GOMORY INTEGER PROGRAMS

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Abstract. The set of all group relaxations of an integer program contains certain special members called Gomory relaxations. A family of integer programs with a fixed coefficient matrix and cost vector but varying right hand sides is a Gomory family if every program in the family can be solved by one of its Gomory relaxations. In this paper, we characterize Gomory families. Every TDI system gives a Gomory family, and we construct Gomory families from matrices whose columns form a Hilbert basis for the cone they generate. The existence of Gomory families is related to the Hilbert covering problems that arose from the conjectures of Sebö. Connections to commutative algebra are outlined at the end.

1. Introduction

Given an integer $d \times n$ matrix A of rank d and a cost vector $c \in \mathbb{Z}^n$, we consider the family $IP_{A,c}$ of all feasible integer programs of the form

$$IP_{A,c}(b) := min \ \{c \cdot x \ : \ Ax = b, \ x \in \mathbb{N}^n\}$$

as the right hand side vector b varies. We will assume that $\{x \in \mathbb{R}^n_{\geq 0} \ : \ Ax = 0\} = \{0\}$ which guarantees that all programs in $IP_{A,c}$ are bounded and that $cone(A)$, the cone generated by the columns of A, is pointed. The symbol $\mathbb{N}$ denotes the set of non-negative integers.

In [10], Gomory defined the *group relaxation* of $IP_{A,c}(b)$:

$$min \ \{\tilde{c}_{\bar{\sigma}} \cdot x_{\bar{\sigma}} \ : A_\sigma x_\sigma + A_{\bar{\sigma}} x_{\bar{\sigma}} = b, \ x_{\bar{\sigma}} \geq 0, (x_\sigma, x_{\bar{\sigma}}) \in \mathbb{Z}^n\}$$

where A_σ is the optimal basis of the linear programming relaxation of $IP_{A,c}(b)$ and non-negativity restrictions on the optimal basic variables x_σ have been dropped. The cost vector $\tilde{c}_{\bar{\sigma}}$ is the restriction of $(c - c_\sigma A_\sigma^{-1} A)$ to the components indexed by $\bar{\sigma}$, the complement of σ in $\{1, \ldots, n\}$. The *extended group relaxations* of $IP_{A,c}(b)$ introduced by Wolsey are the $2^{|\sigma|}$ relaxations obtained by dropping non-negativity restrictions on each subset of the variables in x_σ [26]. The set of d-dimensional simplicial cones $cone(A_\sigma)$, as A_σ varies over the optimal bases of the LP-relaxations of all programs in $IP_{A,c}$, triangulates

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cone(A). This is called the *regular triangulation* of *cone*(A) with respect to c and is denoted as Δ_c. The collection of all sets σ indexing the full dimensional cones of Δ_c along with all their subsets form a simplicial complex on $\{1, \ldots, n\}$ which we also call Δ_c. In this paper we consider the set of all group relaxations of $IP_{A,c}(b)$ obtained by dropping non-negativity restrictions on the variables indexed by each face of Δ_c (Definition 2.7). This is a larger set of group relaxations for $IP_{A,c}(b)$ than the set of extended group relaxations of Wolsey. We show that these are precisely all the bounded group relaxations of $IP_{A,c}(b)$ (Theorem 2.8). Among these group relaxations, the easiest to solve are those indexed by the maximal faces of Δ_c. We call these the *Gomory relaxations* of $IP_{A,c}(b)$. The family of integer programs $IP_{A,c}$ is called a *Gomory family* if all its members can be solved by one of their Gomory relaxations.

Theorem 2.22 characterizes Gomory families. This theorem is a consequence of re-casting algebraic results on *toric initial ideals* in terms of group relaxations of integer programs [14], [15]. These algebraic results come from *Gröbner bases* methods in integer programming [20]. No familiarity with these techniques is assumed in this paper. In Section 3 we relate Gomory families to *total dual integrality* (TDI-ness). Lemma 3.3 shows that $yA \leq c$ is a TDI system if and only if the regular triangulation Δ_c is *unimodular*. This leads to Theorem 3.4 that if $yA \leq c$ is TDI then $IP_{A,c}$ is a Gomory family.

In Sections 4 and 5 we exhibit general classes of Gomory families. A matrix A is said to be *normal* if its columns form a Hilbert basis for *cone*(A). In Section 4 we introduce Δ-normal matrices which form a proper subset of normal matrices. Theorem 4.7 shows that every Δ-normal matrix A gives rise to a Gomory family $IP_{A,c}$. While we do not know if every normal matrix gives rise to a Gomory family, we show that for small values of d every regular triangulation of *cone*(A) supports Gomory families (Theorem 5.5). Gomory families induce special covers of $\mathbb{N}A$, the semigroup generated by the columns of A, which relates their existence to the Hilbert cover questions found in [3], [8] and [18].

Throughout this paper we consider triangulations of *cone*(A). Every one dimensional face of one of our triangulations must be generated by a column of A, although not all columns of A need generate a one-dimensional face. More precisely, these are triangulations of the vector configuration consisting of the columns of A and not just *cone*(A). However, we will refer to them as triangulations of *cone*(A). All computations in this paper rely on the connections of this material to commutative algebra as described in [14], [15], [20] and [22]. The relevant connections and codes are described briefly in Section 6.

2. Gomory families

In this paper, we fix a matrix $A \in \mathbb{Z}^{d \times n}$ of rank d and a cost vector $c \in \mathbb{Z}^n$ and consider the family $IP_{A,c}$ of all integer programs

$$IP_{A,c}(b) := min \ \{c \cdot x \ : \ Ax = b, \ x \in \mathbb{N}^n\}$$

as b varies in the semigroup $\mathbb{N}A := \{Au \ : \ u \in \mathbb{N}^n\} \subseteq \mathbb{Z}^d$. This semigroup is contained in the intersection of the pointed polyhedral cone $cone(A) := \{Ax \ : \ x \in \mathbb{R}^n_{\geq 0}\}$, and $\mathbb{Z}A := \{Az \ : \ z \in \mathbb{Z}^n\}$, the lattice generated by the columns of A. We may assume without loss of generality that $\mathbb{Z}A = \mathbb{Z}^d$.

The feasible *linear programs* from A and c are of the form

$$LP_{A,c}(b) := min \ \{ \ c \cdot x \ : \ Ax = b, \ x \geq 0\}$$

where $b \in cone(A)$. We denote this family as $LP_{A,c}$. For $\sigma \subseteq \{1, \dots, n\}$, let A_σ be the submatrix of A whose set of column indices is σ.

Definition 2.1. For $\sigma \subseteq \{1, \dots, n\}$, $cone(A_\sigma)$ is a face of the *regular subdivision* Δ_c of $cone(A)$ if and only if there exists a vector $y \in \mathbb{R}^d$ such that $y \cdot a_j = c_j$ for all $j \in \sigma$ and $y \cdot a_j < c_j$ for all $j \notin \sigma$.

Remark 2.2. The regular subdivision Δ_c is gotten by taking the cone in $\mathbb{R}^{d+1}$ generated by the lifted vectors $(a_i, c_i) \in \mathbb{R}^{d+1}$ where a_i is the i-th column of A and c_i is the i-th component of c, and then projecting the *lower* facets of this lifted cone back onto $cone(A) \subset \mathbb{R}^d$. See [1]. The most efficient way of computing Δ_c is by using a fast implementation for computing convex hulls in order to identify the lower facets of the lifted cone.

We assume that c is *generic* in the sense that Δ_c is a *triangulation* of $cone(A)$. All cost vectors except those lying on a finite set of hyperplanes of $\mathbb{R}^n$ are generic [1]. Using σ to label $cone(A_\sigma)$, the triangulation Δ_c can be denoted as a set of subsets of $\{1, \dots, n\}$. This set of sets is closed under inclusion since Δ_c is a simplicial complex, and hence it is specified completely by its maximal elements.

Example 2.3. Suppose $A = \begin{pmatrix} 1 & 2 & 2 & 3 & 2 \\ 3 & 3 & 1 & 1 & 0 \end{pmatrix}$ and $c = (9, 9, 4, 0, 9)$. Figure 1 illustrates how the regular triangulation Δ_c of $cone(A)$ is obtained via Remark 2.2. The columns of A are the five vectors in $\mathbb{R}^2$ shown in the top picture, with column a_i labeled by $i = 1, \dots, 5$. These five vectors are then replaced by the lifted vectors (a_i, c_i) in $\mathbb{R}^3$. The cone generated by the lifted vectors is shown in the bottom left of Figure 1. The shaded facets are the *lower* facets of this cone. Note that the liftings of columns 2 and 3 are in the interior of the cone. The

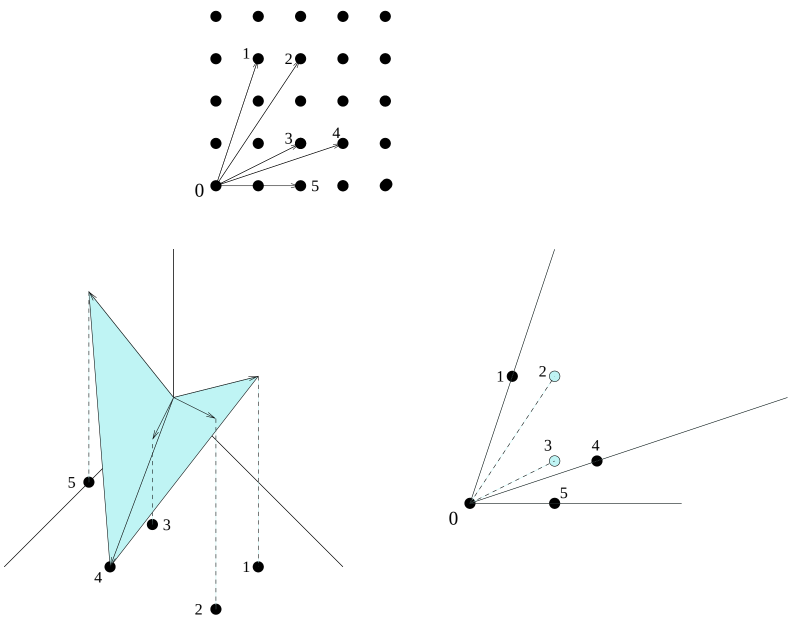

FIGURE 1. The regular triangulation of $cone(A)$ from Example 2.3.

bottom right picture shows the regular triangulation Δ_c of $cone(A)$ obtained by projecting the lower facets of the lifted cone onto $cone(A)$. The full simplicial complex $\Delta_c = \{\{1,4\},\{4,5\},\{1\},\{4\},\{5\},\emptyset\}$. Since it suffices to list its maximal faces, we write $\Delta_c = \{\{1,4\},\{4,5\}\}$.

For a vector $x \in \mathbb{R}^n$, let $supp(x) := \{i : x_i \neq 0\}$ denote the *support* of x. The significance of regular triangulations for linear programming is summarized in the following proposition.

Proposition 2.4. [21, Lemma 1.4] *The optimal solution of $LP_{A,c}(b)$ is the feasible solution x^* with $supp(x^*) = \tau$ where τ is the smallest face of the regular triangulation Δ_c such that $b \in cone(A_\tau)$.* ☐

Proposition 2.4 implies that $\sigma \subseteq \{1, \dots, n\}$ is a maximal face of Δ_c if and only if A_σ is an optimal basis for all $LP_{A,c}(b)$ with b in $cone(A_\sigma)$. Given a polyhedron $P \subset \mathbb{R}^n$ and a face F of P, the *normal cone* of F at P is the cone $N_P(F) := \{\omega \in \mathbb{R}^n \, : \, \omega \cdot x' \geq \omega \cdot x, \ \forall x' \in F \text{ and } x \in P\}$. The set of all normal cones of P form the *normal fan* of P in $\mathbb{R}^n$.

Proposition 2.5. *The regular triangulation Δ_c of $cone(A)$ is the normal fan of the polyhedron $P_c := \{y \in \mathbb{R}^d \, : yA \leq c\}$.*

Proof. The polyhedron P_c is the feasible region of max $\{y \cdot b \, : \, yA \leq c, \ y \in \mathbb{R}^d\}$, the dual program to $LP_{A,c}(b)$. The normal fan of P_c is supported on $cone(A)$, i.e. the union of the normal cones of P_c is *cone*(A), since this is the polar cone of the recession cone $\{y \in \mathbb{R}^d : yA \leq 0\}$ of P_c. Suppose b is any vector in the interior of a maximal face *cone*(A_σ) of Δ_c. Then by Proposition 2.4, $LP_{A,c}(b)$ has an optimal solution x^* with support σ. By complementary slackness, an optimal solution y^* to the dual of $LP_{A,c}(b)$ satisfies $y^* \cdot a_j = c_j$ for all $j \in \sigma$ and $y^* \cdot a_j \leq c_j$ otherwise. Since σ is a maximal face of Δ_c, in fact, $y^* \cdot a_j < c_j$ for all $j \notin \sigma$. Further, y^* is unique since A_σ is nonsingular. As $y \cdot b$ is maximized over P_c at y^*, it follows that b (and hence *cone*(A_σ)) is contained in the normal cone of P_c at the vertex y^*. If b lies in the interior of another maximal face *cone*(A_τ) then y', the dual optimal solution to $LP_{A,c}(b)$ satisfies $y'A_\tau = c_\tau$ and $y'A_{\bar{\tau}} < c_{\bar{\tau}}$ where $\tau \neq \sigma$. Hence y' is distinct from y^* and each maximal cone in Δ_c lies in a distinct maximal cone in the normal fan of P_c. Since Δ_c and the normal fan of P_c have the same support, they must therefore coincide. $\square$

Corollary 2.6. *The polyhedron P_c is simple if and only if the regular subdivision Δ_c is a triangulation of cone(A), i.e., c is generic.* $\square$

Regular subdivisions were introduced in [9] and have since been studied from various points of view. They play a central role in the algebraic study of integer programming ([20], [21]). We use them here to define *group relaxations* of $IP_{A,c}(b)$.

A subset τ of $\{1, \dots, n\}$ partitions $x = (x_1, \dots, x_n)$ as x_τ and $x_{\bar{\tau}}$ where x_τ consists of the variables indexed by τ, and $x_{\bar{\tau}}$ the variables indexed by the complementary set $\bar{\tau}$. Similarly, the matrix A is partitioned as $A = [A_\tau, A_{\bar{\tau}}]$ and the cost vector as $c = (c_\tau, c_{\bar{\tau}})$. If σ is a maximal face of Δ_c then A_σ is nonsingular and $Ax = b$ can be written as $x_\sigma = A_\sigma^{-1}(b - A_{\bar{\sigma}} x_{\bar{\sigma}})$. Then $c \cdot x = c_\sigma(A_\sigma^{-1}(b - A_{\bar{\sigma}} x_{\bar{\sigma}})) + c_{\bar{\sigma}} x_{\bar{\sigma}} = c_\sigma A_\sigma^{-1} b + (c_{\bar{\sigma}} - c_\sigma A_\sigma^{-1} A_{\bar{\sigma}}) x_{\bar{\sigma}}$. Let $\tilde{c}_{\bar{\sigma}} := c_{\bar{\sigma}} - c_\sigma A_\sigma^{-1} A_{\bar{\sigma}}$ and for any face τ of σ, let $\tilde{c}_{\bar{\tau}}$ be the extension of $\tilde{c}_{\bar{\sigma}}$ to a vector in $\mathbb{R}^{|\bar{\tau}|}$ by adding zeros.

Definition 2.7. The group relaxation of the integer program $IP_{A,c}(b)$ with respect to the face τ of Δ_c, denoted as $G^\tau(b)$, is the program

$$min \ \{\tilde{c}_{\bar{\tau}} \cdot x_{\bar{\tau}} \ : A_\tau x_\tau + A_{\bar{\tau}} x_{\bar{\tau}} = b, \ x_{\bar{\tau}} \geq 0, (x_\tau, x_{\bar{\tau}}) \in \mathbb{Z}^n\}.$$

The group relaxation $G^\tau(b)$ solves $IP_{A,c}(b)$ if its optimal solution is non-negative. These relaxations contain among them the usual group relaxations of $IP_{A,c}(b)$ found in the literature. The program $G^\sigma(b)$ where A_σ is the optimal basis of the linear relaxation $LP_{A,c}(b)$ is precisely Gomory's group relaxation of $IP_{A,c}(b)$ [10]. The set of relaxations $G^\tau(b)$ as τ varies among the subsets of this σ are the extended group relaxations of $IP_{A,c}(b)$ defined by Wolsey [26]. Since $\emptyset \in \Delta_c$, $G^\emptyset(b) = IP_{A,c}(b)$ is a group relaxation of $IP_{A,c}(b)$ by Definition 2.7, and hence $IP_{A,c}(b)$ will certainly be solved by one of its extended group relaxations. However, one can construct examples where a group relaxation $G^\tau(b)$ solves $IP_{A,c}(b)$, but $G^\tau(b)$ is neither Gomory's group relaxation of $IP_{A,c}(b)$ nor one of its nontrivial extended Wolsey relaxations (see Example 2.21). Theorem 2.8 will show that the relaxations in Definition 2.7 are precisely all the bounded group relaxations of $IP_{A,c}(b)$. Hence this definition considers all the group relaxations of each integer program in $IP_{A,c}$ that can possibly solve the program.

For our purposes it is convenient to reformulate $G^\tau(b)$ as follows. Let $B \in \mathbb{Z}^{n\times(n-d)}$ be any matrix such that the columns of B generate the $(n-d)$-dimensional lattice $\mathcal{L} = \{x \in \mathbb{Z}^n : Ax = 0\} \subset \mathbb{Z}^n$ and let u be a feasible solution of $IP_{A,c}(b)$. Then

$$\begin{aligned} IP_{A,c}(b) &= min\ \{c\cdot x\ :\ Ax = b,\ x \in \mathbb{N}^n\} \\ &= min\ \{c\cdot x\ :\ x \equiv u\,(mod\ \mathcal{L}),\ x \geq 0\} \\ &= min\ \{c\cdot x\ :\ x = u - Bz,\ x \geq 0,\ z \in \mathbb{Z}^{n-d}\}. \end{aligned}$$

The last problem is equivalent to $min\ \{c\cdot(u - Bz)\ :\ Bz \leq u,\ z \in \mathbb{Z}^{n-d}\}$ and hence, $IP_{A,c}(b)$ is equivalent to the problem

$$min\ \{(-cB)\cdot z\ :\ Bz \leq u,\ z \in \mathbb{Z}^{n-d}\}. \tag{1}$$

There is a bijection between the set of feasible solutions of (1) and the set of feasible solutions of $IP_{A,c}(b)$ via the isomorphism $z \mapsto u - Bz$. In particular, $0 \in \mathbb{R}^{n-d}$ is feasible for (1) since it is the pre-image of u.

Let π_τ be the projection map from $\mathbb{R}^n \to \mathbb{R}^{|\bar\tau|}$ that kills all coordinates indexed by τ. If $B^{\bar\tau}$ denotes the $|\bar\tau| \times (n-d)$ submatrix of B obtained by deleting the rows indexed by τ, then we denote by $\mathcal{L}_\tau$ the lattice $\pi_\tau(\mathcal{L}) = \{B^{\bar\tau}z\ :\ z \in \mathbb{Z}^{n-d}\}$. Suppose x is a feasible solution of $G^\tau(b)$, and u is a feasible solution of $IP_{A,c}(b)$. Clearly, u is also a feasible solution for $G^\tau(b)$, and we deduce that $x \equiv u\ (mod\,\mathcal{L})$ where $x_{\bar\tau} \geq 0$. After applying the projection map π_τ we obtain that $x_{\bar\tau} \equiv \pi_\tau(u)\ (mod\,\mathcal{L}_\tau),\ x_{\bar\tau} \geq 0$ (see also [23]). So the group relaxation $G^\tau(b)$ is equivalent to the *lattice program*

$$min\ \{\tilde c_{\bar\tau}\cdot x_{\bar\tau}\ :\ x_{\bar\tau} \equiv \pi_\tau(u)\ (mod\,\mathcal{L}_\tau),\ x_{\bar\tau} \geq 0\}.$$

The same reasoning for the equivalence of the second and third lines in the reformulation of $IP_{A,c}(b)$ above is valid here as well, and hence $G^\tau(b)$ is reformulated to be

$$min\ \{(-\tilde c_{\bar\tau}B^{\bar\tau})\cdot z\ :\ B^{\bar\tau}z \leq \pi_\tau(u),\ z \in \mathbb{Z}^{n-d}\}.$$

Since $\tilde c_{\bar\tau} = \pi_\tau(c - c_\sigma A_\sigma^{-1}A)$ for any maximal face σ of Δ_c containing τ, and the support of $c - c_\sigma A_\sigma^{-1}A$ is contained in $\bar\tau$, we get that $\tilde c_{\bar\tau}B^{\bar\tau} = (c - c_\sigma A_\sigma^{-1}A)B = cB$ since $AB = 0$. Hence $G^\tau(b)$ is equivalent to

$$min\ \{(-cB)\cdot z\ :\ B^{\bar\tau}z \leq \pi_\tau(u),\ z \in \mathbb{Z}^{n-d}\}. \tag{2}$$

The feasible solutions to (1) are the lattice points in the rational polyhedron $P_u := \{z \in \mathbb{R}^{n-d} : Bz \leq u\}$ and those to (2) are the lattice points in the relaxation $P_u^{\bar{\tau}} := \{z \in \mathbb{R}^{n-d} : B^{\bar{\tau}} z \leq \pi_\tau(u)\}$ of P_u obtained by deleting the inequalities indexed by τ. In theory, one could define group relaxations of $IP_{A,c}(b)$ with respect to any $\tau \subseteq \{1, \dots, n\}$. The following result justifies Definition 2.7.

Theorem 2.8. *The group relaxation $G^\tau(b)$ of $IP_{A,c}(b)$ has a finite optimal solution if and only if $\tau \subseteq \{1, \dots, n\}$ is a face of Δ_c.*

Proof. Since all data are integral it suffices to prove that the linear relaxation $min\ \{(-cB) \cdot z \ : \ z \in P_u^{\bar{\tau}}\}$ is bounded if and only if $\tau \in \Delta_c$.

If τ is a face of Δ_c then there exists $y \in \mathbb{R}^d$ such that $yA_\tau = c_\tau$ and $yA_{\bar{\tau}} < c_{\bar{\tau}}$. Using the fact that $A_\tau B^\tau + A_{\bar{\tau}} B^{\bar{\tau}} = 0$ we see that $cB = c_\tau B^\tau + c_{\bar{\tau}} B^{\bar{\tau}} = yA_\tau B^\tau + c_{\bar{\tau}} B^{\bar{\tau}} = y(-A_{\bar{\tau}} B^{\bar{\tau}}) + c_{\bar{\tau}} B^{\bar{\tau}} = (c_{\bar{\tau}} - yA_{\bar{\tau}}) B^{\bar{\tau}}$. This implies that cB is a positive linear combination of the rows of $B^{\bar{\tau}}$ since $c_{\bar{\tau}} - yA_{\bar{\tau}} > 0$. Hence cB lies in the polar of $\{z \in \mathbb{R}^{n-d} : B^{\bar{\tau}} z \leq 0\}$ which is the recession cone of $P_u^{\bar{\tau}}$ proving that the linear program $min\ \{(-cB) \cdot z \ : \ z \in P_u^{\bar{\tau}}\}$ is bounded.

The linear program $min\ \{(-cB) \cdot z \ : \ z \in P_u^{\bar{\tau}}\}$ is feasible since 0 is a feasible solution. If it is bounded as well then $min\ \{c_\tau x_\tau + c_{\bar{\tau}} x_{\bar{\tau}} \ : \ A_\tau x_\tau + A_{\bar{\tau}} x_{\bar{\tau}} = b,\ x_{\bar{\tau}} \geq 0\}$ is feasible and bounded. Hence the dual of the latter program $max\ \{y \cdot b \ : \ yA_\tau = c_\tau,\ yA_{\bar{\tau}} \leq c_{\bar{\tau}}\}$ is feasible. This shows that a superset of τ is a face of Δ_c which implies that $\tau \in \Delta_c$ since Δ_c is a simplicial complex. □

The reformulations (1) and (2) imply that $G^\tau(b)$ solves $IP_{A,c}(b)$ if and only if both programs have the same optimal solution $z^* \in \mathbb{Z}^{n-d}$. If $G^\tau(b)$ solves $IP_{A,c}(b)$ then $G^{\tau'}(b)$ also solves $IP_{A,c}(b)$ for every τ' *contained in* τ. We say that $\tau \in \Delta_c$ is *associated* to $IP_{A,c}$ if for some $b \in \mathbb{N}A$, $G^\tau(b)$ solves $IP_{A,c}(b)$ but $G^{\tau'}(b)$ does not for all faces $\tau' \neq \tau$ of Δ_c *containing* τ. Several results about the structure of the subposet of faces of Δ_c that are associated to $IP_{A,c}$ can be found in [14]. For instance, the associated sets of $IP_{A,c}$ occur in saturated chains [14, Theorem 3.1]. For a given $b \in \mathbb{N}A$, the most easily solved group relaxations of $IP_{A,c}(b)$ are those $G^\sigma(b)$ where σ is a maximal face of Δ_c. In this case, the lattice $\mathcal{L}_\sigma$ is of finite index. In other words, the group $\mathbb{Z}^{n-d}/\mathcal{L}_\sigma$ is a finite group. For all lower dimensional faces $\tau \in \Delta_c$ the group $\mathbb{Z}^{n-d}/\mathcal{L}_\tau$ is infinite. In the finite case, Gomory showed that $G^\sigma(b)$ can be solved by a dynamic programming algorithm [10]. Algebraically, $G^\tau(b)$ is solved by computing a Gröbner basis of the *lattice ideal* defined by $\mathcal{L}_\tau$. This ideal is *zero-dimensional* only if σ is a maximal face of Δ_c. Gröbner bases computations for zero-dimensional ideals are easier than for general polynomial ideals [7]. Motivated by these observations, we make the following definitions.

Definition 2.9. A group relaxation $G^\sigma(b)$ of $IP_{A,c}(b)$ is called a *Gomory relaxation* if σ is a maximal face of Δ_c.

Definition 2.10. The family of integer programs $IP_{A,c}$ is a *Gomory family* if, for *every* $b \in \mathbb{N}A$, $IP_{A,c}(b)$ is solved by one of its Gomory relaxations.

Our goal in the rest of this section is to characterize Gomory families of integer programs (Theorem 2.22). We will assume from now on that every integer program in $IP_{A,c}$ has a unique optimal solution which is a stricter notion of genericity of c than Δ_c being a triangulation [21]. Let $\mathcal{O}_c \subseteq \mathbb{N}^n$ be the set of all optimal solutions of the programs in $IP_{A,c}$. The set $\mathcal{O}_c$ is known to be a *down set* or *order ideal* in $\mathbb{N}^n$, i.e. $u \in \mathcal{O}_c$ and $v \leq u$, $v \in \mathbb{N}^n$ implies that $v \in \mathcal{O}_c$ [25]. For a given A, there are only finitely many sets $\mathcal{O}_c$ as c varies. Two generic cost vectors c and c' are equivalent if $\mathcal{O}_c = \mathcal{O}_{c'}$ and all equivalence classes of generic cost vectors are open full dimensional polyhedral cones in $\mathbb{R}^n$ [21]. Since c is generic, $\mathcal{O}_c$ is in bijection with $\mathbb{N}A$ via the map $\phi_A : \mathcal{O}_c \to \mathbb{N}A$ such that $u \mapsto Au$. Let $Q_u := \{z \in \mathbb{R}^{n-d} \ : \ Bz \leq u, \ (-cB) \cdot z \leq 0\}$ and $Q_u^{\bar\tau} := \{z \in \mathbb{R}^{n-d} \ : \ B^{\bar\tau} z \leq \pi_\tau(u), \ (-cB) \cdot z \leq 0\}$.

Lemma 2.11. *(i) A vector u is in $\mathcal{O}_c$ if and only if $Q_u \cap \mathbb{Z}^{n-d} = \{0\}$.*
(ii) For $u \in \mathcal{O}_c$, the group relaxation $G^\tau(Au)$ solves the integer program $IP_{A,c}(Au)$ if and only if $Q_u^{\bar\tau} \cap \mathbb{Z}^{n-d} = \{0\}$.

Proof. (i) The lattice point u belongs to $\mathcal{O}_c$ if and only if u is the optimal solution to $IP_{A,c}(Au)$ which is equivalent to $0 \in \mathbb{Z}^{n-d}$ being the optimal solution to the reformulation (1) of $IP_{A,c}(Au)$. Since c is generic, the last statement is equivalent to $Q_u \cap \mathbb{Z}^{n-d} = \{0\}$. The second statement follows from (i) and the fact that (2) solves (1) if and only if they have the same optimal solution. $\square$

Example 2.12. Consider $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ and the cost vector $c = (0, 1, 3, 0)$. Then the columns of

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}$$

form a basis for $\mathcal{L}$. We will denote the rows of B by $b_1, \ldots, b_4$. If $b = (3, 2)^t$, then the optimal solution of $IP_{A,c}(b)$ is $u = (1, 2, 0, 0)$. The

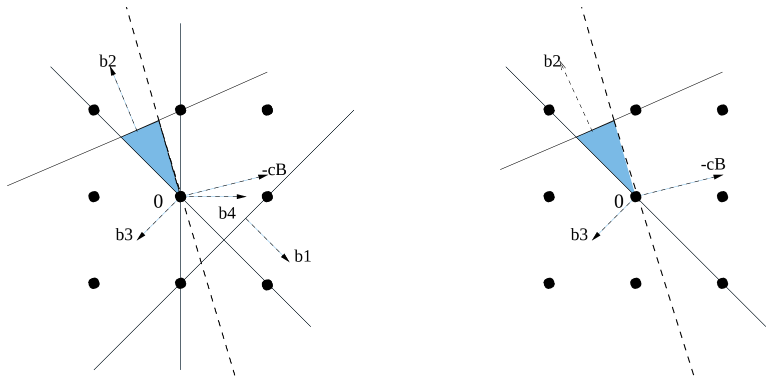

FIGURE 2. Q_u and $Q_u^{\bar{\tau}}$ of Example 2.12.

first picture in Figure 2 shows $Q_u = \{z \in \mathbb{R}^2 : \ Bz \leq u, \ (-cB) \cdot z \leq 0\}$. The optimality of u is evident from the fact that the only integral point in Q_u is the origin. The second picture in Figure 2 shows $Q_u^{\bar{\tau}} = \{z \in \mathbb{R}^2 : \ b_2 \cdot z \leq 2, \ b_3 \cdot z \leq 0, \ (-cB) \cdot z \leq 0\}$ for $\tau = \{1, 4\}$. The origin is again the only lattice point in $Q_u^{\bar{\tau}}$. Hence $G^\tau(b)$ solves $IP_{A,c}(b)$.

From now on, we will use the optimal solution to $IP_{A,c}(b)$ as the vector u in (1) and (2). For an element $u \in \mathcal{O}_c$ and a face τ of Δ_c let $S(u, \tau)$ be the affine semigroup $u + \mathbb{N}(e_i : i \in \tau)$ in $\mathbb{N}^n$ where e_i denotes the i-th unit vector in $\mathbb{R}^n$. In other words, $S(u, \tau)$ is the translation by u of the semigroup consisting of all vectors in $\mathbb{N}^n$ with support in τ.

Lemma 2.13. *Suppose u lies in $\mathcal{O}_c$. If $G^\tau(Au)$ solves $IP_{A,c}(Au)$, then $G^\tau(Av)$ solves $IP_{A,c}(Av)$ for all $v \in S(u, \tau)$.*

Proof. If $v \in S(u, \tau)$, then $\pi_\tau(u) = \pi_\tau(v)$, and this implies that $Q_u^{\bar{\tau}} = Q_v^{\bar{\tau}}$. If $G^\tau(Au)$ solves $IP_{A,c}(Au)$, then $\{0\} = Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = Q_v^{\bar{\tau}} \cap \mathbb{Z}^{n-d}$ for all $v \in S(u, \tau)$. This implies the result by Lemma 2.11 (ii). □

Proposition 2.14. *The affine semigroup $S(u, \tau)$ is contained in $\mathcal{O}_c$ if and only if $G^\tau(Au)$ solves $IP_{A,c}(Au)$.*

Proof. Suppose $S(u, \tau) \subseteq \mathcal{O}_c$. Then for all $v \in S(u, \tau)$, $Q_v = \{z \in \mathbb{R}^{n-d} \ : \ B^\tau z \leq \pi_{\bar{\tau}}(v), \, B^{\bar{\tau}} z \leq \pi_\tau(u), \, (-cB) \cdot z \leq 0\} \cap \mathbb{Z}^{n-d} = \{0\}$. Since $\pi_{\bar{\tau}}(v)$ can be any vector in $\mathbb{N}^{|\tau|}$, and $Q_u^{\bar{\tau}}$ is bounded by Theorem 2.8, $Q_u^{\bar{\tau}} = \{z \in \mathbb{R}^{n-d} \ : \ B^{\bar{\tau}} z \leq \pi_\tau(u), \, (-cB) \cdot z \leq 0\} \cap \mathbb{Z}^{n-d} = \{0\}$. Hence, by Lemma 2.11 (ii), $G^\tau(Au)$ solves $IP_{A,c}(Au)$.

Conversely, if $G^\tau(Au)$ solves $IP_{A,c}(Au)$, then $\{0\} = Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = Q_v^{\bar{\tau}} \cap \mathbb{Z}^{n-d}$ for all $v \in S(u, \tau)$. Since $Q_v^{\bar{\tau}}$ is a relaxation of Q_v, $Q_v \cap \mathbb{Z}^{n-d} = \{0\}$ for all $v \in S(u, \tau)$ and hence by Lemma 2.11 (i), $S(u, \tau) \subseteq \mathcal{O}_c$. □

Definition 2.15. For $\tau \in \Delta_c$ and $u \in \mathcal{O}_c$, the pair (u, τ) is called an *admissible pair* of $\mathcal{O}_c$ if

(i) $G^\tau(Au)$ solves $IP_{A,c}(Au)$ or equivalently, $S(u,\tau) \subseteq \mathcal{O}_c$, and
(ii) the support of u is contained in $\bar{\tau}$.
An admissible pair (u,τ) is a *standard pair* of $\mathcal{O}_c$ if the affine semigroup $S(u,\tau)$ is not properly contained in another affine semigroup $S(v,\tau')$ where (v,τ') is also an admissible pair of $\mathcal{O}_c$.

Definition 2.16. For $u \in \mathbb{N}^n$ and a face τ of Δ_c, we say that the polytope $Q_u^{\bar{\tau}}$ is a *standard polytope* of $IP_{A,c}$ if $Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = \{0\}$ and every relaxation of $Q_u^{\bar{\tau}}$ obtained by removing an inequality in $B^{\bar{\tau}} z \leq \pi_\tau(u)$ contains a non-zero lattice point.

Theorem 2.17. *The following are equivalent:*
(i) An admissible pair (u,τ) is a standard pair of $\mathcal{O}_c$.
(ii) The polytope $Q_u^{\bar{\tau}}$ is a standard polytope of $IP_{A,c}$.
(iii) The face τ of Δ_c is associated to $IP_{A,c}$.

Proof. The proof of $(i) \Leftrightarrow (ii)$ was first presented in Theorem 2.5 in [14]. A proof in the language used here, as well as a proof of $(i) \Leftrightarrow (iii)$ can be found in Theorem 3.10 in [24]. For the first statement, observe that the pair (u,τ) is standard if and only if for every $i \notin \tau$ there exists $v \in S(u,\tau)$ and a positive integer m_i such that $v + m_i e_i$ is *not* in $\mathcal{O}_c$. Because $Q^{\bar{\tau}}_{u+m_ie_i} = Q^{\bar{\tau}}_{v+m_ie_i}$, equivalently, for every $i \notin \tau$ there exists $v \in S(u,\tau)$ and a positive integer m_i such that $Q^{\bar{\tau}}_{u+m_ie_i}$ has more than one lattice point. This means, if we drop the inequality indexed by i in $B^{\bar{\tau}} z \leq \pi_\tau(u)$, we will introduce a lattice point into $Q_u^{\bar{\tau}}$. Hence, $Q_u^{\bar{\tau}}$ is a standard polytope. For the second statement, note that if (u,τ) is standard then $S(u,\tau)$ is contained in $\mathcal{O}_c$. Proposition 2.14 implies that $G^\tau(Au)$ solves $IP_{A,c}(Au)$. It also follows that for no $\tau' \supset \tau$, the group relaxation $G^{\tau'}(Au)$ can solve $IP_{A,c}(Au)$, since otherwise $S(u,\tau)$ would be contained in $S(u,\tau')$. This shows that τ is associated to $IP_{A,c}$. The converse follows by tracing this reasoning backwards. $\square$

Under the map $\phi_A : \mathcal{O}_c \to \mathbb{N}A$ such that $u \mapsto Au$, the affine semigroup $S(u,\tau)$ where (u,τ) is a standard pair of $\mathcal{O}_c$ maps to the affine semigroup $Au + \mathbb{N}A_\tau$ in $\mathbb{N}A$. Since every integer program in $IP_{A,c}$ is solved by one of its group relaxations, $\mathcal{O}_c$ is covered by its standard pairs. We call this cover and its image in $\mathbb{N}A$ under ϕ_A, the *standard pair decomposition* of $\mathcal{O}_c$ and $\mathbb{N}A$ respectively. The terminology used above has its origins in [22] which introduced the *standard pair decomposition* of a *monomial ideal.* The specialization to integer programming appears in [20, §12.D], [15] and [14].

Lemma 2.18. *(i) The standard pair decomposition of $\mathcal{O}_c$ is unique.*
(ii) For each $\tau \in \Delta_c$, there are only finitely many standard pairs of $\mathcal{O}_c$ that are indexed by τ.

Proof. The standard pairs of $\mathcal{O}_c$ are determined by the standard polytopes of $IP_{A,c}$. For each $\tau \in \Delta_c$, there is a unique finite list of standard polytopes indexed by $\bar{\tau}$. □

Borrowing terminology from [22], we call the number of standard pairs of the form $(\cdot, \tau)$ the *multiplicity* of τ in $\mathcal{O}_c$. The total number of standard pairs is called the *arithmetic degree* of $\mathcal{O}_c$. By Theorem 2.16, multiplicity of τ in $\mathcal{O}_c$ is the number of distinct standard polytopes of $IP_{A,c}$ indexed by $\bar{\tau}$ and the arithmetic degree of $\mathcal{O}_c$ is the total number of standard polytopes of $IP_{A,c}$. Lemma 2.19 shows that the maximal faces of Δ_c play a special role in the standard pair decomposition of $\mathcal{O}_c$. Part (ii) can also be deduced from the work of Gomory [10].

Lemma 2.19. [20, §12.D] *(i)* $(0, \sigma)$ *is a standard pair of* $\mathcal{O}_c$ *if and only if* σ *is a maximal face of* Δ_c.
(ii) If σ *is a maximal face of* Δ_c*, the multiplicity of* σ *in* $\mathcal{O}_c$ *is the index of the sublattice* $\mathbb{Z}A_\sigma$ *in* $\mathbb{Z}A$.

The index of $\mathbb{Z}A_\sigma$ in $\mathbb{Z}A$ is also the determinant of A_σ divided by the g.c.d of the maximal minors of A. In contrast to Lemma 2.19, if τ is a lower dimensional face of Δ_c, it may not index any standard pair of $\mathcal{O}_c$. This makes the structure of $\mathcal{O}_c$ complicated. (See [14] for further results.) We give two examples below.

Example 2.20. We will use the matrix A and the cost vector c of Example 2.3. The standard pair decomposition of $\mathcal{O}_c$ consists of 23 standard pairs:

τ	first argument of standard pairs of the form$(\cdot, \tau)$
$\{1,4\}$	$1,\ e_2,\ 2e_2,\ 3e_2,\ 4e_2,\ 5e_2,\ e_3,\ 2e_3$
$\{4,5\}$	$1,\ e_3$
$\{4\}$	$3e_3,\ e_1+3e_3,\ 2e_1+3e_3,\ 3e_1+3e_3,\ 2e_1+4e_3,$ $e_1+4e_3,\ 4e_3,\ e_1+e_5,\ e_1+e_2+e_5,\ e_2+e_5$
$\emptyset$	$2e_2+e_5,\ e_1+2e_2+e_5,\ e_1+2e_5$

Figure 3 shows the covering of $\mathbb{N}A$ by the corresponding affine semigroups. For instance, the eight standard pairs of the form $(\cdot, \{1,4\})$ are illustrated in the first drawing. The "outermost" cone represents the standard pair $(0, \{1,4\})$, or equivalently the affine semigroup $S(0, \{1,4\})$ which consists of the non-negative integer combinations of the first and fourth columns of A. The elements of this semigroup are indicated by dotted circles. The seven translations of this semigroup shown in the first drawing correspond to the other seven standard pairs of the form $(\cdot, \{1,4\})$. The two standard pairs of the form $(\cdot, \{4,5\})$ can be seen in the second drawing. The ten standard pairs $(\cdot, \{4\})$ are depicted by the ten "half-lines" of lattice points in the same drawing. We invite

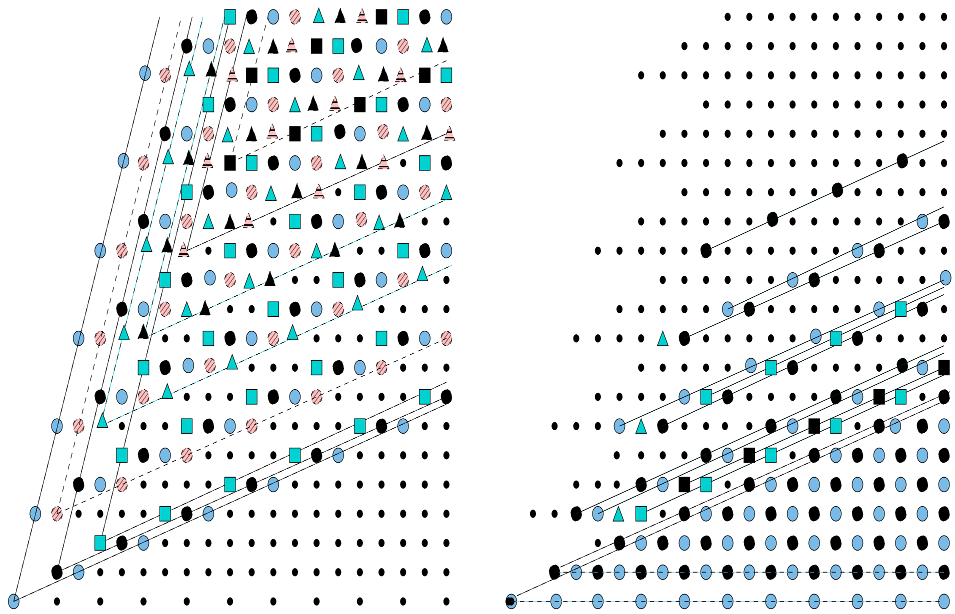

FIGURE 3. Standard pair decomposition of $\mathbb{N}A$.

the reader to also locate the three standard pairs $(\cdot, \emptyset)$ given by the three isolated triangles in the second drawing. Note that not all lower dimensional faces of Δ_c index standard pairs of $\mathcal{O}_c$ in this example.

Example 2.21. Consider the following $A \in \mathbb{Z}^{3\times 6}$ of rank three:

$$\begin{pmatrix} 5 & 0 & 0 & 2 & 1 & 0 \\ 0 & 5 & 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & 2 & 0 & 3 \end{pmatrix}.$$

The first three columns of A generate $cone(A)$. If $c = (21, 6, 1, 0, 0, 0)$ then $\Delta_c = \{\{1,3,4\}, \{1,4,5\}, \{2,5,6\}, \{3,4,6\}, \{4,5,6\}\}$. The arithmetic degree of $\mathcal{O}_c$ is 70. The 70 standard pairs are listed in Figure 4.

Observe that the integer program $IP_{A,c}(b)$ where $b = A(e_1+e_2+e_3)$ is solved by $G^\tau(b)$ with $\tau = \{1,4,5\}$. By Proposition 2.4, Gomory's relaxation of $IP_{A,c}(b)$ is indexed by $\sigma = \{4,5,6\}$ since b lies in the interior of the face $cone(A_\sigma)$ of Δ_c. However, neither this relaxation nor any nontrivial extended relaxation solves $IP_{A,c}(b)$ since the optimal solution $e_1+e_2+e_3$ is not covered by any standard pair $(\cdot, \tau)$ where τ is a non-empty subset of $\{4,5,6\}$.

The results stated above give characterizations of Gomory families.

Theorem 2.22. *The following conditions are equivalent:*
(i) $IP_{A,c}$ is a Gomory family.

τ	standard pairs $(\cdot, \tau)$
$\{1,3,4\}$	$(0,\cdot)$, $(e_5,\cdot)$, $(e_6,\cdot)$, $(e_5+e_6,\cdot)$, $(2e_6,\cdot)$
$\{1,4,5\}$	$(0,\cdot)$, $(e_2,\cdot)$, $(e_3,\cdot)$, $(e_6,\cdot)$, $(e_2+e_3,\cdot)$, $(2e_2,\cdot)$, $(3e_2,\cdot)$, $(2e_2+e_3,\cdot)$
$\{2,5,6\}$	$(0,\cdot)$, $(e_3,\cdot)$, $(2e_3,\cdot)$
$\{3,4,6\}$	$(0,\cdot)$, $(e_5,\cdot)$, $(2e_5,\cdot)$, $(3e_5,\cdot)$
$\{4,5,6\}$	$(0,\cdot)$, $(e_3,\cdot)$, $(2e_3,\cdot)$, $(3e_3,\cdot)$, $(4e_3,\cdot)$
$\{1,4\}$	$(e_3+2e_5+e_6,\cdot)$, $(2e_3+2e_5+e_6,\cdot)$, $(2e_3+2e_5,\cdot)$, $(2e_3+3e_5,\cdot)$, $(2e_3+4e_5,\cdot)$
$\{1,5\}$	$(e_2+e_6,\cdot)$, $(2e_2+e_6,\cdot)$, $(3e_2+e_6,\cdot)$
$\{2,5\}$	$(e_3+e_4,\cdot)$, $(e_4,\cdot)$, $(2e_4,\cdot)$
$\{3,4\}$	$(e_2,\cdot)$, $(e_1+e_2,\cdot)$, $(e_1+2e_5,\cdot)$, $(e_1+2e_5+e_6,\cdot)$, $(e_2+e_5,\cdot)$,
$\{3,6\}$	$(e_2,\cdot)$, $(e_2+e_5,\cdot)$
$\{4,5\}$	$(e_2+2e_3,\cdot)$, $(e_2+3e_3,\cdot)$, $(2e_2+2e_3,\cdot)$, $(3e_2+e_3,\cdot)$, $(4e_2,\cdot)$
$\{5,6\}$	$(e_2+3e_3,\cdot)$
$\{1\}$	$(e_2+e_3+e_6,\cdot)$, $(e_2+e_3+e_5+e_6,\cdot)$, $(e_2+2e_6,\cdot)$, $(e_2+e_3+2e_6,\cdot)$, $(2e_2+2e_6,\cdot)$, $(e_2+e_3+2e_5+e_6,\cdot)$
$\{3\}$	$(e_1+e_2+e_6,\cdot)$, $(e_1+e_2+2e_6,\cdot)$
$\{4\}$	$(e_1+e_2+2e_3+e_5,\cdot)$, $(e_1+e_2+2e_3+2e_5,\cdot)$, $(e_1+e_2+2e_3+3e_5,\cdot)$, $(e_1+e_2+2e_3+4e_5,\cdot)$, $(e_1+3e_3+3e_5,\cdot)$, $(e_1+3e_3+4e_5,\cdot)$
$\{\emptyset\}$	$(e_1+e_2+2e_3+e_5+e_6,\cdot)$, $(e_1+e_2+2e_3+2e_5+e_6,\cdot)$, $(e_1+2e_2+e_3+e_6,\cdot)$, $(e_1+2e_2+e_3+e_5+e_6,\cdot)$, $(e_1+2e_2+e_3+2e_5+e_6,\cdot)$, $(e_1+2e_2+e_3+2e_6,\cdot)$, $(e_1+3e_2+2e_6,\cdot)$

FIGURE 4. Standard Pairs of Example 2.21.

(ii) The associated sets of $IP_{A,c}$ are precisely the maximal faces of Δ_c.
(iii) $(\cdot,\tau)$ is a standard pair of $\mathcal{O}_c$ if and only if τ is a maximal face of the regular triangulation Δ_c.
(iv) All standard polytopes of $IP_{A,c}$ are simplices.

Proof. Definition 2.10 implies that $IP_{A,c}$ is a Gomory family if and only if the associated sets of $IP_{A,c}$ are the maximal faces of Δ_c. By $(i) \Leftrightarrow (iii)$ of Theorem 2.17 this is equivalent to the statement (iii) above. The equivalence $(ii) \Leftrightarrow (iv)$ follows from $(i) \Leftrightarrow (ii)$ of Theorem 2.17, and that each standard polytope will have exactly $n-d+1$ facets. $\square$

If there is a generic cost vector c such that for a triangulation Δ of $cone(A)$, $\Delta = \Delta_c$, then we say that Δ *supports* the order ideal $\mathcal{O}_c$ and the family of integer programs $IP_{A,c}$. No regular triangulation of the matrix A in Example 2.21 supports a Gomory family while there are six

Gomory families for the matrix in Example 2.3. Here is another matrix with a Gomory family. We will use this example again in Section 4.

Example 2.23. Consider the 3×6 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

In this case, $cone(A)$ has 14 distinct regular triangulations and 48 distinct sets $\mathcal{O}_c$ as c varies among all generic cost vectors. Ten of these triangulations support Gomory families; one for each triangulation. For instance, if $c = (0, 0, 1, 1, 0, 3)$, then

$$\Delta_c = \{\sigma_1 = \{1,2,5\},\ \sigma_2 = \{1,4,5\},\ \sigma_3 = \{2,5,6\},\ \sigma_4 = \{4,5,6\}\}$$

and $IP_{A,c}$ is a Gomory family since the standard pairs of $\mathcal{O}_c$ are: $(0,\sigma_1)$, (e_3,σ_1), (e_4,σ_1), $(0,\sigma_2)$, $(0,\sigma_3)$, and $(0,\sigma_4)$.

3. Total dual integrality and Gomory families

We now relate the notion of *total dual integrality* [17, §22] to Gomory families. Recall that $\mathbb{Z}A = \mathbb{Z}^d$ by assumption and that $c \in \mathbb{Z}^n$.

Definition 3.1. The system $yA \leq c$ is totally dual integral (TDI) if $LP_{A,c}(b)$ has an integral optimal solution for each $b \in cone(A) \cap \mathbb{Z}^d$.

Definition 3.2. The regular triangulation Δ_c is *unimodular* if $\mathbb{Z}A_\sigma = \mathbb{Z}^d$ for every maximal face $\sigma \in \Delta_c$.

Lemma 3.3. *The system $yA \leq c$ is TDI if and only if the regular triangulation Δ_c is unimodular.*

Proof. The regular triangulation Δ_c is the normal fan of P_c by Proposition 2.5, and it is unimodular if and only if $\mathbb{Z}A_\sigma = \mathbb{Z}^d$ for every maximal face $\sigma \in \Delta_c$. This is equivalent to saying that every $b \in cone(A_\sigma) \cap \mathbb{Z}^d$ lies in $\mathbb{N}A_\sigma$ for every maximal face σ of Δ_c. By Lemma 2.4, this happens if and only if $LP_{A,c}(b)$ has an integral optimum for all $b \in cone(A) \cap \mathbb{Z}^d$. □

Theorem 3.4. *If $yA \leq c$ is TDI then $IP_{A,c}$ is a Gomory family.*

Proof. By Lemma 2.19, $(0,\sigma)$ is a standard pair of $\mathcal{O}_c$ for every maximal face σ of Δ_c. Lemma 3.3 implies that $cone(A_\sigma)$ is unimodular (i.e., $\mathbb{Z}A_\sigma = \mathbb{Z}^d$), and therefore $\mathbb{N}A_\sigma = cone(A_\sigma) \cap \mathbb{Z}^d$ for every maximal face σ of Δ_c. Hence the semigroups $\mathbb{N}A_\sigma$ arising from the standard pairs $(0,\sigma)$ as σ varies over the maximal faces of Δ_c cover $\mathbb{N}A$. Therefore the only standard pairs of $\mathcal{O}_c$ are $(0,\sigma)$ as σ varies over the maximal faces of Δ_c. The result then follows from Theorem 2.22. □

When $yA \leq c$ is TDI, the multiplicity of any maximal face σ of Δ_c in $\mathcal{O}_c$ is one, and all other faces have multiplicity zero. While this is sufficient for $IP_{A,c}(b)$ to be a Gomory family, it is far from necessary. TDI-ness guarantees that $LP_{A,c}(b)$ has an integral optimum for every integral b in $cone(A)$. In contrast, if $IP_{A,c}(b)$ is a Gomory family, the linear optima of the programs in $LP_{A,c}$ may not be integral.

If A is *unimodular* (i.e., $\mathbb{Z}A_\sigma = \mathbb{Z}^d$ for every nonsingular maximal submatrix A_σ of A), then the feasible regions of the programs in $LP_{A,c}$ have integral vertices for each $b \in cone(A) \cap \mathbb{Z}^d$, and $yA \leq c$ is TDI *for all* c. Therefore, if A is unimodular, then $IP_{A,c}$ is a Gomory family for all generic cost vectors c. However, just as TDI-ness of $yA \leq c$ is not necessary for $IP_{A,c}$ to be a Gomory family, unimodularity of A is not necessary for $IP_{A,c}$ to be a Gomory family *for all* c.

Example 3.5. Consider the seven by twelve integer matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

of rank seven. The maximal minors of A have absolute values zero, one and two and hence A is not unimodular. This matrix has 376 distinct regular triangulations supporting 418 distinct order ideals $\mathcal{O}_c$. In each case, the standard pairs of $\mathcal{O}_c$ are indexed by just the maximal simplices of the regular triangulation Δ_c that supports it. Hence $IP_{A,c}$ is a Gomory family for all generic c.

4. Δ-normal matrices

In Section 3 we saw that unimodularity of A or more generally, unimodularity of a regular triangulation of $cone(A)$, gives rise to Gomory families of integer programs. In this section, we identify a larger set of matrices and cost vectors that give rise to Gomory families. A common property of unimodular matrices and matrices with a unimodular triangulation is that they form a *Hilbert basis* for $cone(A)$. In other words, $\mathbb{N}A$ equals $cone(A) \cap \mathbb{Z}^d$ for such matrices. Borrowing a term from commutative algebra we make the following definition.

Definition 4.1. A $d \times n$ integer matrix A is *normal* if the semigroup $\mathbb{N}A$ equals $cone(A) \cap \mathbb{Z}^d$.

We first note that if A is not normal, then $IP_{A,c}$ need not be a Gomory family for *any* cost vector c.

Example 4.2. The non-normal matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ gives rise to 10 distinct order ideals $\mathcal{O}_c$ supported on its four regular triangulations $\{\{1,4\}\}$, $\{\{1,2\},\{2,4\}\}$, $\{\{1,3\},\{3,4\}\}$ and $\{\{1,2\},\{2,3\},\{3,4\}\}$. Each $\mathcal{O}_c$ has at least one standard pair that is indexed by a lower dimensional face of Δ_c.

The matrix in Example 2.21 was also not normal and recall that it had no Gomory families. Therefore, we will now focus our attention on normal matrices. Although we do not know at this time whether *every* normal matrix A has some generic cost vector c such that $IP_{A,c}$ is a Gomory family, we will now show that under certain additional conditions, normal matrices do give rise to Gomory families.

Definition 4.3. A $d \times n$ integer matrix A is Δ-normal if $cone(A)$ has a triangulation Δ such that for *every* maximal face $\sigma \in \Delta$, the columns of A in $cone(A_\sigma)$ form a Hilbert basis for $cone(A_\sigma)$.

Remark 4.4. If A is Δ-normal for some triangulation Δ, then it is normal. To see this note that every lattice point in $cone(A)$ lies in $cone(A_\sigma)$ for some maximal face $\sigma \in \Delta$. Since A is Δ-normal, this lattice point also lies in the semigroup generated by the columns of A in $cone(A_\sigma)$ and hence in $\mathbb{N}A$.

Observe that A is Δ-normal with respect to all the unimodular triangulations of $cone(A)$. Hence triangulations Δ with respect to which A is Δ-normal generalize unimodular triangulations of $cone(A)$.

Examples 4.5 and 4.6 show that the set of matrices where $cone(A)$ has a unimodular triangulation is a proper subset of the set of Δ-normal matrices which in turn is a proper subset of the set of normal matrices.

Example 4.5. Examples of normal matrices with no unimodular triangulations can be found in [2] and [8]. If $cone(A)$ is simplicial for such a matrix, A will be Δ-normal with respect to its coarsest (regular) triangulation Δ consisting of the single maximal face with support $cone(A)$. For instance, consider the following example taken from [8]:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Here $cone(A)$ has 77 regular triangulations none of which are unimodular. Since $cone(A)$ is the simplicial cone generated by a_1, a_2, a_3, a_8, A is Δ-normal with respect to its coarsest regular triangulation $\{\{1,2,3,8\}\}$.

Example 4.6. There are normal matrices A that are not Δ-normal with respect to any triangulation of $cone(A)$. To see such an example, consider the following modification of the matrix in Example 4.5 that appears in [20, Example 13.17] :

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This matrix is again normal and each of its nine columns generate an extreme ray of $cone(A)$. Hence the only way for this matrix to be Δ-normal for some Δ would be if Δ is a unimodular triangulation of $cone(A)$. However, $cone(A)$ has no unimodular triangulations.

Theorem 4.7. *If A is Δ-normal for some regular triangulation Δ then there exists a generic cost vector $c \in \mathbb{Z}^n$ such that $\Delta = \Delta_c$ and $IP_{A,c}$ is a Gomory family.*

Proof. Before we prove the theorem we explain the ideas in the proof. Since Δ is a regular triangulation, $\Delta = \Delta_w$ for some $w \in \mathbb{R}^n$. The maximal faces of Δ are the projections of the lower facets of the cone C generated by the lifted vectors (a_i, w_i). See Remark 2.2 and Example 2.3 for this procedure. In the first step of the proof, we will define a new vector c' such that $\Delta_w = \Delta_{c'}$. We will choose c' so that each (a_i, c'_i) (the i-th lifted column of A) will lie on a lower facet of C. In the second step, we will use the fact the A is Δ-normal, and we will show that if $b \in \mathbb{N}A$ lies in $cone(A_\sigma)$ for a maximal face $\sigma \in \Delta_{c'}$ then there is an optimal solution of $IP_{A,c'}(b)$ whose support is contained in the set of indices corresponding to the columns of A that are in $cone(A_\sigma)$. In fact, by perturbing c' to a generic c, we will make sure that there is a unique such optimal solution. The last step of the proof will show that this unique optimal solution is covered by a standard pair of the form $(\cdot, \sigma)$. We will conclude that only maximal faces of Δ are associated to $IP_{A,c}$, and hence $IP_{A,c}$ is a Gomory family.

Without loss of generality we can assume that the columns of A in $cone(A_\sigma)$ form a minimal Hilbert basis of $cone(A_\sigma)$ for any maximal face σ of Δ. If there were a redundant element, the smaller matrix obtained by removing this column from A would still be Δ-normal.

For a maximal face $\sigma \in \Delta$, let $\sigma_{in} \subset \{1, \ldots, n\}$ be the set of indices of all columns of A lying in $cone(A_\sigma)$ that are different from the columns of A_σ. Suppose $a_{i_1}, \ldots, a_{i_k}$ are the columns of A that generate the one dimensional faces of Δ, and $\omega \in \mathbb{R}^n$ a cost vector such that $\Delta = \Delta_\omega$. We modify ω to obtain a new cost vector $c' \in \mathbb{R}^n$ such that we still have $\Delta = \Delta_{c'}$ as follows. For $j = 1, \ldots, k$, let ${c'}_{i_j} := \omega_{i_j}$. If $j \in \sigma_{in}$ for some maximal face $\sigma \in \Delta$, then $a_j = \sum_{i\in\sigma} \lambda_i a_i$, $0 \leq \lambda_i < 1$ and we define ${c'}_j := \sum_{i\in\sigma} \lambda_i \omega_i$. Hence, for all $j \in \sigma_{in}$, $(a_j, {c'}_j) \in \mathbb{R}^{d+1}$ lies in $C_\sigma := cone((a_i, {c'}_i) : i \in \sigma) = cone((a_i, \omega_i) : i \in \sigma)$ which is a facet of $C = cone((a_i, \omega_i) : i = 1, \ldots, n)$. If $y \in \mathbb{R}^d$ is a vector as in Definition 2.1 showing that σ is a maximal face of $\Delta_{c'}$ then $y \cdot a_i = {c'}_i$ for all $i \in \sigma \cup \sigma_{in}$ and $y \cdot a_j < {c'}_j$ otherwise. Since $cone(A_\sigma) = cone(A_{\sigma\cup\sigma_{in}})$, we conclude that $cone(A_\sigma)$ is a maximal face of $\Delta_{c'}$.

If $b \in \mathbb{N}A$ lies in $cone(A_\sigma)$ for a maximal face $\sigma \in \Delta_{c'}$, then $IP_{A,c'}(b)$ has at least one feasible solution u with support in $\sigma \cup \sigma_{in}$ since A is Δ-normal and the columns $A_{\sigma\cup\sigma_{in}}$ form a Hilbert basis of $cone(A_\sigma)$. Further, $(b, c' \cdot u) = (Au, c' \cdot u)$ lies in C_σ and all feasible solutions of $IP_{A,c'}(b)$ with support in $\sigma \cup \sigma_{in}$ have the same cost value by construction. Suppose $v \in \mathbb{N}^n$ is any feasible solution of $IP_{A,c'}(b)$ with support not in $\sigma \cup \sigma_{in}$. Then $c' \cdot u < c' \cdot v$ since $(a_i, {c'}_i) \in C_\sigma$ if and only if $i \in \sigma \cup \sigma_{in}$ and C_σ is a facet among the lower facets of C. Hence the optimal solutions of $IP_{A,c'}(b)$ are precisely those feasible solutions with support in $\sigma \cup \sigma_{in}$. The vector b can be expressed as $b = b' + \sum_{i\in\sigma} z_i a_i$ where $z_i \in \mathbb{N}$ are unique and $b' \in \{\sum_{i\in\sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ is also unique. The vector $b' = \sum_{j\in\sigma_{in}} r_j a_j$ where $r_j \in \mathbb{N}$. Setting $u_i = z_i$ for all $i \in \sigma$, $u_j = r_j$ for all $j \in \sigma_{in}$ and $u_k = 0$ otherwise, we obtain all feasible solutions u of $IP_{A,c'}(b)$ with support in $\sigma \cup \sigma_{in}$.

The arguments above imply that the optimal solutions to $IP_{A,c'}(b)$ are to be found among u with support in $\sigma \cup \sigma_{in}$. If there is more than one such optimal solution, then c' is not generic. In this case, we can perturb c' to a generic cost vector $c = c' + \epsilon\omega'$ by choosing $1 \gg \epsilon > 0$, ${\omega'}_j \ll 0$ whenever $j = i_1, \ldots, i_k$ and ${\omega'}_j = 0$ otherwise. Since all data is integral, $c \in \mathbb{Q}^n$ and hence can be scaled to lie in $\mathbb{Z}^n$. Suppose $u_1, \ldots, u_t$ are the optimal solutions of the integer programs $IP_{A,c}(b')$ where $b' \in \{\sum_{i\in\sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$. (Note that $t = |\{\sum_{i\in\sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d|$ is the index of $\mathbb{Z}A_\sigma$ in $\mathbb{Z}A$.) The support of each such u_i is contained in σ_{in}. For any $b \in cone(A_\sigma) \cap \mathbb{Z}^d$, the optimal solution of $IP_{A,c}(b)$ is hence $u = u_i + z$ for some $i \in \{1, \ldots, t\}$ and $z \in \mathbb{N}^n$ with support in σ. This shows that $\mathbb{N}A$ is covered by the affine semigroups $\phi_A(S(u_i, \sigma))$ where σ is a maximal face of Δ and u_i as above for each

σ. By construction, the corresponding admissible pairs (u_i, σ) are all standard for $\mathcal{O}_c$. We conclude that $IP_{A,c}$ is a Gomory family. □

Corollary 4.8. *Let A be any normal matrix such that $cone(A)$ is simplicial, and let Δ be the coarsest triangulation whose single maximal face has support $cone(A)$. Then there exists a cost vector $c \in \mathbb{Z}^n$ such that $\Delta = \Delta_c$ and $IP_{A,c}$ is a Gomory family.*

Example 4.9. Consider the normal matrix in Example 2.23. Here $cone(A)$ is generated by the first, second and sixth columns of A and hence A is Δ-normal with respect to the regular triangulation $\Delta = \{\{1, 2, 6\}\}$. There are 13 distinct sets $\mathcal{O}_c$ supported on Δ. Among the 13 corresponding families of integer programs, only one is a Gomory family. A representative cost vector for this $IP_{A,c}$ is $c = (0, 0, 4, 4, 1, 0)$. The standard pair decomposition of $\mathcal{O}_c$ is the one constructed in Theorem 4.7. The affine semigroups $S(\cdot, \sigma)$ from this decomposition are:

$$S(0, \sigma),\ S(e_3, \sigma),\ S(e_4, \sigma),\ \text{and } S(e_5, \sigma).$$

Note that A is not Δ-normal with respect to the regular triangulation supporting the Gomory family $IP_{A,c}$ in Example 2.23. The columns of A in $cone(A_{\sigma_1})$ are the columns of A_{σ_1} and A_3. The vector $(1, 2, 2)$ is in the minimal Hilbert basis of $cone(A_{\sigma_1})$ but is not a column of A. This example shows that a regular triangulation Δ of $cone(A)$ can support a Gomory family even if A is not Δ-normal. The Gomory families in Theorem 4.7 have a very special standard pair decomposition.

5. Hilbert covers and Gomory families

The results in the previous section lead to the following problem.

Problem 5.1. *If $A \in \mathbb{Z}^{d\times n}$ is a normal matrix, does there exist a generic cost vector $c \in \mathbb{Z}^n$ such that $IP_{A,c}$ is a Gomory family?*

We do not know the answer to this question. However, in this section we answer a stronger version of this question for small values of d and state our observations for general d. We begin with the following result.

Theorem 5.2. *If $A \in \mathbb{Z}^{d\times n}$ is a normal matrix and $d \leq 3$, then there exists a generic cost vector $c \in \mathbb{Z}^n$ such that $IP_{A,c}$ is a Gomory family.*

Proof. It is known that if $d \leq 3$ then $cone(A)$ has a regular unimodular triangulation Δ_c [18]. The result then follows from Theorem 3.4. □

Before we proceed, we rephrase Problem 5.1 in terms of covering properties of $cone(A)$ and $\mathbb{N}A$ along the lines of [2], [3], [4], [8] and [18]. To obtain the same set up as in these papers we assume in this section that the columns of A form the unique minimal Hilbert basis of *cone*(A). Using the terminology in [3], the *free Hilbert cover* problem asks whether there exists a covering of $\mathbb{N}A$ by semigroups $\mathbb{N}A_\tau$ where the columns of A_τ are linearly independent. The *unimodular Hilbert cover* problem asks whether *cone*(A) can be covered by full dimensional unimodular subcones *cone*(A_τ) (i.e., $\mathbb{Z}A_\tau = \mathbb{Z}^d$), while the stronger *unimodular Hilbert partition* problem asks whether *cone*(A) has a unimodular triangulation. (Note that if *cone*(A) has a unimodular Hilbert cover or partition using subcones *cone*(A_τ), then $\mathbb{N}A$ is covered by the semigroups $\mathbb{N}A_\tau$.) All these problems have positive answers if $d \leq 3$ since *cone*(A) admits a unimodular Hilbert partition in this case [2], [18]. Normal matrices (with $d = 4$) such that *cone*(A) has no unimodular Hilbert partition can be found in [2] and [8]. Examples (with $d = 6$) that admit no free Hilbert cover and hence no unimodular Hilbert cover can be found in [3] and [4].

When $yA \leq c$ is TDI, the standard pair decomposition of $\mathbb{N}A$ induced by c gives a unimodular Hilbert partition of *cone*(A) by Lemma 3.3. An important difference between Problem 5.1 and the Hilbert cover problems is that *affine* semigroups cannot be used in Hilbert covers. Moreover, affine semigroups that are allowed in standard pair decompositions come from integer programming. If there are no restrictions on the affine semigroups that can be used in a cover, $\mathbb{N}A$ can always be covered by full dimensional affine semigroups: for any triangulation Δ of *cone*(A) with maximal subcones *cone*(A_σ), the affine semigroups $b + \mathbb{N}A_\sigma$ cover $\mathbb{N}A$ as b varies in $\{\sum_{i\in\sigma} \lambda_i a_i : 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ and σ varies among the maximal faces of the triangulation. A *partition* of $\mathbb{N}A$ derived from this idea can be found in [19, Theorem 5.2].

In order to state the main theorem of this section, we recall the notion of *supernormality* which was introduced in [12].

Definition 5.3. A matrix $A \in \mathbb{Z}^{d\times n}$ is *supernormal* if for every submatrix A' of A, the columns of A that lie in *cone*(A') form a Hilbert basis for *cone*(A').

Proposition 5.4. *For $A \in \mathbb{Z}^{d\times n}$, the following are equivalent:*

(i) *A is supernormal,*
(ii) *A is Δ-normal for every regular triangulation Δ of cone(A),*
(iii) *Every triangulation of cone(A) in which every column of A generates a one-dimensional face, is unimodular.*

Proof. The equivalence of (i) and (iii) was established in [12, Proposition 3.1]. Definition 5.3 shows that (i) $\Rightarrow$ (ii). Hence we just need to show that (ii) $\Rightarrow$ (i). Suppose that A is Δ-normal for every regular triangulation of *cone*(A). In order to show that A is supernormal we only need to check submatrices A' where the dimension of $cone(A')$ is d. Choose a cost vector c with $c_i \gg 0$ if the i-th column of A does not generate an extreme ray of $cone(A')$, and $c_i = 0$ otherwise. This gives a polyhedral subdivision of $cone(A)$ in which $cone(A')$ is a maximal face. There are standard procedures that will refine this subdivision to a regular triangulation Δ of $cone(A)$. Let T be the set of maximal faces σ of Δ such that $cone(A_\sigma)$ lies in $cone(A')$. Since A is Δ-normal, the columns of A that lie in $cone(A_\sigma)$ form a Hilbert basis for $cone(A_\sigma)$ for each $\sigma \in T$. However, since their union is the set of columns of A that lie in $cone(A')$, this union forms a Hilbert basis for $cone(A')$. $\square$

It is easy to catalog all Δ-normal and supernormal matrices, of the type considered in this paper, for small values of d. We say that the matrix A is *graded* if its columns span an affine hyperplane in $\mathbb{R}^d$. If $d = 1$, $cone(A)$ has n triangulations $\{\{i\}\}$ each of which has the unique maximal subcone $cone(A_i)$ whose support is $cone(A)$. If we assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then A is normal if and only if either $a_1 = 1$, or $a_n = -1$. Also, A is normal if and only if it is supernormal. If $d = 2$ and the columns of A are ordered counterclockwise around the origin, then A is normal if and only if $det(a_i, a_{i+1}) = 1$ for all $i = 1, \ldots, n-1$. Such an A is supernormal since it is Δ-normal for every triangulation Δ — the Hilbert basis of a maximal subcone of Δ is precisely the set of columns of A in that subcone. If $d = 3$ then as mentioned before, $cone(A)$ has a unimodular triangulation with respect to which A is Δ-normal. However, not every such A needs to be supernormal: we saw that the matrix in Example 2.23 is not Δ-normal for the Δ supporting the Gomory family in that example. If $d = 3$ and A is graded, then without loss of generality we can assume that the columns of A span the hyperplane $x_1 = 1$. If A is normal as well, then its columns are precisely all the lattice points in the convex hull of A. Conversely, every graded normal A with $d = 3$ arises this way — its columns are all the lattice points in a polygon in $\mathbb{R}^2$ with integer vertices. In particular, every triangulation of $cone(A)$ that uses all the columns of A is unimodular. Hence, by Proposition 5.4, A is supernormal, and therefore Δ-normal for any triangulation of A.

Theorem 5.5. *Let $A \in \mathbb{Z}^{d \times n}$ be a normal matrix of rank d.*

(i) *If $d = 1, 2$ or A is graded and $d = 3$, every regular triangulation of $cone(A)$ supports at least one Gomory family.*
(ii) *If $d = 2$ and A is graded, every regular triangulation of $cone(A)$ supports exactly one Gomory family.*
(iii) *If $d = 3$ and A is not graded, or if $d \geq 4$, then not all regular triangulations of $cone(A)$ may support a Gomory family. In*

particular, A may not be Δ-normal with respect to every regular triangulation of $cone(A)$.

Proof. (i) If $d = 1, 2$ or A is graded and $d = 3$, A is supernormal and hence by Proposition 5.4 and Theorem 4.7, every regular triangulation of $cone(A)$ supports at least one Gomory family.
(ii) If $d = 2$ and A is graded, then we may assume that

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n-1 \end{pmatrix}.$$

In this case, A is supernormal and hence every regular triangulation Δ of $cone(A)$ supports a Gomory family by Theorem 4.7. Suppose the maximal cones of Δ, in counter-clockwise order, are $C_1, \dots, C_r$. Assume the columns of A are labeled such that $C_i = cone(a_{i-1}, a_i)$ for $i = 1, \dots, r$, and the columns of A in the interior of C_i are labeled in counter-clockwise order as $b_{i1}, \dots, b_{ik_i}$. Hence we may assume that the n columns of A from left to right are:

$$a_0, b_{11}, \cdots, b_{1k_1}, a_1, b_{21}, \cdots, a_{r-1}, b_{r1}, \cdots, b_{rk_r}, a_r.$$

Indexing the columns of A by their labels, the maximal faces of Δ are $\sigma_i = \{i-1, i\}$ for $i = 1, \dots, r$. Let e_i be the unit vector of $\mathbb{R}^n$ indexed by the true column index of a_i in A and e_{ij} be the unit vector of $\mathbb{R}^n$ indexed by the true column index of b_{ij} in A. Since the columns of A form a minimal Hilbert basis of $cone(A)$, e_i is the unique solution to $IP_{A,c}(a_i)$ for all c and e_{ij} is the unique solution to $IP_{A,c}(b_{ij})$ for all c. Hence the standard pairs of Theorem 4.7 are $(0, \sigma_i)$ and (e_{ij}, σ_i) for $i = 1, \dots, r$ and $j = 1, \dots, k_i$.

Suppose Δ supports a second Gomory family $IP_{A,\omega}$. Then every standard pair of $\mathcal{O}_w$ is also of the form $(\cdot, \sigma_i)$ for $\sigma_i \in \Delta$, and r of them are $(0, \sigma_i)$ for $i = 1, \dots, r$. The remaining standard pairs are of the form (e_{ij}, σ_k). To see this, consider the semigroups in $\mathbb{N}A$ arising from the standard pairs of $\mathcal{O}_w$. The total number of standard pairs of $\mathcal{O}_c$ and $\mathcal{O}_w$ are the same. Since the columns of A all lie on $x_1 = 1$, no two b_{ij}s can be covered by a semigroup coming from the same standard pair and none of them are covered by a semigroup $(0, \sigma_i)$. We show that if (e_{ij}, σ_k) is a standard pair of $\mathcal{O}_w$ then $k = i$ and thus $\mathcal{O}_w = \mathcal{O}_c$.

If $r = 1$, the standard pairs of $\mathcal{O}_w$ are $(0, \sigma_1), (e_{11}, \sigma_1), \dots, (e_{1k_1}, \sigma_1)$ as in Theorem 4.7. If $r > 1$, consider the last cone $C_r = cone(a_{r-1}, a_r)$. If a_{r-1} is the second to last column of A, then C_r is unimodular and the semigroup from $(0, \sigma_r)$ covers $C_r \cap \mathbb{Z}^2$. The subcomplex comprised of $C_1, \dots, C_{r-1}$ is a regular triangulation Δ' of $cone(A')$ where A' is obtained by dropping the last column of A. Since A' is a normal graded matrix with $d = 2$ and Δ' has less than r maximal cones, the standard pairs supported on Δ' are as in Theorem 4.7 by induction. If a_{r-1} is not the second to last column of A then b_{rk_r}, the second to last column of A is in the Hilbert basis of C_r but is not a generator of C_r. So $\mathcal{O}_w$ has a standard pair of the form (e_{rk_r}, σ_i). If $\sigma_i \neq \sigma_r$, then the lattice point $b_{rk_r} + a_r$ cannot be covered by the semigroup from this or any other standard pair of $\mathcal{O}_w$ (this is most easily seen by drawing a picture). Hence $\sigma_i = \sigma_r$. By a similar argument, the remaining standard pairs indexed by σ_r are $(e_{r(k_r-1)}, \sigma_r), \ldots, (e_{r1}, \sigma_r)$ along with $(0, \sigma_r)$. These are precisely the standard pairs of $\mathcal{O}_c$ indexed by σ_r. Again we are reduced to considering the subcomplex comprised of $C_1, \ldots, C_{r-1}$ and by induction, the remaining standard pairs of $\mathcal{O}_w$ are as in Theorem 4.7.

(iii) The 3×6 normal matrix A of Example 2.23 has 10 distinct Gomory families supported on 10 out of the 14 regular triangulations of $cone(A)$. Furthermore, the normal matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

has 11 distinct Gomory families supported on 11 out of the 19 regular triangulations of $cone(A)$. $\square$

6. Computations and Commutative Algebra

As mentioned in the introduction, the computations in this paper were done using the connections of this material to commutative algebra (see [20], [21] and [25]). In this section we give a brief description of our methods.

Our general feeling is that Problem 5.1 has a negative answer. To check whether a matrix A has a Gomory family, we first need to compute all the distinct sets of optimal solutions $\mathcal{O}_c$ (to the programs in $IP_{A,c}$) that arise as c varies among the generic cost vectors with respect to A. As mentioned in Section 2, there are only finitely many such sets for a fixed A. To check whether $IP_{A,c}$ is a Gomory family, we need to compute the standard pair decomposition of $\mathcal{O}_c$: $IP_{A,c}$ is a Gomory family if and only if all the standard pairs of $\mathcal{O}_c$ are indexed by maximal faces of Δ_c.

A *monomial* x^u in the *polynomial ring* $S := \mathbb{Q}[x_1, \ldots, x_n]$ is the product $x^u = x_1^{u_1} \cdots x_n^{u_n}$ where $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$. An ideal M in S is called a *monomial ideal* if it is generated by monomials. Since every ideal in S is finitely generated, $M = \langle x^{v_1}, \ldots, x^{v_t} \rangle$ for a set of minimal generators $x^{v_1}, \ldots, x^{v_t}$. The *toric ideal* of A, denoted as I_A is the *binomial ideal* in S defined as:

$$\langle x^u - x^v : \ u, v \in \mathbb{N}^n \text{ and } Au = Av \rangle.$$

The *cost* of a monomial x^u with respect to a cost vector $c \in \mathbb{R}^n$ is the dot product $c \cdot u$ and the *initial term* of a polynomial $f = \sum \lambda_u x^u \in S$ is the sum of all terms in f of highest cost. For any ideal $I \subset S$, the *initial ideal* of I with respect to c, denoted as $in_c(I)$, is the ideal generated by all the initial terms of all polynomials in I. These concepts come from the theory of *Gröbner bases* for polynomial ideals [6]. The toric ideal I_A provides the algebraic link between integer programming and Gröbner basis theory. For an introduction to this connection see [20].

Proposition 6.1. *(i) A cost vector $c \in \mathbb{R}^n$ is generic for A if and only if the initial ideal $in_c(I_A)$ is a monomial ideal.*
(ii) For a generic c, the vector u belongs to $\mathcal{O}_c$ if and only if x^u is not in the initial ideal $in_c(I_A)$. □

There are only finitely many distinct initial ideals for a polynomial ideal [20], and hence there are only finitely many distinct sets $\mathcal{O}_c$ as c varies among the generic cost vectors. We say that two cost vectors c and c' in $\mathbb{R}^n$ are *equivalent* if $in_c(I_A) = in_{c'}(I_A)$.

Theorem 6.2. [21, Theorem 3.10] *(i) Each equivalence class of generic cost vectors form an open full dimensional polyhedral cone in $\mathbb{R}^n$. The closure of this cone is called the* Gröbner cone *of c where c is any vector in the interior of the cone.*
(ii) The collection of all Gröbner cones of A form a complete polyhedral fan in $\mathbb{R}^n$ called the Gröbner fan *of A.*
(iii) The Gröbner fan of A is the normal fan of an $(n-d)$-dimensional polytope called the state polytope *of A.* □

By the above results, $\mathcal{O}_c$ can be computed implicitly by computing the monomial ideal $in_c(I_A)$. This can be done using a computer algebra package like Macaulay 2 [11]. In order to find all initial ideals of I_A, we use the software package TiGERS [16] for enumerating the vertices of the state polytope of A. At each vertex, TiGERS returns the initial ideal induced by a vector in the interior of the normal cone at that vertex. The standard pair decomposition of the set of monomials outside a monomial ideal described in terms of its minimal generators can be calculated using Macaulay 2. See [13] for details.

To obtain a normal matrix of the type discussed in this paper, it suffices to start with an arbitrary set of vectors $a_1, \ldots, a_p \in \mathbb{Z}^d$ such that $cone(a_1, \ldots, a_p)$ is pointed and full dimensional and then to compute the Hilbert basis of $cone(a_1, \ldots, a_p)$. The elements in the Hilbert basis form the columns of a normal matrix. We used the package Normaliz by Bruns and Koch [5] to compute Hilbert bases.

The regular triangulation Δ_c of $cone(A)$ is a pure d-dimensional complex of cones and hence every maximal face $\sigma \in \Delta_c$ has cardinality d. Hence we get the following algorithm.

Algorithm 6.3. *How to check if $A \in \mathbb{Z}^{d\times n}$ gives a Gomory family.*
(i) Compute all the initial ideals of the toric ideal I_A using TiGERS.
(ii) For each initial ideal $in_c(I_A)$:
(a) Use Macaulay 2 to find its standard pairs.
(b) $IP_{A,c}$ is a Gomory family if and only if every set τ in a standard pair $(\cdot, \tau)$ has cardinality d.

A monomial x^u is *square-free* if $u \in \{0,1\}^n$ and a monomial ideal M is *square-free* if all its minimal generators are square-free. The *radical* of a monomial ideal M in S is the ideal $\sqrt{M} := \langle f : f^r \in M \text{ for some } r \in \mathbb{N}\rangle$. The radical $\sqrt{M}$ is a square-free monomial ideal. The *Stanley Reisner* ideal of the regular triangulation Δ_c is the square-free monomial ideal

$$\langle \Pi_{i\in\tau} x_i : \tau \text{ is a minimal non-face of } \Delta_c\rangle.$$

Theorem 6.4. [20, Theorem 8.3] *The Stanley-Reisner ideal of the regular triangulation Δ_c is the radical of $in_c(I_A)$.* ☐

If two distinct initial ideals $in_c(I_A)$ and $in_{c'}(I_A)$ have the same radical, then $\Delta_c = \Delta_{c'}$ and we say that Δ_c supports these initial ideals. Several initial ideals of I_A may have the same radical. Using the above theorem, the initial ideals of I_A output by TiGERS can be grouped according to their radicals or equivalently, the regular triangulations supporting them. This, in combination with Algorithm 6.3, allows us to check whether a regular triangulation supports a Gomory family.

Recall from Lemma 3.3 that $yA \leq c$ is TDI if and only if Δ_c is unimodular. Using the following result, we obtain an algebraic check for TDI-ness of $yA \leq c$.

Theorem 6.5. [20, Corollary 8.9] *The regular triangulation Δ_c is unimodular if and only if the initial ideal $in_c(I_A)$ is square-free.* ☐

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