup" means semigroup with identity, and subsemigroups are assumed to contain the original identity. Thus, a semigroup of monomials in $x_1, \cdots, x_n$

must contain $1 = x_1^0 \cdots x_n^0$, and must be closed under multiplication. We shall say that a semigroup M of monomials in $x_1, \cdots, x_n$ is *full* if whenever p, p', p'' are monomials such that $pp' = p''$ and $p', p'' \in M$, then $p \in M$. This condition depends not just on the semigroup structure of M, but also on how M lies in the semigroup of all monomials. We shall say that a semigroup M of monomials is *normal* if M is finitely generated, and whenever there are monomials $p, p', p'' \in M$ and a positive integer n such that $p(p')^n = (p'')^n$, then there is a monomial $p_1 \in M$ such that $p = p_1^n$. This condition depends only on the semigroup structure of M. Clearly, full semigroups of monomials are normal if they are finitely generated, and it does turn out that full semigroups are finitely generated.

Our main result is Theorem 1 below. The preceding proposition gives the correct perspective in which to view normal subrings of $K[x_1, \cdots, x_n]$ which are generated by monomials. (A domain is *normal* if it is Noetherian and integrally closed in its fraction field.)

PROPOSITION 1. *Let M be a semigroup of monomials in the variables $x_1, \cdots, x_n$. Then the following conditions are equivalent:*

(1) *For some field K, the subring $K[M] \subset K[x_1, \cdots, x_n]$ is normal.*

(2) *M is normal.*

(3) *M is isomorphic as a semigroup to a full semigroup of monomials in a (possibly) different finite set of variables.*

(4) *M is finitely generated as a semigroup and for every integrally closed domain D, the subring $D[M] \subset D[x_1, \cdots, x_n]$ is integrally closed in its fraction field.*

The proof of Proposition 1, which is not particularly hard, is given in the next section.

THEOREM 1. *Let M be a normal semigroup of monomials in the variables $x_1, \cdots, x_n$. Then $R[M]$ is Cohen-Macaulay for every Cohen-Macaulay ring R.*

Theorem 1 yields the desired result on rings of invariants of tori at once:

COROLLARY 1. *If $G = GL(1, K)^m$ acts rationally on a finite-dimensional K-vector space V, then R^G, the ring of invariants of the induced action of G on the symmetric algebra R of V, is Cohen-Macaulay.*

Proof. The set M of monomials whose exponents satisfy a system of equations like the system (*) discussed above is evidently a full semigroup of monomials.

Almost all of the rest of this paper is devoted to proving Theorem 1.

Remark 1. The corollary is a nontrivial theorem even if $m = 1$. One can reduce to the case where $G = GL(1, K)$ acts on $K[x_1, \cdots, x_r, y_1, \cdots, y_s]$ by $a: x_i \mapsto a x_i$, $1 \leqq i \leqq r$, $a: y_j \mapsto a^{-1} y_j$, $1 \leqq j \leqq s$. See [12], proof of Proposition 8. In this case, $R^G = K[x_i y_j]_{i,j}$, the Segre product of $K[x_1, \cdots, x_r]$ and $K[y_1, \cdots, y_s]$. Then $R^G \cong K[u_{ij}]/I$, where the u_{ij} are rs new indeterminates and I is the ideal generated by the 2 by 2 minors of the matrix $[u_{ij}]$. The result in this case was first obtained by Chow [4] and Sharpe [16], [17]. New proofs follow from [12], Corollary 4, [10], Corollary 3.13, and, of course, the main result here.

Remark 2. Chow's methods cannot be used to get the general case of Corollary 1: the rings R^G considered are not, in general, obtainable by iterated formation of Segre products (even if we allow the trick, used in [1], of changing the grading at each stage) and/or adjunction of indeterminates. In fact we shall associate a polytope with each R^G: under this association, adjoining an indeterminate corresponds to forming the cone over the polytope, and taking the Segre product of two of the rings (even with altered gradings) corresponds (up to isomorphism) to taking the Cartesian product of their polytopes. Even in the plane, we already miss the convex n-gons for $n \geqq 5$. We return to this point in § 4: see Remark 9. It would be interesting if our results here could be unified with Chow's results.

Remark 3. When K is finite, we consider only those elements of R which are invariant under even non-rational points of $GL(1, K)^m$, so that our treatment is uniform in the field K. If, in the case where K is finite, we regard $G = GL(1, K)^m$ instead as a finite discrete group, the theorem is still true, because the order of G is not divisible by the characteristic of K: see Proposition 13 of [12]. (Of course, R^G is much larger for this G.)

Remark 4. Let $\mathbf{N}$ be the class of Noetherian rings which are normal domains, $\mathbf{C}$ the class of those which are Cohen-Macaulay, and $\mathbf{P}$ the class of those which have principal ideals unmixed. Then $\mathbf{N} \cup \mathbf{C} \subset \mathbf{P}$ but $\mathbf{N}$ and $\mathbf{C}$ are incomparable. Let $\mathbf{M}$ be the class of rings finitely generated over the field K by monomials. The surprising result $\mathbf{N} \cap \mathbf{M} \subset \mathbf{C}$ might lead one to hope that $\mathbf{P} \cap \mathbf{M} = \mathbf{C} \cap \mathbf{M}$. This is false: all the inclusions

$$\mathbf{N} \cap \mathbf{M} \subset \mathbf{C} \cap \mathbf{M} \subset \mathbf{P} \cap \mathbf{M} \subset \mathbf{M}$$

are proper. $A = K[x_1^2, x_1^3, x_2, x_1 x_2]$ is in $\mathbf{M}$ but not in $\mathbf{P}$. Note the relation $x_1 x_2 (x_1^2) = x_1^3 (x_2)$ on the system of parameters x_1^2, x_2. Let $B = K[M]$, where $M =$

$$\{x_1^{h_1} x_2^{h_2} x_3^{h_3} x_4^{h_4} \colon h_1, h_2, h_3, h_4 \in Z_+,\ h_1 + h_2 = h_3 + h_4,\ h_1, h_2 \neq 1\}\,.$$

Z_+ is the nonnegative integers. M is easily seen to be finitely generated. Then B is in $\mathbf{P} \cap \mathbf{M}$ but not in $\mathbf{C}$. To see this, first note that the integral closure B' of B is $K[M']$, where $M' =$

$$\{x_1^{h_1}x_2^{h_2}x_3^{h_3}x_4^{h_4}: h_1, h_2, h_3, h_4 \in Z_+, \ h_1 + h_2 = h_3 + h_4\} .$$

A system of parameters for $K[M]$ is $x_1^2x_3^2, \ x_2^2x_4^2, \ x_1^2x_4^2 + x_2^2x_3^2$. This system of parameters is not an R-sequence, for we have the relation

$$x_1^3x_2^3x_3^3x_4^3(x_1^2x_4^2 + x_2^2x_3^2) = x_1^3x_2^3x_3x_4^5(x_1^2x_3^2) + x_1^3x_2^3x_3^5x_4(x_2^2x_4^2) .$$

That $B \in \mathbf{P}$ may be seen as follows: if not, there are elements $f, \ g \in B$ such that $fB: gB$ is a rank 2 or 3 prime of B. Then $g/f \in B' - B$, for if $g/f \in B'$, then fB' is mixed. Let J_i, $i = 1, 2$, be the rank one prime of B generated by the monomials which involve x_i. Since $g/f \in B' - B$, it involves a monomial linear in x_1 or x_2. If g/f contains a monomial linear in x_i but not a monomial linear in x_{3-i}, then $fB: gB = J_i$. If g/f contains a monomial linear in x_1 and a monomial linear in x_2, then $fB: gB = J_1 \cap J_2$.

Finally, note that $K[x_1^2, x_1^3]$ is in $\mathbf{C}$ but not in $\mathbf{N}$.

2. Normal rings generated by monomials

Our goal in this section is the proof of Proposition 1. We first introduce some notation to which we shall adhere throughout. As before, Q denotes the rational numbers. $Q_+ = \{q \in Q: q \geqq 0\}$. We define Q_+^n, the first orthant in Q^n, as $(Q_+)^n$. Z is the integers. If $S \subset Q^n$, we write S_+ for $S \cap Q_+^n$. If $x_1, \cdots, x_n$ are indeterminates and $h = (h_1, \cdots, h_n) \in Z_+^n$, we write x^h for $x_1^{h_1} \cdots x_n^{h_n}$. This gives an isomorphism of the semigroup Z_+^n under addition onto the semigroup of monomials in the variables $x_1, \cdots, x_n$. We denote the inverse isomorphism by $\log_x$ or, simply, log. We shall say that a subsemigroup S of Q_+^n is *full* if whenever $s, \ s' \in S$ and $s - s' \in Q_+^n$, then $s - s' \in S$. If it happens that $S \subset Z_+^n$, we can replace Q_+^n by Z_+^n in this definition without changing the meaning. Thus, M is a full semigroup of monomials if and only if $\log_x M$ is a full subsemigroup of Z_+^n. We also make the obvious definition of *normal* for additive semigroups, by switching from multiplicative to additive notation. Thus, M is normal if and only if $\log M$ is normal.

Let H be an additive, torsion-free, cancellative semigroup. Let $Z(H)$ denote the universal abelian group generated by H. Then $Q \otimes_Z Z(H)$ is, in an obvious sense, the universal Q-vector space spanned by H. We shall denote this vector space by $Q(H)$ and we shall denote by $Q_+(H)$ the smallest subsemigroup of $Q(H)$ which contains H and is closed under multiplication by elements of Q_+. It is easy to see that H is normal if and only if $Q_+(H) \cap Z(H) = H$. Note that if H is a subsemigroup of a Q-vector space W, then

$\phi: H \subset W$ extends uniquely to an isomorphism ϕ' of $Q(H)$ with the span of H in W, and that ϕ' induces natural isomorphisms of $Z(H)$ with $H - H \subset W$ and of $Q_+(H)$ with the set of Q_+-linear combinations of elements of H in W (which is identical with the set $\{h/n: h \in H,\ n \in Z_+ - \{0\}\}$). Given an embedding $H \subset W$ we shall identify $Q(H)$, $Z(H)$, and $Q_+(H)$ with their uniquely isomorphic counterparts in W.

We now show that (1) $\Rightarrow$ (2) in Proposition 1. Assume (1). Since $K[M]$ is a graded Noetherian K-algebra, we can choose a finite set of monomials $p_1, \cdots, p_k \in M$ which generate the irrelevant maximal ideal of $K[M]$. It follows that $K[M] = K[p_1, \cdots, p_k]$ (see Lemma 5.7 and its proof in [8]), and hence that M is generated by $p_1, \cdots, p_k$. Now suppose that $p, p', p'' \in M$ and that $p(p')^n = (p'')^n$. Then $(p''/p')^n = p$, so that p''/p' is in the integral closure of $K[M]$ and hence is in $K[M]$. Since $K[M]$ is a free module over K with M as basis, $p''/p' \in M$, and we let $p_1 = p''/p'$.

We next show that (2) $\Rightarrow$ (3). We identify $W = Q(\log_z M)$, $Z(\log_z M)$, and $Q_+(\log_z M)$ with their counterparts in Q^n. Let $W^* = \mathrm{Hom}_Q(W, Q)$ be the vector space dual of W. Let

$$U = \{w^* \in W^*: w^*(w_i) \geqq 0,\ 1 \leqq i \leqq k\}\,.$$

Since U is a finite intersection of half-spaces in W^*, there are finitely many functionals $w_1^*, \cdots, w_r^* \in U$ such that U is the Q_+-span of $w_1^*, \cdots, w_r^*$. Since $Q_+(\log_z M)$ is the Q_+-span of $w_1, \cdots, w_k$, it is a finite intersection of half-spaces, and it follows that

$$Q_+(\log_z M) = \bigcap_j \{w \in W: w_j^*(w) \geqq 0\}\,.$$

By replacing the w_j^* by suitable positive integer multiples, we can assume that $w_j^*(w_i) \in Z_+$ for all i, j. Let $T: W \to Q^r$ be defined by setting $T = (w_1^*, \cdots, w_r^*)$. By construction, $T(\log_z M) \subset Z_+^r$. To complete the proof, we need only show that T is one-one on $\log_z M$ and that $T(\log_z M)$ is a full subsemigroup of Z_+^r. Suppose that $h \neq h'$ in $\log_z M$: say their i^{th} components disagree. Let w^* be the linear functional on W obtained by projection on the i^{th} component in Q^n. Since w^* is nonnegative on $w_1, \cdots, w_k$, we have that $w^* \in U$. Hence, we can write $w^* = \sum_j q_j w_j^*$, where the $q_j \in Q_+$. Since $w^*(h) \neq w^*(h')$, we can choose j such that $w_j^*(h) \neq w_j^*(h')$, and so $T(h) \neq T(h')$.

It remains to show that $T(\log_z M)$ is full. Suppose that $T(h) - T(h') \in Z_+^r$. Then each of $w_1^*, \cdots, w_r^*$ is nonnegative on $h - h'$, so that $h - h' \in Q_+(\log_z M)$. Since $h, h' \in \log_z M$ and $\log_z M$ is normal, $h - h' \in \log_z M$, as required.

We shall next show that (3) $\Rightarrow$ (4). Let K be the fraction field of D. Since

$$D[M[= K[M] \cap D[x_1, \cdots, x_n] ,$$

it suffices to show that M is finitely generated and that for each field K, $K[M]$ is integrally closed in its fraction field. These facts follow from Lemmas 1 and 2 below. We first need to recall the definition of a Reynolds operator (see [12], Propositions 9-12). If $A \subset R$ are rings (where "ring" always means commutative, associative ring with identity), by a *Reynolds operator* from R to A we mean an A-module homomorphism $\rho: R \to A$ such that $\rho(a) = a$ for each $a \in A$. Thus, to give a Reynolds operator $\rho: R \to A$ is the same as to give an A-module complement E for A in R.

LEMMA 1. *Let M be a full semigroup of monomials in the variables $x_1, \cdots, x_n$. Then for each ring R there is a Reynolds operator from $R[x_1, \cdots x_n]$ to $R[M]$. Hence, if R is Noetherian, so is $R[M]$. It follows that M is finitely generated as a semigroup.*

Proof. $R[M]$ is a free R-module with M as basis. As a complement for $R[M]$ we can take the (free) R-submodule E of $R[x_1, \cdots, x_n]$ spanned by the monomials not in M. E is an $R[M]$-submodule of $R[x_1, \cdots, x_n]$ precisely because the product of a monomial not in M and one in M is not in M (M is full). That $R[M]$ is Noetherian if R is Noetherian now follows from ([12], Proposition 10, part 3). In particular, when $R = K$, a field, $K[M]$ is Noetherian, and it follows just as in the proof of (1) $\Rightarrow$ (2) that M must be finitely generated.

LEMMA 2. *Let M be a full semigroup of monomials in the variables $x_1, \cdots, x_n$ and let K be a field. Let F be the fraction field of $K[M]$. Then $F \cap K[x_1, \cdots, x_n] = K[M]$. Hence, $K[M]$ is integrally closed in F.*

Proof. Suppose $g \in (F \cap K[x_1, \cdots, x_n]) - K[M]$. We can assume without loss of generality that none of the monomials occurring in g is in M. Then $g = f/f'$, where f, f' are in $K[M] - \{0\}$. Let p be any monomial which occurs in $f = gf'$. Then p can be written $p''p'$, where p'' and p' are monomials which occur in g and f', respectively. Since p, $p' \in M$ and M is full, $p'' \in M$, a contradiction.

The proof of (3) $\Rightarrow$ (4) is now complete once we observe that if M and M' are semigroups of monomials which are isomorphic as semigroups, then $R[M] \cong R[M']$ as R-algebras for any ring R. This is a consequence of the fact that $R[M]$ is the semigroup ring of M with coefficients in R.

Since (4) $\Rightarrow$ (1) is obvious, we have proved Proposition 1.

Note that if M is full it still need not be true that $K[M]$ is integrally closed in $K[x_1, \cdots, x_n]$: e.g. let $n = 1$ and let $M = \{x_1^{2k}: k \in Z_+\}$. We show next that this is the only type of example of what can go wrong.

If M is a full semigroup of monomials (respectively, if S is a full subsemigroup of Z_+^n), let the *expansion* of M (of S), M^e (S^e), be the set of monomials p (elements $s \in Z_+^n$) such that $p^k \in M$ $(ks \in S)$ for some integer $k>0$. It is easy to see that since M (S) is full, so is M^e (S^e). If $M = M^e$ $(S = S^e)$ we say that M (S) is *expanded*. Clearly, the integral closure of $K[M]$ in $K[x_1, \cdots, x_n]$ contains $K[M^e]$, so that $K[M]$ cannot be integrally closed in $K[x_1, \cdots, x_n]$ unless M is expanded. On the other hand, if M is expanded, $K[M]$ is, indeed, integrally closed in $K[x_1, \cdots, x_n]$. To prove this we first observe:

LEMMA 3. *$S \subset Z_+^n$ is an expanded ($\Rightarrow$ full) subsemigroup if and only if one of the following two equivalent conditions holds:*

(1) *$S = W \cap Z_+^n$, where W is a vector subspace of Q^n.*

(2) *S is the set of solutions in the nonnegative integers of a system of homogeneous linear equations with rational coefficients.*

If M is an expanded semigroup of monomials then $K[M]$ is the ring of invariants of a torus as in Corollary 1. *Conversely, every ring of invariants of a torus of the type described in Corollary* 1 *is K-isomorphic to $K[M]$ for some expanded semigroup of monomials M.*

Proof. That $W \cap Z_+^n$ is expanded is clear. On the other hand, if S is expanded, let W be the span of S. It is then easy to see that $S = W \cap Z_+^n$. The equivalence of (1) and (2) is clear. The remaining statements of the lemma now follow from characterization (2) of expanded subsemigroups of Z_+^n and our reformulation in § 1 of Corollary 1 in terms of monomials whose exponents satisfy a system (*) of equations.

Remark 5. If M is an expanded semigroup of monomials, so that $K[M] = R^G$, the Reynolds operator from $R = K[x_1, \cdots, x_n]$ to $K[M]$ described earlier is the same one given by invariant theory (see [8], Definition 5.3 or [13], Definition 1.5).

LEMMA 4. *If M is a full semigroup of monomials in $x_1, \cdots, x_n$ and K is a field, the integral closure of $K[M]$ in $K[x_1, \cdots, x_n]$ is $K[M^e]$, and so $K[M]$ is integrally closed in $K[x_1, \cdots, x_n]$ if and only if M is expanded.*

Proof. We need only show that $K[M]$ is integrally closed in $R = K[x_1, \cdots x_n]$ if M is expanded. But then $K[M]$ is R^G, where G is a torus acting rationally. If $f \in R$ is integral over R^G (or even algebraic) then the orbit of f under G is finite: it is contained in the set of roots in R of the equation of dependence for f. The stabilizer of f in G is therefore a closed subgroup of finite index: a contradiction, since G is connected.

We recall Proposition 12 of [12]:

LEMMA 5. *Suppose that R is integral over a subring A and that there is a Reynolds operator from R to A. Then if R is Cohen-Macaulay, A is Cohen-Macaulay.*

From this lemma we can show that the question of whether $K[M]$ is Cohen-Macaulay for a normal semigroup M of monomials is determined by the semigroup structure of $Q_+(\log M)$. To be precise:

LEMMA 6. *If M, M' are normal semigroups of monomials and $Q_+(\log M)$ and $Q_+(\log M')$ are isomorphic as semigroups, then $K[M]$ is Cohen-Macaulay if and only if $K[M']$ is Cohen-Macaulay.*

In particular, if M is any full subsemigroup then $K[M]$ is Cohen-Macaulay if and only if $K[M^e]$ is Cohen-Macaulay. (For we have that $Q_+(\log M) = Q_+(\log M^e)$).

Proof. We shall assume that $K[M']$ is Cohen-Macaulay and prove that $K[M]$ is Cohen-Macaulay. By Proposition 1, we can assume that M' is a full subsemigroup of monomials in, say, $y_1, \cdots, y_r$, and that M is a semigroup in $x_1, \cdots, x_n$. Suppose that $\phi: Q_+(\log_x M) \to Q_+(\log_y M')$ is the isomorphism. By replacing ϕ by a suitable positive integer multiple we can assume without loss of generality that $\phi(\log_x M) \subset \log_y M'$. Note that ϕ extends uniquely to an isomorphism of $Q(\log_x M)$ with $Q(\log_y M')$, which we shall also denote by ϕ. We claim that $\phi(\log_x M)$ is a full subsemigroup of Z^r_+. For suppose $h, h' \in \log_x M$ and $\phi(h) - \phi(h') \in Z^r_+$. Then $\phi(h - h') \in (\log_y M' - \log_y M) \cap Z^r_+$, and since $\log_y M'$ is full, $\phi(h - h') \in \log_y M'$. Since ϕ is an isomorphism of $Q_+(\log_x M)$ with $Q_+(\log_y M')$ and is one-one on all of $Q(\log_x M)$, we must have that $h - h' \in Q_+(\log_x M)$. But $h - h' \in \log_x M - \log_x M$ and $\log_x M$ is normal. It follows that $h - h' \in \log_x M$, as required.

We replace M by $M_1 = y^H$, where $H = \phi(\log_x M)$. M_1 is a full semigroup of monomials in $y_1, \cdots, y_r$ which is isomorphic to M. We want to show that $K[M_1]$ is Cohen-Macaulay. Since M_1 is full, we have a Reynolds operator from $K[y_1, \cdots, y_r]$ to $K[M_1]$, and hence, by restriction, from $K[M']$ to $K[M_1]$. By hypothesis, $K[M']$ is Cohen-Macaulay. By Lemma 5, to complete the proof we need only show that $K[M']$ is integral over $K[M_1]$. But since each element of $Q_+(\log_y M') = Q_+(\log_y M_1)$ has a positive integer multiple in $\log_y M_1$, each element of M' has a power in M_1. This completes the proof.

Remark 6. The general case of Theorem 1 for R a field therefore follows from Corollary 1 of Theorem 1, by Lemma 6 above and Lemma 3.

3. Polytopes of ideals

Let P be a (real or rational) polytope of dimension d. We refer to the $(d-1)$-faces of P as the *facets* of P. We denote by ∂P the union of the facets of P. Let $L(P)$ be the set of subsets of ∂P which are unions of faces. $L(P)$ is a lattice under $\bigcap$, $\bigcup$. Every element B of $L(P)$ has a unique irredundant representation as a nonempty union of faces: we call these faces the components of B. (We regard the empty set $\varnothing$ itself as a face of dimension -1.) The dimension of an element of $L(P)$ is the largest dimension of any component. We regard two polytopes as isomorphic if their lattices are isomorphic. Note that the dimension function is recoverable from the lattice structure.

By a *polytope of ideals* in a ring R we mean a family $\mathcal{I}$ of proper ideals of R which form a lattice under $+$, $\bigcap$ together with a lattice isomorphism α of $\mathcal{I}$ onto $L(P)$ for some polytope P (α reverses order) such that for some constant $c \in Z$ and for every ideal $I \in \mathcal{I}$,

$$\dim I - \dim \alpha(I) = c \; .$$

Here, $\dim I$ is the Krull dimension of R/I. We refer to $\alpha: \mathcal{I} \to L(P)$ as *giving* the polytope of ideals. In this situation we agree to denote $\alpha^{-1}: L(P) \to \mathcal{I}$ by β. (We shall even speak of a "P of ideals", e.g. a "cube of ideals" or a "simplex of ideals".) We refer to the ideals $\beta(B)$, where B is a face of P, as the face ideals of $\mathcal{I}$. Every element of $\mathcal{I}$ has a unique irredundant decomposition as an intersection of face ideals. It is easy to see that if $\dim I - \dim \alpha(I)$ is constant on the face ideals, then it is constant on $\mathcal{I}$: $\dim(I_1 \cap I_2) = \max\{\dim I_1, \dim I_2\}$. We note that a polytope of ideals is a distributive lattice, for $L(P)$ is distributive. We also note that $L(P)$ for a given rational polytope P is the same as $L(P)$ for the corresponding real polytope (i.e. the real polytope with the same vertices as P which is, moreover, the closure of P). For this reason we shall only state the theorems below for real polytopes, although in the applications all the polytopes we consider are rational.

We recall that an ideal I of a ring R is *semiregular* if R/I is Cohen-Macaulay. To prove Theorem 1 we shall need:

THEOREM 2. *Let $\alpha: \mathcal{I} \to L(P)$ give a polytope of homogeneous ideals in a finitely generated graded K-algebra R. Suppose that the face ideals of $\mathcal{I}$ are semiregular. Then $\beta(\partial P)$ is semiregular.*

Remark 7. This seemingly strange result turns out to be quite useful in practice: one is trying to prove that the rings in a certain class $\mathbf{R}$ of finitely generated graded K-algebras are Cohen-Macaulay. It then happens that given $R \in \mathbf{R}$ one can choose a form of positive degree $f \in R$, not a zerodivisor, such that either (1) R/fR is again in the class, or (2) fR is $\beta(\partial P)$ for a polytope of homogeneous ideals given by $\alpha: \mathcal{I} \to L(P)$ such that for each face ideal I of $\mathcal{I}$, R/I is in the class. It then follows by Noetherian induction, Theorem 2, and Lemma 8 of § 4 that all the rings $R \in \mathbf{R}$ are Cohen-Macaulay. This is the idea of the proof of the main result of [12] (where only the case when P is a 1-simplex was needed), and also, of the main result of [10] (where, although this point of view was not taken, it was necessary to consider arbitrary n-simplices of ideals: cf. Proposition 1.4 of [10]). Here, we shall actually need Theorem 2 at least for all rational polytopes P. The inductive step here will be a little more complicated than described above: we need to perturb things a bit by means of Lemma 6 to set up the right situation.

In order to prove Theorem 2 we introduce the notion of a "constructible" subset of ∂P. We prove the result recursively for all constructible subsets of ∂P. Finally, we make use of the results of [2] to show that ∂P is itself constructible.

We define the *constructible* sets in $L(P)$ recursively thus:

(i) Every face in $L(P)$ is constructible.

(ii) If B_1 and B_2 are constructible sets of dimension k and $B_1 \cap B_2$ is a constructible set of dimension $k - 1$, then $B_1 \cup B_2$ is constructible.

THEOREM 2°. *Let $\alpha: \mathcal{I} \to L(P)$ give a polytope of ideals in a Noetherian ring R and let B be a constructible subset of ∂P. Suppose that one of the following conditions holds*:

(a) *R is local and the face ideals of $\mathcal{I}$ are semiregular.*

(b) *R is a finitely generated graded K-algebra for some field K, the ideals of $\mathcal{I}$ are homogeneous, and the face ideals of $\mathcal{I}$ are semiregular.*

Then $\beta(B)$ is semiregular.

Moreover, ∂P is constructible, so that we can always apply this result with $B = \partial P$.

Proof. We first note that case (b) follows from case (a). For localizing at the irrelevant maximal ideal will not affect whether a finitely generated graded K-algebra is Cohen-Macaulay [12], Proposition 19, nor will it affect the dimension. Moreover, the map from the lattice of homogeneous ideals of the graded ring to their expansions in the localization will be a lattice isomorphism.

To handle case (a), we observe that we can replace R by its completion without affecting the relevant issues. Hence, we may assume that $R = R'/I$, where R' is a regular local ring. Let $\phi: R' \to R$. Then it is easy to see that we need only prove the result for $\phi^{-1}(\mathcal{I})$ in R': hence, we may assume without loss of generality that R is a regular local ring. But then an ideal J of R is semiregular if and only if it is perfect, i.e. $dh_R\, R/J = \text{grade}\, J$, where dh denotes homological (projective) dimension and the *grade* of a proper ideal of a Noetherian ring R is the length of the longest R-sequence it contains. This follows, for example, from Theorem 28.2 of [14]. Now, utilizing the recursive definition of constructible set, we need only make the following observations:

(i) Each face ideal is perfect, by hypothesis.

(ii) If I_1, I_2 are perfect ideals of $\mathcal{I}$ of the same dimension, k, and $I_1 + I_2$ is a perfect ideal of dimension $k - 1$, then $I_1 \cap I_2$ is perfect. (This follows from Proposition 18 of [12] once it is noted that grade $I = \dim R - \dim I$ for any proper ideal I of a regular local ring.)

These two observations make it evident that the ideal corresponding to any constructible subset of ∂P is perfect. We are left with the question of the constructibility of ∂P. But this is immediate from the recursive definition of shellability (Definition 1) used in [2] and the Corollary to Proposition 2 of [2], which asserts that the boundary ∂P of any (convex) real polytope is shellable.

This completes the proof of Theorem 2° and hence of Theorem 2.

Remark 8. It is of some interest to determine the class of constructible sets $B \in L(P)$ in some nonrecursive fashion. Even better would be a purely topological description of the largest class of sets $B \in L(P)$ for which it can be inferred that $\beta(B)$ must be semiregular under the hypotheses of Theorem 2°. It is not clear that membership in this class is a purely topological matter. We do note the following topological facts about constructible sets:

(i) A constructible set is a union of faces of the same dimension.

(ii) In dimensions -1, 0, 1 we can characterize the constructible sets thus: in dimension -1, $\varnothing$ is the only set and the only constructible set; in dimension 0, every set of vertices is constructible; and in dimension 1, a union of edges is constructible if and only if it is connected.

(iii) By using the Mayer-Vietoris sequence in homology, the fact that the union of two simply connected complexes is simply connected if the intersection is connected, the Hurewicz theorem, and other standard facts from algebraic topology, it is not difficult to show that a constructible set B of dimension $k \geqq 2$ has the following property:

If C is a subcomplex of B of dimension i, $-1 \leqq i \leqq k-3$, then $\pi_j(B-C) = 0$, $1 \leqq j \leqq k-i-2$, and $B-C$ is connected if $\dim C \leqq k-2$. Here, π_j is the j^{th} homotopy group.

It is natural to ask whether, under some reasonable additional hypothesis, conditions (i), (ii), and (iii) characterize the sets B in $L(P)$ such that $\beta(B)$ is semiregular (under the hypothesis of Theorem 2°). The author does not know the answer even if $\mathcal{J}$ is the simplex of radical ideals in $K[x_1, \cdots, x_n]$ whose face ideals are the primes generated by the nonempty subsets of $\{x_1, \cdots, x_n\}$ (or even whether, given that B is a subcomplex homeomorphic to a k-cell for some k, $\beta(B)$ must be semiregular).

4. The main result

In this section, we shall prove Theorem 1. We shall first prove:

THEOREM 1°. *Let M be an expanded semigroup of monomials in $x_1, \cdots, x_n$, and let K be a field. Then $K[M]$ is Cohen-Macaulay.*

The general case of Theorem 1 when $R = K$, a field, then follows from Lemma 6 and we can use the results of [7] to obtain Theorem 1 for arbitrary Cohen-Macaulay rings R once the case where R is a field is known.

We shall prove Theorem 1° by induction on n, but we need some preliminaries. Let M be an expanded semigroup of monomials in $x_1, \cdots, x_n$. Let $R = K[x_1, \cdots, x_n]$. Let $\mathcal{J}$ be the $(n-1)$-simplex of ideals in R whose face ideals are the ideals generated by the nonempty subsets of $\{x_1, \cdots, x_n\}$. $\mathcal{J}$ may also be described as the set of ideals in $K[x_1, \cdots, x_n]$ which can be generated by a nonempty family of square-free monomials in $x_1, \cdots, x_n$. Let $\mathcal{I} = \mathcal{J} \cap K[M]$, i.e.

$$\mathcal{I} = \{J \cap K[M] \colon J \in \mathcal{J}\} .$$

We shall want to show that $\mathcal{I}$ is a polytope of ideals in $K[M]$. We first note that if $J \in \mathcal{J}$, then $J \cap K[M]$ is generated, in fact, spanned, by those monomials in M which are multiples (in R) of the generating monomials of J. It easily follows that $J \cap K[M] = \rho(J)$, where ρ is the Reynolds operator from R to $K[M]$ described in the proof of Lemma 1. In other words, $\mathcal{I} = \rho(\mathcal{J})$. It is clear that ρ preserves sums, while from the fact that $\rho(J) = J \cap K[M]$ it is clear that ρ preserves intersections. Hence, ρ gives a lattice homomorphism of $\mathcal{J}$ onto $\mathcal{I}$. Thus, $\mathcal{I}$ is a distributive lattice, and is generated as a lattice by the images of $x_1R, \cdots, x_nR$.

We next want to associate a polytope $P = P(M)$ with M (it will turn out that $\mathcal{I}$ is a "P of ideals"). Let X be the hyperplane in Q^n defined by the equation $t_1 + \cdots + t_n = 1$. Let $P = P(M) = X \cap Q_+(\log_x M)$. P is a bounded intersection of half-spaces, hence, a polytope in Q^n. Note that, up to isomorphism, P is determined by the semigroup structure of $S = Q_+(\log_x M)$. In fact, it is clear that each ray emanating from the origin and lying in S meets P in precisely one point. It follows that $L(P)$ is isomorphic to the lattice L of nonempty unions of nonempty faces of the polyhedral set S. (Of course, in passing from $L(P)$ to L dimensions are increased by 1.) It is evident that the lattice L depends only on the semigroup structure of $S = Q_+(\log_x M)$: it is the same as the lattice of nonempty unions of nonempty faces of $Q_+(\log_x M)$ regarded as a polyhedral set in $Q(\log_x M)$. Thus, it makes sense to write $P = P(S)$. Note that we could have defined P as the intersection of S with any hyperplane $q_1t_1 + \cdots + q_nt_n = 1$ $(q_i \in Q)$ such that the linear functional $\sum_i q_it_i$ is positive on $S - \{0\}$. In fact, the linear functionals $\sum_i q_it_i$ on Q^n which are positive on $S - \{0\}$ correspond to the choices of grading for $K[M]$. (Two functionals give the same grading if they agree on S. One must take a suitable positive integer multiple to get the grading function to have integer values. We think of gradings which are rational multiples of one another as essentially the same.) Thus, we can think of the different polytopes P obtained from S by various hyperplane sections as corresponding to various gradings on $K[M]$. We can therefore paraphrase some of our observations by saying that the isomorphism class of the polytope is independent of the grading.

Remark 9. It is easy to see that if we form the Segre product of $K[M]$ and $K[M']$ with respect to gradings d and d' and grade the Segre product by $d + d'$, the new polytope is the Cartesian product of the original ones. It is also easy to see that if we adjoin a new indeterminate x to $K[M]$ the new polytope (corresponding to the semigroup

$$\{px^j: p \in M,\ \ j \in Z_+\})$$

is the cone over P. This completes our justification of Remark 2 in § 1.

Remark 10. It is not difficult to show that every rational polytope P is isomorphic to $P(M)$ for some expanded semigroup M.

If u is a subset of $\{1, \cdots, n\}$, let

$$I_u = (\{x_i: i \in u\}R) \cap K[M]$$

which, if $u \neq \varnothing$, is in $\mathcal{I}$. $(I_\varnothing = (0).)$ Let M_u be the set of monomials in the variables

$$\{x_i\colon i \in \{1, \cdots, n\} - u\}\ .$$

Let X_u be the set of points $(q_1, \cdots, q_n)$ in Q^n such that $q_i = 0$ if $i \in u$. Then $\log_x M_u = \mathbb{Z}^n \cap X_u$. If $n(u)$ is the number of elements of $\{1, \cdots, n\}$ not in u, then X_u can be naturally identified with $Q^{n(u)}$, and we can think of the log function for the polynomial ring $K[M_u]$ as taking values in X_u instead of in $Q^{n(u)}$.

LEMMA 7. *Let M be an expanded semigroup of monomials in the variables $x_1, \cdots, x_n$. Then for each $u \subset \{1, \cdots, n\}$, $M \cap M_u$ is an expanded semigroup of monomials in the variables*

$$\{x_i\colon i \in \{1, \cdots, n\} - u\}\ .$$

The faces of $P = P(M)$ are precisely the sets $P \cap X_u$. (P itself is $P \cap X_\varnothing$.) The facets of P lie among the sets $P \cap X_{\{i\}}$, $1 \leqq i \leqq n$.

For each u, $K[M]/I_u \cong K[M \cap M_u]$, and $P(M \cap M_u)$, regarded as a subset of X_u, is $P(M) \cap X_u$.

Moreover, for each $u \subset \{1, \cdots, n\}$,

$$\dim I_u = \dim K[M]/I_u = \dim (P \cap X_u) + 1\ .$$

In particular, $\dim K[M] = \dim S = \dim P + 1$.

Proof. It is obvious that $M \cap M_u$ is expanded if M is. The statements about the faces and facets follow from the fact that $S = W \cap Q^n_+$, where W is a vector space. I_u is spanned as a K-vector space by those monomials in M which involve an x_i for some $i \in u$. It is consequently clear that $K[M]/I_u \cong K[M \cap M_u]$. What is more, it is obvious that

$$\begin{aligned} X \cap Q_+(\log_x (M \cap M_u)) &= X \cap Q_+(\log_x M) \cap Q_+(\log_x M_u) \\ &= P \cap (X_u)_+ = P \cap X_u\ . \end{aligned}$$

The first statement about dimension follows from the observations above and the second statement about dimension. To see that $\dim K[M]$ (Krull dimension, of course) $= \dim S$, note that if $s_1, \cdots, s_k$ is a basis for the vector space $S - S$ consisting of elements of $\log_x M$, so that $k = \dim S$, then $x^{s_1}, \cdots, x^{s_k}$ are algebraically independent elements of $K[M]$ which form a transcendence basis for its fraction field.

PROPOSITION 2. *Let M be an expanded semigroup of monomials in $x_1, \cdots, x_n$. Assume that each of the x_i actually occurs in some element of M. Define a map α from $\mathcal{I}$ to the set of subsets of $P = P(M)$ by*

$$\alpha(I) = P - Q_+(\log_x (I \cap M))\ .$$

Then α is a lattice isomorphism of $\mathcal{I}$ with $L(P)$ such that for each $u \subset$

$\{1, \cdots, n\}$, $\alpha(I_u) = P \cap X_u$. Moreover, for each $I \in \mathcal{I}$, $\dim I = \dim \alpha(I) + 1$. Thus, $\alpha: \mathcal{I} \to L(P)$ gives $\mathcal{I}$ as a polytope of ideals.

Proof. Each $I \in \mathcal{I}$ is spanned as a K-vector space by $I \cap M$. It is clear then that $\alpha_1: I \mapsto I \cap M$ is an injection of $\mathcal{I}$ into the set of subsets of M which takes $+$ to $\bigcup$ and $\bigcap$ to $\bigcap$. Since $\log_x$ injects M into Q_+^n, $\log_x \alpha_1$ gives an injection of $\mathcal{I}$ into the subsets of Q_+^n which takes $+$ to $\bigcup$ and $\bigcap$ to $\bigcap$. It is easy to see that

$$Q_+(\log_x (I \cap M)) \cap Z_+^n = \log_x (I \cap M) .$$

It follows at once that $Q_+ \log_x \alpha_1$ is a lattice isomorphism of $\mathcal{I}$ into the subsets of Q_+^n which are unions of rays. Since each element of $\log_x (I \cap M)$ is in $S - \{0\}$, each of these rays is determined by its point of intersection with X (the hyperplane $t_1 + \cdots + t_n = 1$), which is the same as its point of intersection with P. Hence, $\alpha': I \mapsto P \cap (Q_+ \log_x (I \cap M))$ is an injective lattice homomorphism of $\mathcal{I}$, $+$, $\bigcap$ into the set of subsets of P, $\bigcup$, $\bigcap$. If we compose α' with the map which takes each subset of P to its complement, we get α. A routine computation shows that $\alpha(I_u) = P \cap X_u$ for each $u \subset \{1, \cdots, n\}$. It follows that each face P is in the image of α. Hence, $\operatorname{Im} \alpha \supset L(P)$. But $\mathcal{I}$ is a distributive lattice generated by $I_{\{1\}}, \cdots, I_{\{n\}}$, so that $\operatorname{Im} \alpha$ is generated by the proper faces of P (proper faces because of the condition that each x_i actually occur in some element of M), and it follows that $\operatorname{Im} \alpha = L(P)$.

Finally, to show that $\dim I = \dim \alpha(I) + 1$ for every $I \in \mathcal{I}$, it suffices to show this when $I = I_u$, $u \subset \{1, \cdots, n\}$. But we know this from the preceding lemma.

Proof of Theorem 1°. We use induction on n. If $n = 1$, it is easy to see that $\dim K[M] \leqq 1$, so that $K[M]$ is Cohen-Macaulay. Now suppose that $n > 1$ and that the result is true for expanded semigroups of monomials in fewer than n variables. Let M be a given expanded semigroup of monomials in $x_1, \cdots, x_n$. If one of the variables x_i does not occur in any $p \in M$, then the given semigroup can be regarded as a semigroup in fewer than n variables. Hence, we need not consider this case, and we can assume that for each i, $1 \leqq i \leqq n$, there is a monomial $p_i \in M$ which involves x_i. Let $p = p_1 \cdots p_n$. Then $p = x^h$, where $h = (h_1, \cdots, h_n)$ has positive integer entries, and $p \in M$. Define T, a linear transformation from Q^n to Q^n which takes the first orthant onto itself by

$$T(q_1, \cdots, q_n) = (q_1/h_1, \cdots, q_n/h_n) .$$

Let $S = Q_+(\log_x M)$. The $T(S)$ is a subsemigroup of Q_+^n isomorphic to S.

Moreover, $T(S)$ is full:

$$(T(S) - T(S)) \cap Q_+^n = (T(S) - T(S)) \cap T(Q_+^n)$$
$$= T((S - S) \cap Q_+^n) = T(S) ,$$

as required. Let $H = T(S) \cap Z_+^n$. It is then easy to see that H is an expanded subsemigroup of Z_+^n such that $T(S) = Q_+(H)$. By Lemma 6, $K[M]$ is Cohen-Macaulay if and only if $K[x^H]$ is Cohen-Macaulay, for

$$Q_+(H) = T(S) \cong S = Q_+(\log_x M) .$$

By construction, $(1, \cdots, 1) \in H$, so that we can assume without loss of generality that $x_1 \cdots x_n \in M$.

We now apply Proposition 2. The map $\alpha\colon \mathscr{I} \to L(P)$ gives a polytope of ideals in $K[M]$. The face ideals are the I_u for $u \neq \varnothing$, and each I_u is semiregular by the induction hypothesis: $K[M]/I_u \cong K[M \cap M_u]$, and $M \cap M_u$, is an expanded semigroup in fewer than n variables, hence, $K[M \cap M_u]$ is Cohen-Macaulay. By Theorem 2, $\beta(\partial P)$ is semiregular, i.e.

$$\beta(P \cap (\textstyle\bigcup_i X_{(i)})) = \textstyle\bigcap_i (x_i R \cap K[M])$$
$$= (x_1 \cdots x_n R) \cap K[M] = (x_1 \cdots x_n) K[M]$$

is semiregular, where $R = K[x_1, \cdots, x_n]$. But then the fact that $K[M]$ is Cohen-Macaulay is immediate from the Corollary to Proposition 19 of [12], which may be restated as follows:

LEMMA 8. *If S is a Noetherian graded K-algebra over a field K, and f is a form of positive degree not a zerodivisor in S, then S is Cohen-Macaulay if and only if S/fS is Cohen-Macaulay, i.e. if and only if fS is semiregular.*

The proof of Theorem 1° is now complete.

Remark 11. Regard Q^{n-1} as $Q^{n-1} \times \{0\}$ in Q^n and let $S = Q_+(\log_x M)$ for an expanded semigroup of monomials M. Let ϕ be the projection map from Q^n onto Q^{n-1} (where we think of Q^n as $Q^{n-1} \times Q$). It is not hard to see that ϕ gives an isomorphism of S onto a full subsemigroup of Q_+^{n-1} unless $P \cap X_{(n)}$ is a facet of P. Using this fact and Lemma 6, it is possible to reduce the inductive step in the proof to the case where the facets of P are precisely the sets $P \cap X_{(i)}$, $1 \leqq i \leqq n$, so that the ideals $I_{(i)}$ are rank one primes of $K[M]$. It is perhaps a little clearer what is happening when S is, so to speak, minimally embedded in this way, although the argument is no shorter.

Remark 12. Instead of showing that a monomial $p \in M$ involving all the variables generates a semiregular ideal, we might have singled out any monomial (e.g. one involving a minimal subset of the variables), used the same trick to reduce to the case where this monomial is linear in each of the

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Remark 13. If M is a normal semigroup of monomials, then the Segre product of $K[M]$ with $K[x_1, x_2]$ is Cohen-Macaulay (for it is easy to see that this ring is also generated over K by a normal semigroup of monomials). Hence, we can frequently deduce from the converse statements in Chow's main Theorem [4], p. 818, that $K[M]$ is proper. This is useful, for it then follows that the Segre product of $K[M]$ with any proper Cohen-Macaulay homogeneous K-algebra is Cohen-Macaulay if the dimensions are bigger than one.

Proof of Theorem 1. For the rest of this paper we fix a normal semigroup M of monomials in $x_1, \cdots, x_n$ and we let $p_1, \cdots, p_k$ be a fixed set of generators for M, not containing 1. Let $y_1, \cdots, y_k$ be indeterminates. Let R be any ring. We then have a unique R-homomorphism from $R[y] = R[y_1, \cdots, y_k]$ onto $R[M]$ which takes y_i to p_i, $1 \leqq i \leqq k$. We denote the kernel of this homomorphism by $\mathfrak{A}_R$. It is easy to see that $\mathfrak{A}_R$ is generated by the differences

$$y_1^{m_1} \cdots y_k^{m_k} - y_1^{n_1} \cdots y_k^{n_k}$$

such that

$$p_1^{m_1} \cdots p_k^{m_k} = p_1^{n_1} \cdots p_k^{n_k} .$$

We then have that for any ring R, $R \otimes_Z Z[M] = R[M]$, and that $R \otimes_Z \mathfrak{A}_Z = \mathfrak{A}_R$. Moreover, by Theorem 1° and Lemma 7, for any field K, $K \otimes_Z \mathfrak{A}_Z$ is a semiregular homogeneous prime ideal of grade $g = k - \dim Q(\log_z M)$ in $K[y]$. It follows from the main results of [7] or Proposition 20 of [12] that $\mathfrak{A}_Z$ is a generically perfect prime ideal of $Z[y]$ and therefore, by the main result of [9], strongly generically perfect: see [15] for the definition of this notion. It follows that $\mathfrak{A}_R$ is a perfect ideal for every Noetherian ring R, and, hence, a semiregular ideal when R is Cohen-Macaulay. But this means that when R is Cohen-Macaulay, so is $R[y]/\mathfrak{A}_R \cong R[M]$, and the proof of Theorem 1 is complete.

We conclude with the explicit statement of some ideal-theoretic corollaries of the generic perfection of $\mathfrak{A}_Z$. As we have already shown above:

COROLLARY 2. *$\mathfrak{A}_Z$ is a strongly generically perfect homogeneous prime ideal of $Z[y]$ of grade $g = k - \dim Q(\log_\pi M)$.*

Hence, from the main result of [15] (or the main results of [7]):

COROLLARY 3. *Let $\mathcal{K}$ be a graded free minimal resolution of $Z[y]/\mathfrak{A}_Z$. Then $\mathcal{K}$ has length g, and the complex $\mathcal{K}$ is grade-sensitive. That is, if $u_1, \cdots, u_k$ are elements of any Noetherian ring R, and we make R into a $Z[y]$-algebra by means of the homomorphism ϕ which takes y_i to u_i, $1 \leqq i \leqq k$, then if E is any R-module of finite type such that $JE \neq E$, where $J = \phi(\mathfrak{A}_Z)R$, then the grade of J on E is the number of vanishing homology groups, counting from the left, of the complex $\mathcal{K} \otimes E$, where the tensor product is over $Z[y]$. In particular, if the grade of J on E is g, then $\mathcal{K} \otimes E$ is acyclic.*

Thus, it is natural of think of $\mathcal{K}$ as a generalized Koszul complex.

Finally, the results of §§ 5-7 of [7] yield:

COROLLARY 4. *Let R, $u_1, \cdots, u_k$, ϕ, and J be as in Corollary 3. Then every minimal prime of J has rank at most $g = k - \dim Q(\log_\pi M)$. If the grade of J is as large as possible, i.e. g, then J is perfect, hence, grade unmixed, and all the associated primes of J have grade g. If J has grade g and R is Cohen-Macaulay, then the associated primes of J all have rank g, and R/J is again Cohen-Macaulay.*

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