SIMPLEXITY OF THE CUBE

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Abstract

A computer-assisted linear programming approach is used to study minimum-cardinality decompositions of the cube. A triangulation of the 7-cube into 1493 *simplices is given and it is shown that this, and a previously given triangulaton of the* 6*-cube into* 308 *simplices, are the smallest possible for these dimensions. A characterization is given for the numbers of the various types of simplices used in all minimum-cardinality triangulations of the d-cube for* $d = 5, 6, 7$*. It is shown that the minimum of the cardinalities of all corner-slicing triangulations of* I^7 *is 1820. For decompositions of the cube more general than triangulations, it is shown for dimension* 5 *that the minimum cardinality is* 67*, and for dimension* 6 *it is at least* 270.

1. Introduction

We consider finite decompositions of the d-dimensional cube I^d, where $I = [0, 1]$, into simplices with disjoint interiors whose union is I^d. In this paper, all the simplices in I^d have the property that their vertices are also vertices of I^d. Our main interest is in *triangulations*: special decompositions into simplices such that the intersection of any two simplices is a face of each of them. A *corner* at the vertex v of I^d is $\text{conv}\{v, x_1, \ldots, x_d\}$

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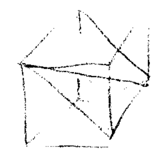
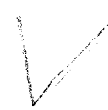
where $x_1, \ldots, x_d$ are the vertices of I^d adjacent to v. A triangulation is called *corner-slicing* if it includes the corners at all the vertices of I^d with odd (or alternately even) coordinate sums. Triangulations are used in simplicial algorithms for finding fixed points, and, in general, smaller cardinality suggests greater computational efficiency. We let $S(d)$ (respectively $T(d)$, $T^c(d)$) be the number of simplices in a minimum-cardinality decomposition (triangulation, corner-slicing triangulation) of I^d. Thus $S(d) \leq T(d) \leq T^c(d)$. The asymptotic behavior of these sequences and their values for small dimensions have been studied by several authors. Our goal is to further specify these sequences for small d.



It is a triviality that $S(2) = T(2) = T^c(2) = 2$. In an early paper, Mara proved that $T(3) = T^c(3) = 5$ and $T^c(4) = 16$ [15]. Cottle established directly that $T(4) = 16$ [7]. Sallee and Lee described corner-slicing triangulations of I^d which show $T^c(d) \leq 67$, 364, 2445 for $d = 5, 6, 7$ [14, 16]. The idea of using linear programming to establish lower bounds for these sequences was introduced by Sallee [17]. His approach shows $S(d) \geq 5$, 16, 60, 250, 1117 for $d = 3, 4, 5, 6, 7$. Sallee then developed the middle-cut triangulations (not corner-slicing for $d \geq 5$) of I^d [18]. These yield the bounds $T(6) \leq 324$ and $T(7) \leq 1962$. Working independently with similar ideas, Böhm introduced decompositions which give improved upper bounds on $S(d)$ for $d \geq 7$. In particular his results include $S(7) \leq 1927$ [3]. Broadie and Cottle developed some special properties of corner-slicing triangulations and established the value $T^c(5) = 67$ [5]. Hughes used linear programming techniques and showed $T(5) = 67$ and $T^c(6) = 324$ [11]. These results can be checked without a computer, but to do so requires an extremely diligent reader. Hughes and Anderson described a triangulation of I^6 with 308 simplices and thus $T(6) \leq 308$ [13]. Verification of this triangulation is helped by computer programs. Using linear programming and a computer to compute exterior-facet tuples, Hughes developed lower bounds for $\{S(d)\}$ as optimal objective values of linear programming problems [12]. Actual numbers were found only for dimensions 3–11 and include $S(5) \geq 61$, $S(6) \geq 259$, and $S(7) \geq 1175$.



Summarizing, in dimensions 3–7 it is known that

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$$S(3) = T(3) = T^c(3) = 5$$
$$S(4) = T(4) = T^c(4) = 16$$
$$61 \leq S(5) \leq T(5) = T^c(5) = 67$$
$$259 \leq S(6) \leq T(6) \leq 308 < T^c(6) = 324$$
$$1175 \leq S(7) \leq 1927,\ 1175 \leq T(7) \leq 1962,\ 1175 \leq T^c(7) \leq 2445$$

In the current paper, we develop the results $S(5) = 67$, $270 \leq S(6)$, $T(6) = 308$,

$T(7) = 1493$, and $T^c(7) = 1820$. We also characterize the numbers of the various types of simplices used in all minimum-cardinality triangulations of I^5, I^6, and I^7 and describe all such triangulations of I^5. The linear programming approach used in the current paper is simpler and far more systematic than that used earlier by Hughes [11]. Unfortunately, even for the same dimension, the new approach leads to larger linear programming problems. All our results rely on many computer programs including linear programming software. To completely confirm our work, considerable programming would be required.

In Section 2, we introduce the concepts which are needed for our linear programming approach. The basic linear programming problems are discussed in Section 3. Section 4 contains some comments about the computer programs used to generate the linear programming problems, and in Section 5 we present our results. In dimensions 6 and 7, space limitations prohibit displaying the linear programming problems or the specific data used to generate them. The appendix displays such data for dimension 5.

2. Definitions and theorems

In this section we lay the foundation necessary for an understanding of the linear programming problems used to obtain the results of this paper. We let c denote the centroid of I^d, $c = (0.5, \ldots, 0.5)$, and let e be the row vector of 1's in R^d. The *complement* of a $\{0,1\}$-matrix is obtained by replacing each 0 by 1 and each 1 by 0.

In R^d the volume of a d-simplex is

$$(1/d!)\left|\det\left[M, e^T\right]\right| \tag{2.1}$$

where M is a matrix whose rows are the coordinates of the vertices.

We let g_n be the maximum of the determinants of all $n \times n$ $\{0,1\}$-matrices. It is known that $g_n = 1, 2, 3, 5, 9, 32$ for $n = 2, 3, \ldots, 7$ [4, 6, 10 ,19]. (See especially [10] for an extensive list of references.) It is easily seen that the largest of the absolute values of the determinants of all $n \times n$ $\{0,1\}$-matrices with a column of 1's is g_{n-1}, and hence the largest volume of a simplex in I^d is $g_d/d!$.

A facet of a simplex contained in I^d is an *exterior facet* if it is contained in a facet of I^d.

A simplex S in I^d with exterior facet F is a pyramid with base F and altitude 1, and consequently

$$(1/d)\big((d-1)\text{-volume of } F\big) = \text{volume of } S. \tag{2.2}$$

Let S be a d-simplex in I^d with $d \geq 3$. Let S_1 be the set of exterior facets of S and note that each of these is a $(d-1)$-simplex in a facet of I^d, which is a copy of I^{d-1}. Then, considering the facets of I^d to be disjoint, we let S_2 be the set of exterior facets of the $(d-1)$-simplices in S_1. Continuing in this manner, let $S_3, \ldots, S_{d-2}$ be similarly defined. Then the $(d-2)$-tuple $(|S_1|, \ldots, |S_{d-2}|)$ is the *exterior-facet tuple* of S.

One of the authors has introduced linear programming problems, with a variable for each volume-exterior-facet-tuple pair, whose optimal objective values provide lower bounds for the $S(d)$'s [12]. These problems were solved only for $d \leq 11$. The constraints reflect the volumes of the k-faces of I^d for $k = d, d-1, \ldots, 2$. In addition to being concerned with all three sequences $\{S(d)\}$, $\{T(d)\}$, and $\{T^c(d)\}$, instead of just $\{S(d)\}$, the current paper differs largely by having the variables depend on more information and including constraints which take advantage of the additional structure.

We turn to the key definition of a configuration class of simplices in I^d. Our definition uses mappings in R^d of a special form; each is the composition of a mapping which permutes the coordinates followed by reflections across the hyperplanes $\{ x_i = 1/2 : i \in J \}$ for some subset J of the coordinates of R^d. These mappings are of the form

$$A(x) = \frac{1}{2}(e - eD) + xPD \tag{2.3}$$

where $x = (x_1, \ldots, x_d)$, P is a $d \times d$ permutation matrix, and D is a $d \times d$ diagonal matrix whose diagonal entries are in $\{-1, 1\}$. They are affine isometries. Applied to vertices of I^d, such a mapping permutes the coordinates and then, for all i in some subset J of $\{1, 2, \ldots, d\}$, replaces the possibly new i^{th} coordinate by its complement. If S is a d-simplex in I^d with vertices $v_1, \ldots, v_{d+1}$, then the image of S under a map A of the form (2.3) is the congruent simplex $A(S)$ with vertices $A(v_1), \ldots, A(v_{d+1})$. The set of such ordered pairs $(S, A(S))$ is an equivalence relation and we call the equivalence classes ***configuration classes***. If S and T are simplices in the same configuration class, then they have the same volume, the same exterior-facet tuple, and the barycentric coordinates of c relative to the vertices of S are the same as those relative to the vertices of T.

There are four configuration classes for I^3 and these classes can be distinguished by their volume-exterior-facet-tuple pairs: $(1/3!, f)$ for $f \in \{1, 2, 3\}$, and $(2/3!, 0)$. We note, however, that in I^4, the simplices described by the matrices

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in different configuration classes even though they both have volume $1/4!$, exterior-facet tuple $(2,4)$, and the sequence barycentric coordinates of c relative of the vertices of either is $(0.5, 0.5, 0, 0, 0)$.

Theorem 2.1. Let F be a facet of a simplex whose vertices are vertices of I^d. Then the hyperplane H in R^d containing F has a normal vector with all integer components.

Proof. Let $F = \text{conv}\{v_1, \ldots, v_d\}$. An equation for H is

$$\det \begin{bmatrix} v_1 & 1 \\ v_2 & 1 \\ \vdots & \vdots \\ v_d & 1 \\ x & 1 \end{bmatrix} = 0,$$

where $x = (x_1, \ldots, x_d)$. If we write this in the form $n_1x_1 + n_2x_2 + \cdots + n_dx_d = b$, then the normal vector $(n_1, \ldots, n_d)$ has all integer components. Moreover $|b| \leq g_d$ and $|n_i| \leq g_{d-1}$ for $i = 1, \ldots, d$.

□

Definition. If a normal $n = (n_1, \ldots, n_d)$ for a hyperplane H that contains a facet of a simplex in I^d has $0 \leq n_1 \leq n_2 \leq \cdots \leq n_d$ with all integer components and $\gcd(n_1, \ldots, n_d) = 1$, then we call n a *fundamental normal* for I^d and H a *fundamental hyperplane* for I^d.

Definition. A hyperplane H is called *central* or *noncentral* for I^d according as the centroid c of I^d is in H or is not in H.

Definition. Suppose H is a noncentral hyperplane for I^d. The half-space of R^d determined by H which contains c is the *long half-space* determined by H, and the other half-space is the *short half-space*.

Definition. In R^d the nonzero normal vector n is *equivalent* to the vector m in case the coordinates of n can be permuted and then, perhaps, the signs of some of the coordinates changed to arrive at a real multiple of m.

This is an equivalence relation, and any hyperplane containing a facet of a simplex in I^d has a normal equivalent to a fundamental normal for I^d.

Definition. The hyperplanes $n \cdot x = g$ and $m \cdot x = h$ containing facets of simplices of I^d are called *equivalent* for I^d if n and m are equivalent normals and the ratios

$$\Big(\big(\max_{x \in I^d} n \cdot x \big) - g \Big) : \Big(g - \big(\min_{x \in I^d} n \cdot x \big) \Big)$$

and

$$\Big(\big(\max_{x \in I^d} m \cdot x \big) - h \Big) : \Big(h - \big(\min_{x \in I^d} m \cdot x \big) \Big)$$

are the same or reciprocals.

Intuitively, equivalent hyperplanes cut the cube in symmetric ways. The hyperplanes $x_1 + x_2 + x_3 = 1$ and $-2x_1 + 2x_2 - 2x_3 = 0$ in R^3 are equivalent for I^3.

We turn to the concept of a base which plays an important role in our linear programming problems.

Definition. A set of d vertices of I^d is called a *base* for I^d if it is affinely independent. A base is *central* if the hyperplane it determines is central for I^d.

Thus a base for I^d is the same as the set of vertices of a facet of a simplex in I^d. If the base B consists of the vertices of a facet of the simplex S, we often refer to B as a base of S.

Suppose $S = \text{conv}\{v_1, \ldots, v_{d+1}\}$ is a simplex in I^d with facets $F_1, F_2, \ldots, F_{d+1}$ where $v_i \notin F_i$. Let $\lambda_1, \ldots, \lambda_{d+1}$ be the sequence of barycentric coordinates of c relative to the vertices of S. Then F_i determines a central hyperplane for I^d if and only if $\lambda_i = 0$. Also S is in the short (respectively long) half-space determined by F_i if and only if $\lambda_i < 0$ ($\lambda_i > 0$). We call the sign of λ_i the *half-space indicator* of S for F_i. Thus if the common facet F of two adjacent simplices in I^d is contained in a noncentral hyperplane for I^d, then the simplices have opposite half-space indicators for F.

Definition. The bases B_1 and B_2 for I^d are called *equivalent* if there is a mapping of the form (2.3) which carries B_1 onto B_2.

This is an equivalence relation and we refer to the equivalence classes as *base classes.* Clearly all bases B in the same base class have the same value for the $(d-1)$-volume of $\text{conv}(B)$. Later we will note a helpful connection between this volume and $|\det(M_B)|$ where M_B is a matrix whose rows are the elements of B.

Lemma 2.2. Suppose the vertices $v^1 \dots, v^k$ of I^d, satisfy $n \cdot v^i = f$ for $i = 1, \dots, k$. Suppose $w^1, \dots, w^k$ are obtained from these vertices by complementation in coordinate p: thus

$$w_p^i = \begin{cases} 0, & \text{if } v_p^i = 1; \\ 1, & \text{if } v_p^i = 0; \end{cases}$$

and $w_j^i = v_j^i$ for $j \neq p$. Then for $i = 1, \dots, k$,

$$(n_1, \dots, n_{p-1}, -n_p, n_{p+1}, \dots, n_d) \cdot w^i = f - n_p,$$

where $n = (n_1, \dots, n_d)$. Moreover, if $v^1, \dots, v^k$ are affinely independent, then so are $w^1, \dots, w^k$.

The proof is straightforward and is left to the reader.

Theorem 2.3. Suppose the bases B_1 and B_2 for I^d lie in the fundamental hyperplane H given by $n \cdot x = f$ with fundamental normal $n = (n_1, \dots, n_d)$. Suppose M_1 and M_2 are $d \times d$ matrices such that the rows of each M_i are the vertices of B_i. Then B_1 and B_2 are equivalent if and only if M_2 can be obtained from M_1 by successive use of matrix operations of the following types (only 1, 2, and 3 in case H is noncentral for I^d):

(1) Permute the rows of the matrix.
(2) If $n_i = n_{i+1} = \cdots = n_{i+k}$, then permute columns $i, i+1, \dots, i+k$ of the matrix.
(3) For some i with $n_i = 0$, replace column i of the matrix by its complement.
(4) For all i with $n_i \neq 0$, replace column i of the matrix by its complement.

Proof. ($\Rightarrow$) Suppose B_1 and B_2 are equivalent. Then M_2 can be obtained from M_1 by first permuting the rows of M_1 to obtain $\widehat{M}$; then using some permutation P on the columns of $\widehat{M}$ to obtain $\widetilde{M}$; and finally, for some subset C of the set of d columns of $\widetilde{M}$, replacing each column of C by its complement. Then n is a normal for the hyperplane containing the rows of $\widehat{M}$ and some permutation of its coordinates, say $m = (m_1, \dots, m_d)$, is a normal for the hyperplane containing the rows of $\widetilde{M}$. Also by Lemma 2.2, $(s_1 m_1, \dots, s_d m_d)$ where

$$s_i = \begin{cases} 1, & \text{if } i \notin C; \\ -1, & \text{if } i \in C; \end{cases}$$

is a normal for H. There are two possibilities:

(i) For $i \in C$, $m_i = 0$ and $m = n$.

(ii) For $m_i \neq 0$, $i \in C$ and $(s_1 m_1, \ldots, s_d m_d) = -n$. In this case, according to Lemma 2.2, H is also given by

$$-n \cdot x = f - \sum_{i=1}^{d} n_i$$

which implies $2f = \sum_{i=1}^{d} n_i$; thus, H is a central hyperplane for I^d.

In both cases $m = n$, and consequently P can be expressed as a composition of permutations of type 2.

($\Leftarrow$) The rest of the proof now follows easily by showing that each of the four matrix operations (three if H is noncentral for I^d) applied to a matrix M representing a base contained in H yields a matrix representing a base which is also contained in H.

□

Matrices representing equivalent bases may have determinants with different absolute values. We show this is not possible for bases contained in the same fundamental hyperplane for I^d.

Theorem 2.4. Suppose the bases B_1 and B_2 for I^d are equivalent and are contained in the fundamental hyperplane H given by $n \cdot x = f$ where $n = (n_1, \ldots, n_d)$ is a fundamental normal. Suppose M_1 and M_2 are $d \times d$ matrices such that the rows of each M_i are the vertices of B_i. Then $|\det(M_1)| = |\det(M_2)|$.

Proof. It suffices to show that each of the operations 1–4 (1–3 if H is noncentral) of Theorem 2.3 does not change the absolute value of the determinant of a matrix whose rows are a base in H. This is clear for operations 1 and 2. The proofs for 3 and 4 are similar and we consider only 4 with H central.

It is enough to show $|\det(N_1)| = |\det(N_2)|$ where N_2 is the complement of N_1 and the rows of N_1 form a base in H. Since the simplices described by $\begin{bmatrix} N_1 \\ e \end{bmatrix}$ and $\begin{bmatrix} N_2 \\ 0 \end{bmatrix}$ are in the same configuration class, we have

$$|\det(N_2)| = \left|\det \begin{bmatrix} N_2 & e^T \\ 0 & 1 \end{bmatrix}\right| = \left|\det \begin{bmatrix} N_1 & e^T \\ e & 1 \end{bmatrix}\right| = \left|(-f)^d \det \begin{bmatrix} (-1/f)N_1 & e^T \\ (-1/f)e & 1 \end{bmatrix}\right|.$$

To the last column of this last matrix, for all $i \in \{1, \ldots, d\}$, we add n_i times column i. Using $f = (\sum_{i=1}^{d} n_i)/2$, we thus have

$$|\det(N_2)| = \left|(-f)^d \det \begin{bmatrix} (-1/f)N_1 & 0 \\ (-1/f)e & -1 \end{bmatrix}\right| = |\det(N_1)|.$$

□

Suppose the set of rows of the $d \times d$ matrix M is a base B which is contained in the fundamental hyperplane $n \cdot x = f$ where $f > 0$. Letting V_B be the $(d-1)$-volume of $\text{conv}(B)$ and S be the simplex $\text{conv}(B \cup \{0\})$, we compute the volume of S in two ways to obtain

$$(1/d!)\left|\det \begin{bmatrix} M & e^T \\ 0 & 1 \end{bmatrix}\right| = (1/d)V_B(f/\sqrt{n \cdot n}).$$

Consequently

$$V_B = \left(\frac{\sqrt{n \cdot n}}{f(d-1)!}\right) |\det(M)|, \tag{2.4}$$

which we record for later use in developing constraints for our linear programming problem for $S(d)$.

There is a natural one-to-one correspondence between the configuration classes of simplices for I^d and the base classes for I^{d+1} which have representatives on the hyperplane $x_{d+1} = 0$.

Every base class has a representative which is contained in a fundamental hyperplane for I^d, in fact a hyperplane with equation $n \cdot x = f$ where $0 \leq f \leq \left(\sum_{i=1}^d n_i\right)/2$ and n is the fundamental normal. We will single out one of these representatives as our canonical representative for the base class. Toward this end we agree on an ordering for $\{0,1\}$-matrices of the same size. We define $[r_1^T, \ldots, r_k^T]^T \leq [s_1^T, \ldots, s_k^T]^T$, where the r_i's and s_i's are row vectors, to mean that the row concatenation $r_1 r_2 \cdots r_k$, as a nonnegative binary number, is not larger than that for $s_1 s_2 \cdots s_k$.

Definition. Let $[B]$ be a base class for I^d. Let the hyperplane H given by $n \cdot x = f$, where n is a fundamental normal and $0 \leq f \leq \left(\sum_{i=1}^d n_i\right)/2$, contain a representative of $[B]$. Let $\overline{M}$ be the maximum of all $d \times d$ matrices whose rows constitute a base in $[B]$ and are contained in H. The set B^* of rows of $\overline{M}$ is our *canonical representative* of the class $[B]$.

We include some information about the bases in dimensions 3 and 4 to help the reader confirm his or her understanding of these concepts.

There are three base classes for I^3 and they have representatives on the three hyperplanes $x_3 = 1$, $x_2 + x_3 = 1$, and $x_1 + x_2 + x_3 = 1$.

In I^4 there are five fundamental normals: 0001, 0011, 0111, 1111, and 1112. There are 15 base classes. For each of these we give the maximum matrix whose rows thus form the canonical representative of the base class:

$$\underset{1}{\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\\0&1&1&0\end{bmatrix}}\quad\underset{2}{\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\\0&1&0&0\end{bmatrix}}\quad\underset{3}{\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\\0&0&0&0\end{bmatrix}}\quad\underset{4}{\begin{bmatrix}1&1&1&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}}\quad\underset{5}{\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&1&1&0\end{bmatrix}}$$

$$\underset{6}{\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}}\quad\underset{7}{\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&0&1&0\end{bmatrix}}\quad\underset{8}{\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&0&0&1\end{bmatrix}}\quad\underset{9}{\begin{bmatrix}1&1&1&0\\1&0&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}}\quad\underset{10}{\begin{bmatrix}1&1&1&0\\1&0&0&1\\0&1&0&1\\0&0&1&0\end{bmatrix}}$$

$$\underset{11}{\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}}\quad\underset{12}{\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix}}\quad\underset{13}{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}\quad\underset{14}{\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&1&1&0\end{bmatrix}}\quad\underset{15}{\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}}.$$

The canonical representatives of base classes 1–4 lie on $x_4 = 0$ which intersects I^4 in a copy of I^3, and these base classes correspond to the four configuration classes for I^3. For 5–10, they lie on the central hyperplane $x_3 + x_4 = 1$ which intersects I^4 in a parallelepiped having squares for two facets and nonsquare rectangles for the other four. The canonical representatives of base classes 11 and 12 lie on the hyperplane $x_2 + x_3 + x_4 = 1$ which intersects I^4 in a prism having two equilateral triangles as opposite facets. For base class 13, the canonical representative lies on $x_1+x_2+x_3+x_4 = 1$ which intersects I^4 in a tetrahedron. For base class 14, it lies on the central hyperplane $x_1 + x_2 + x_3 + x_4 = 2$ which intersects I^4 in a regular octahedron. Finally, for base class 15, it lies on $x_1 + x_2 + x_3 + 2x_4 = 2$ which intersects I^4 in a tetrahedron.

If S_1 and S_2 are simplices in I^d which are in the same configuration class, then there is a one-to-one correspondence between the facets of S_1 and the facets of S_2 such that each pair of corresponding facets determine the same base class. Moreover if F and G are corresponding facets of S_1 and S_2 respectively, then S_1 is in the short (respectively long) half-space determined by F if and only if S_2 is in the short (long) half-space determined by G.

Theorem 2.5. Let $B_1 = \{v_1, \ldots, v_d\}$ and $B_2 = \{w_1, \ldots, w_d\}$ be equivalent bases for I^d. Let H_1 and H_2 be the hyperplanes containing B_1 and B_2, respectively. Let U and V be the open half-spaces determined by H_1. Let $\mathcal{C}_1$ be the set of configuration classes of simplices

$\mathrm{conv}\{v_1, \ldots, v_d, u\}$ with u a vertex of I^d in U and let $\mathcal{C}_2$ be similarly defined for V. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be similarly defined for H_2. Then $\{\mathcal{C}_1, \mathcal{C}_2\} = \{\mathcal{D}_1, \mathcal{D}_2\}$.

Proof. Let A be an affine isometry of the form (2.3) with $A(B_1) = B_2$. The result is easily seen since A carries the simplex $\mathrm{conv}\{v_1, \ldots, v_d, u\}$ to a simplex in the same configuration class and if the vertices u_1 and u_2 of the cube are on the same side of H_1, then $A(u_1)$ and $A(u_2)$ are on the same side of H_2.

□

Of course, for noncentral bases, the sets of configuration classes on the long (short) half-spaces correspond. In the notation of Theorem 2.5, for some central bases we have $\mathcal{C}_1 = \mathcal{C}_2$ and for others $\mathcal{C}_1 \neq \mathcal{C}_2$. We turn to the development of a concept which allows us to build constraints exploiting $\mathcal{C}_1 \neq \mathcal{C}_2$ when possible.

Definition. The central bases B_1 and B_2 whose elements lie on a fundamental hyperplane H with fundamental normal $(n_1, \ldots, n_d)$ are called *superequivalent* if B_2 can be obtained from B_1 by applying to all the elements of B_1, in the same order, the same operations of the following types:

(2′) If $n_i = n_{i+1} = \cdots = n_{i+k}$, then permute coordinates $i, i+1, \ldots, i+k$.

(3′) For some i with $n_i = 0$, replace coordinate i by its complement.

For bases whose elements lie on a central fundamental hyperplane, this is the same as equivalence except that here complementation of all coordinates with nonzero normal coefficients is not allowed. If some operations of type 2′ or 3′ are performed on a vertex of I^d, w, not in H, the resulting point is on the same side of H as w.

Consider a base class $[\widehat{B}]$ relative to equivalence which has representatives contained in the central fundamental hyperplane H. Let $[\widehat{B}]_H$ be the set of such representatives. Either $[\widehat{B}]_H$ is a single equivalence class under superequivalence or it is the union of two such classes. To learn if the former obtains, it suffices to take any $B \in [\widehat{B}]_H$ and see if B is superequivalent to the set $\widetilde{B}$ of complements of elements of B.

Definition. Let $[B]$ be a central base class, determined by equivalence, with representatives contained in the fundamental hyperplane H. If any two bases in $[B]$ whose members lie in H are superequivalent, we call $[B]$ and its elements *selfcomplementing.*

For example, the bases in I^5 represented by the matrices

$$\begin{bmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&1&0&0&1\\ 1&0&1&1&0\\ 0&1&1&0&0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&1&0&0&1\\ 1&0&1&1&0\\ 0&1&0&0&1 \end{bmatrix}$$

and contained in the central hyperplane $x_2 + x_3 + x_4 + x_5 = 2$, are, respectively, selfcomplementing and non-selfcomplementing.

Theorem 2.6. Let $B = \{v_1, \dots, v_d\}$ be a selfcomplementing base whose elements lie on the fundamental hyperplane H. Then the set of configuration classes of the simplices $\text{conv}\{v_1, \dots, v_d, u\}$ with $u \in U$ is the same as that for $\text{conv}\{v_1, \dots, v_d, v\}$ with $v \in V$.

Proof. Let $S = \text{conv}\{v_1, \dots, v_d, u\}$ where $u \in U$. Let $\widetilde{B}$ be the set of complements of the elements of B and let w be the complement of u. Then $w \in V$. Some sequence of operations of types $2'$ and $3'$ from the definition of superequivalence can be applied to $\widetilde{B}$ to yield B. Applied to w, this sequence yields some $v \in V$. Then S and $T = \text{conv}(B \cup \{v\})$ are in the same configuration class and the result is clear.

□

Typically the conclusion of Theorem 2.6 does not hold if B is non-selfcomplementing.

Definition. Consider a central non-selfcomplementing base class $[B^*]$ where the canonical representative B^* is contained in a fundamental hyperplane H. Let B be a base in $[B^*]$. Let A be a mapping of the form (2.3) which carries B onto B^*. Let $p \in R^d$ with p not in the hyperplane determined by B. If $A(p)$ is on the same side of H as 0, we say p is on the *zero side* of B relative to B^*.

One can easily show that p being on the zero side of B relative to B^* does not depend on the choice for A in this definition. Also note that if p is on the zero side of some base B relative to the representative B^*, then p is not on the zero side of the base $\widetilde{B}$, consisting of the complements of the elements of B, relative to B^*. This is in spite of the fact that B and $\widetilde{B}$ are equivalent and contained in the same hyperplane.

Definition. Suppose S is a simplex in I^d. Let F be a facet of S. Let v be the vertex of S not in F and let B be the base determined by F. Then, relative to our canonical representative B^* for the base class $[B]$, we define the *side indicator* i for the pair (S, F)

by

$$i(S,F) = \begin{cases} 0, & \text{if } B \text{ is not central, or is central and selfcomplementing;} \\ -1, & \text{if } v \text{ is on the zero side of } B \text{ relative to } B^* \text{ ;} \\ +1, & \text{otherwise.} \end{cases}$$

Thus we obtain a $(d+1)$-tuple of side indicators for S. One can show that simplices in the same configuration class, relative to the canonical representatives of the base classes, yield the same family of side indicators.

Suppose the simplices S_1 and S_2 in I^d are adjacent with common facet F. Suppose F determines the central non-selfcomplementing base B in the base class with canonical representative B^*. Then, relative to B^*, we have $i(S_1,F)i(S_2,F) = -1$. We will use this to form constraints for our linear programming problem for $T(d)$.

We now have enough groundwork for an understanding of the linear programming problems.

One drawback of our linear programming approach is that the information it provides about creating minimum-cardinality decompositions is very dimension specific. To the extent we are able, we will make some more global comments about the triangulations we produce. To set the stage for this, we close this section by mentioning some general ways of obtaining triangulations of the cube.

The first method is that of coning off to a vertex [16, 3].

Definition. A *d-complex* is a finite set $\mathcal{C}$ of d-polytopes such that $P \cap Q$ is a face of both P and Q for all $P, Q \in \mathcal{C}$.

Definition. Let $\mathcal{C}$ be a d-complex. A *triangulation* $\mathcal{S}$ of $\mathcal{C}$ is a complex of d-simplices such that for $P \in \mathcal{C}$ there is a subset $\mathcal{S}_P$ of $\mathcal{S}$ for which $P = \bigcup \mathcal{S}_P$.

One can show that if v is a vertex of a polytope P and $\mathcal{S}$ is a triangulation of the complex of facets of P opposite v, then $\mathcal{S}_v = \{\text{conv}(\{v\} \cup S) : S \in \mathcal{S}\}$ is a triangulation of P. It is the triangulation obtained by *coning off* $\mathcal{S}$ to v.

The next method is by way of convex enlargements [11]. Let P be a d-polytope contained in a polytope Q in R^d. Suppose $\mathcal{T}$ is a triangulation of P and $v_1, v_2, \ldots, v_n$ is an ordering of the vertices of Q. The first enlargement of $\mathcal{T}$ is carried out by including all simplices $\text{conv}(\{v_1\} \cup F)$ where F is a facet of a simplex S in $\mathcal{T}$ such that two conditions hold: (1) S and v_1 are on opposite sides of the hyperplane containing F, and (2) no simplex of $\mathcal{T}$ is adjacent to S with common facet F. This process is continued for $v_2, \ldots, v_n$ and culminates in a triangulation of Q.

Finally we mention triangulations of I^d arising from middle cuts [3, 18]. Let $H_-(d,m) = \{ x \in R^d : \sum_{i=1}^d x_i \le m \}$ and $H_+(d,m) = \{ x \in R^d : \sum_{i=1}^d x_i \ge m \}$. Then let $A(d,m) = H_-(d,m) \cap I^d$ and $B(d,m) = H_+(d,m) \cap I^d$. Suppose $\mathcal{S}$ is a triangulation of the complex

$$\mathcal{C} = \{ F : F \text{ is a facet of } A(d,m) \text{ opposite } 0 \} \cup \{ F : F \text{ is a facet of } B(d,m) \text{ opposite } e \}.$$

Coning off the simplices of $\mathcal{S}$ in $A(d,m)$ to 0 and the simplices of $\mathcal{S}$ in $B(d,m)$ to e yields a triangulation of the complex $\{A(d,m), B(d,m)\}$ which is a triangulation of I^d. Sallee introduced a recursive method of obtaining triangulations of $\mathcal{C}$ [18]. (The middle-cut decompositions studied by Böhm are similar [3]. The differences are that Böhm works with decompositions which aren't necessarily triangulations and that he slices off some of the corners of the cube before coning off which results in fewer simplices for $d \ge 7$. Böhm's use of the word "triangulation" coincides with our use of "decomposition".)

3. Linear programming problems

In this section we develop the linear programming problems whose solutions yield lower bounds for $S(d)$, $T(d)$, and $T^c(d)$.

The linear programming problem for $T(d)$

Here we introduce a linear programming problem for each dimension d such that every triangulation of I^d determines a feasible solution whose objective value is the cardinality of the triangulation. Thus the minimum objective value provides a lower bound for $T(d)$.

We first describe the variables of the problem. For each configuration class, we select a representative and compute the $d+1$ bases, the exterior-facet tuple, and the volume. Then, for some ordering of the facets, we compute the corresponding side indicators; and, for the induced ordering of the vertices, we compute the signs of the barycentric coordinates of the centroid of the cube relative to these vertices. Another representative of the same configuration class, possibly by reordering its bases, would produce the same information. We associate a problem variable with all the configuration classes which produce the same such information disregarding differences in selfcomplementing bases and bases with fundamental normal $(0, \ldots, 0, 1)$. (Here we are motivated by wanting to eliminate the situation where two or more variables enter into the linear programming problem in exactly the same way.) As an example, consider the dimension-6 configuration classes represented by the simplices

$$\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&1&0&0&0 \\ 0&1&1&0&1&0 \\ 0&1&1&0&1&1 \\ 1&0&1&1&1&1 \\ 1&1&1&1&1&1 \\ 0&0&0&0&0&0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1&0&0&0&0&0 \\ 1&0&0&0&1&0 \\ 0&1&0&1&0&1 \\ 1&0&1&1&1&0 \\ 1&0&1&1&1&1 \\ 1&1&1&1&1&1 \\ 0&0&0&0&0&0 \end{bmatrix}.$$

As is easily checked, these simplices are metrically different and hence in different configuration classes. For each of these the base obtained by deleting any one of the first five vertices is selfcomplementing. Deleting either of the last two vertices from either simplex, yields a base from the same base class. Both simplices have volume 1/6!, exterior-facet tuple (0,0,0,0), zero side indicators, and in both cases the sequence of barycentric coordinates of c relative to the vertices is 0, 0, 0, 0, 0, 0.5, 0.5. Thus both configuration classes are represented by a common problem variable.

The objective function is just the sum of the variables. Associated with each triangulation $\mathcal{T}$ of I^d is the assignment of values to the variables where the value of a variable x_i is the number of simplices of $\mathcal{T}$ which are in the classes represented by x_i. Note that for this assignment, the objective value is the cardinality of the triangulation. All the constraints of our problem will be satisfied by such assignments. There are $d-1$ constraints based on volume considerations for I^d and its k-faces for $2 \le k < d$. There are other constraints corresponding to central non-selfcomplementing base classes and to noncentral base classes which don't have bases contained in facets of I^d.

We first discuss the constraints based on volume. Let $\mathcal{T}$ be a triangulation of I^d and let $\mathcal{T}_1$ be the set of exterior facets of the simplices in $\mathcal{T}$. Then $\mathcal{T}_1$ can be partitioned into $2d$ collections of $(d-1)$-simplices with each collection a triangulation of one of the facets of I^d. We consider the facets of I^d as disjoint and thus consider two simplices in $\mathcal{T}_1$ as disjoint if they are not contained in the same facet of I^d. Then we let $\mathcal{T}_2$ be the set of exterior facets of the simplices in $\mathcal{T}_1$. $\mathcal{T}_2$ can be partitioned into triangulations of the $2^2d(d-1)$ facets of the $2d$ facets of I^d. We continue in this manner until the set $\mathcal{T}_{d-2}$ of 2-simplices is defined. Let the problem variables be x_i, $i = 1, 2, \ldots, m_d$, representing sets of configuration classes of simplices with volumes $v_i/d!$ and exterior-facet tuples $(f_1^i, \ldots, f_{d-2}^i)$. Then, making use of (2.2) and the fact that the k-volume of any k-face of I^d is 1, we are led to the constraints

$$\sum_{i=1}^{m_d} v_i x_i = d!$$

$$\sum_{i=1}^{m_d} v_i f_j^i x_i = 2^j d! \quad j = 1, 2, \ldots, d-2$$

which must be satisfied for assignments of values to the variables coming from triangulations.

We next turn to the constraints corresponding to the base classes. Let $[B^*]$ be a noncentral base class for I^d with canonical representative B^* where B^* is not contained in a facet of I^d. If $B \in [B^*]$ and $\mathcal{T}$ is a triangulation of I^d, then $\mathcal{T}$ has a simplex with base B with the simplex in the short half-space determined by B, if and only if $\mathcal{T}$ has a simplex with base B with the simplex in the long half-space determined by B. Hence, for any triangulation $\mathcal{T}$, the number of occurrences of B in $[B^*]$ as the vertex set of a facet of a simplex in $\mathcal{T}$ in the short half-space determined by B, must be matched by the same quantity for the long half-space. We use this fact to form a constraint corresponding to $[B^*]$. For each problem variable x_i, for some simplex T in a configuration class represented by x_i, let s_i (respectively l_i) be the number of facets F of T such that F determines a base in $[B^*]$ with T in the short (long) half-space determined by F. The constraint corresponding to $[B^*]$ is

$$\sum_{i=1}^{m_d} (l_i - s_i) x_i = 0.$$

For a central non-selfcomplementing base class $[B^*]$, with canonical base B^*, we have a constraint which is similar but differs in that the definitions of s_i and l_i are modified. For example, in defining s_i, the short half-space determined by the base B in $[B^*]$ is replaced by the zero side of B relative to B^*. Thus here we use side indicators instead of half-space indicators. Notice that, since all terms in the equation refer to the same base class, all side indicators are relative to the same canonical representative.

This, along with non-negativity conditions on all the variables, completes the description of our problem for $T(d)$.

We illustrate this problem for $d = 4$. Here the pertinent information is in Table 3.1. In this and lower dimensions, all the central bases are selfcomplementing. Also, it turns out, in dimensions not larger than 5 each variable represents just one configuration class.

variable	base classes (numbers from §2)					signs of barycentric coordinates of c					exterior-fac.tup.		vol. $\times 4!$
X_1	8	8	12	14	15	0	0	1	0	1	0	0	1
X_2	6	8	8	11	11	0	0	0	1	1	0	0	1
X_3	3	6	6	9	12	1	0	0	0	1	1	1	1
X_4	3	7	8	12	14	1	0	0	1	0	1	1	1
X_5	3	5	11	11	15	1	0	1	1	-1	1	1	1
X_6	2	7	8	9	11	1	0	0	0	1	1	2	1
X_7	1	9	9	9	15	1	0	0	0	1	1	3	1
X_8	3	3	12	12	13	1	1	-1	-1	1	2	2	1
X_9	2	3	5	11	12	1	1	0	-1	1	2	3	1
X_{10}	1	3	7	7	14	1	1	0	0	0	2	4	1
X_{11}	2	2	6	7	7	1	1	0	0	0	2	4	1
X_{12}	1	2	2	5	12	1	1	1	0	-1	3	7	1
X_{13}	1	1	1	1	13	1	1	1	1	-1	4	12	1
X_{14}	10	11	11	12	15	0	1	1	1	1	0	0	2
X_{15}	4	13	14	14	14	1	1	0	0	0	1	0	2
X_{16}	4	12	12	12	15	1	1	1	1	-1	1	0	2
X_{17}	13	15	15	15	15	1	1	1	1	1	0	0	3

Table 3.1. Linear programming variables in dimension 4.

The linear programming problem for $T(4)$ is then

Minimize

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16}+x_{17}$$

subject to

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}+x_{13}+2x_{14}+2x_{15}+2x_{16}+3x_{17}=24$$

$$x_3+x_4+x_5+x_6+x_7+2x_8+2x_9+2x_{10}+2x_{11}+3x_{12}+4x_{13}+2x_{15}+2x_{16}=48$$

$$x_3+x_4+x_5+2x_6+3x_7+2x_8+3x_9+4x_{10}+4x_{11}+7x_{12}+12x_{13}=96$$

$$2x_2+2x_5+x_6-x_9+2x_{14}=0$$

$$x_1+x_3+x_4-2x_8+x_9-x_{12}+x_{14}+3x_{16}=0$$

$$+x_8-x_{13}+x_{15}+x_{17}=0$$

$$x_1-x_5+x_7+x_{14}-x_{16}+4x_{17}=0$$

all $x_i \geq 0$.

The linear programming problem for $S(d)$

Here we consider decompositions of the d-cube into d-simplices that have disjoint interiors and whose union is I^d. In contrast to triangulations, this allows the possibility that the intersection of two simplices in the decomposition has dimension $d-1$ and yet this intersection is not a facet of both of the simplices.

The variables, constraints based on volume considerations, and the objective function of our linear programming problem for $T(d)$ carry over exactly to this setting. The constraints corresponding to base classes are replaced by a less restrictive system of equations corresponding to classes of equivalent noncentral hyperplanes containing facets of simplices in I^d. We do not have an equation for the class of equivalent hyperplanes which contain facets of I^d and we do not have equations corresponding to classes of equivalent central hyperplanes.

Suppose $n = (n_1, \ldots, n_d)$ is a fundamental normal and f is a number such that $0 < f < (\sum_{i=1}^{d} n_i)/2$. Let H be the hyperplane given by $n \cdot x = f$. We form a constraint corresponding to H which reflects the fact that, for any decomposition $\mathcal{D}$ and any hyperplane P equivalent to H, the total $(d-1)$-volume of the facets in P of simplices of $\mathcal{D}$ on one side of P matches the total for the other side. Let $\mathcal{B}$ be the set of base classes having at least one base contained in H. We have the corresponding constraint

$$\sum_{[B]\in\mathcal{B}} \sum_{1\leq i\leq m_d} \left|\det(M_{[B]})\right| (l_i([B]) - s_i([B]))x_i = 0.$$

Here $M_{[B]}$ is a $d \times d$ matrix whose rows make up a base contained in H and in the base class $[B]$. We can take $M_{[B]}$ to be the maximum matrix representing the canonical base in $[B]$. By Theorem 2.4 and equation (2.4), $\left|\det(M_{[B]})\right|$ is proportional to the common $(d-1)$-volume of the convex hulls of the bases in $[B]$. Also $l_i([B])$ and $s_i([B])$ are the counts of the long and short half-space indicators for simplices represented by x_i as before, but now with the relevant base class explicitly indicated.

In dimension 4 we have constraints corresponding to the hyperplanes $x_2+x_3+x_4 = 1$, $x_1+x_2+x_3+x_4 = 1$, and $x_1+x_2+x_3+2x_4 = 2$. Base classes 11 and 12 have representatives contained in $x_2+x_3+x_4 = 1$ and have 1 for the absolute value of the relevant determinant. Thus constraints 4 and 5 of the problem for $T(4)$, which correspond to base classes 11 and 12, are replaced by their sum in the problem for $S(4)$. Constraint 6 of the problem for $T(4)$ remains the same in the problem for $S(4)$, and constraint 7 is replaced by its double. The objective function and the first three constraints are the same in both problems.

The linear programming problem for $T^c(d)$

Without loss of generality, we consider only corner-slicing triangulations of I^d which contain the corners at all the vertices of I^d with odd coordinate sums. If these corners are removed from I^d, the closure of the resulting solid, Q_d, is the convex hull of the set of vertices of I^d with even coordinate sums. Brodie and Cottle showed that if $m/d!$ is the volume of a simplex in I^d whose vertices all have even coordinate sums, then m is an even integer [5]. Any corner-slicing triangulation of I^d consists of a triangulation of Q_d along with the 2^{d-1} corners at the odd vertices. The problem for $T^c(d)$ differs from the preceding problems in that it uses only those variables representing configuration classes which have simplices whose vertices all have even coordinate sums and, of course, a variable for the corners. Moreover, except for the base class representing the single internal facet of a corner of I^d, each base class used to generate constraints has representatives whose elements all have even coordinate sums. We also adjoin an additional constraint reflecting the fact that the number of corners must be 2^{d-1}. There are $d-1$ constraints based on volume considerations as before and the objective function is still the sum of the variables.

The problem for $T^c(4)$ is

$$\begin{array}{lrcl} \text{Minimize} & x_{13} + x_{15} & & \\ \text{subject to} & x_{13} + 2x_{15} & = & 24 \\ & 4x_{13} + 2x_{15} & = & 48 \\ & 12x_{13} \qquad & = & 96 \\ & x_{13} - x_{15} & = & 0 \\ & x_{13} \qquad & = & 8 \\ & x_{13}, x_{15} & \geq & 0. \end{array}$$

For $d \leq 4$, the problem is sufficiently constrained that there is only one feasible solution and the optimization is superfluous. In general the problem for $T^c(d)$ is significantly smaller than the problem for $T(d)$.

4. Computer programs

In this section we give a brief overview of the main programs or algorithms used in producing the linear programming problems. A serious reader wanting to verify our results would probably need similar programs. Most are not difficult and we omit the detailed descriptions.

We first mention a program which carries out a finite search, based on Theorem 2.1 and its proof, to find all the fundamental normals and corresponding fundamental hyperplanes for I^d. (The execution time for this program is reasonable for $d \leq 7$, but, without some new insight, it is too large for $d \geq 8$.) There are 3, 5, 11, 41, 383, respectively, fundamental normals for $d = 3, 4, 5, 6, 7$. Also for $d = 3, 4, 5, 6, 7$, there are, respectively, $3, 6, 15, 63, 623$ hyperplanes $n \cdot x = f$ where n is a fundamental normal and $0 \leq f \leq (\sum_{i=1}^{d} n_i)/2$.

We next describe an algorithm that for a given fundamental hyperplane H as just described produces the canonical representatives of all base classes which have bases contained in H. Theorem 2.3 provides the appropriate operational description of equivalence for bases contained in H.

The key procedure of this algorithm transforms a $d \times d$ matrix $M = [v_1^T, \ldots, v_d^T]^T$ whose rows are vertices of I^d in H into the maximum matrix, $\overline{M}$, which can be produced from M using only operations of types 1 and 2 of Theorem 2.3. For $p \in \{1, 2, \ldots, d\}$, let $\overline{M}_p$ be the largest matrix which can be formed using operations of type 2 on matrices made up of p rows of M. Thus $\overline{M}_d = \overline{M}$. Let $\mathcal{G}$ be the rooted tree with d nodes at level 1, $d(d-1)$ at level 2, etc., and $d!$ endpoints, which is commonly used to illustrate the formation of all the permutations of $1, 2, \ldots, d$. The procedure carries out a breadth-first search of $\mathcal{G}$ to find some permutation $(k_1, \ldots, k_d)$ such that some sequence of type 2 operations applied to $[v_{k_1}^T, \ldots, v_{k_d}^T]^T$ yields $\overline{M}_d$. The tree is pruned during the search; thus, after the current nodes at level p have been traversed, the search is narrowed to all permutations $(k_1, \ldots, k_d)$ such that some sequence of type 2 operations applied to $[v_{k_1}^T, \ldots, v_{k_d}^T]^T$ yields $\overline{M}_p$. Then $\overline{M}_d$ is obtained after stage d.

The algorithm systematically considers all $d \times d$ matrices M whose rows are distinct vertices of I^d in H. A matrix M is discarded if its rows are affinely dependent. Suppose $n = (n_1, \ldots, n_d)$ and k is 0 or satisfies $0 = n_1 = \cdots = n_k < n_{k+1}$. If H is noncentral, the algorithm uses the above mentioned procedure on each of the 2^k matrices obtained by complementing the columns in some subset of the first k columns of M. (If H is central, then, because a type 4 operation is allowed, 2^{k+1} matrices are considered.) In this manner, at most 2^k (2^{k+1} if H is central) matrices are produced, and the largest among these maxima is noted. The set of rows of this maximum matrix gives the canonical representative of a base class which, if new, is then given a number.

For readers contemplating the efficiency of the above algorithm, it may be of interest to note that any algorithm which determines whether two bases on a fundamental hyperplane are equivalent can also be used to determine whether two graphs are isomorphic. For $k = 1, 2$ let A_k be the incidence matrix of a graph G_k with no isolated nodes. Thus, for example, if G_1 has nodes $v_1, \ldots, v_n$ and arcs $e_1, \ldots, e_m$, then

$$A_1(i,j) = \begin{cases} 1, & \text{if } v_i \text{ is incident to } e_j; \\ 0, & \text{otherwise.} \end{cases}$$

Embed each A_k in an $(m+n+1) \times (m+n+1)$ matrix N_k as follows:

$$N_k = \begin{bmatrix} I_n & A_k & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Each N_k represents a base for I^{m+n+1} contained in the hyperplane $(0, \ldots, 0, 1) \cdot x = 0$. Using Theorem 2.3, one easily shows that N_1 and N_2 represent equivalent bases if and only if there are permutation matrices P and Q such that $PA_1Q = A_2$, i.e. G_1 and G_2 are isomorphic.

There are 3, 15, 106, 2445, 171697, respectively, base classes for $d = 3, 4, 5, 6, 7$. By using the appropriate remark in Section 2 and counting the canonical forms of base classes with fundamental normal $(0, 0, \ldots, 0, 1)$, we learn there are 4, 17, 237, 9892, respectively, configuration classes of simplices in dimensions $3, 4, 5, 6$. As described in Section 5, there are 1456318 variables in our problem for $T(7)$, and thus there are at least this many configuration classes of simplices in dimension 7.

Programs are also needed to compute the following items:

(a) the volume of a simplex in I^d,

(b) the exterior-facet tuple of a simplex in I^d [12],

(c) the barycentric coordinates of c relative to the vertices of a simplex in I^d,

(d) the number of the canonical representative of a given base,

(e) the fundamental normal and an equation of the hyperplane containing d given affinely independent vertices of I^d, and

(f) the side indicator of a simplex in I^d for a given facet.

Using the above programs, it is now easy to produce the information needed to form the linear programming problems. We form at least one simplex from each configuration class and compute the relevant information. More explicitly, for each canonical representative of a base class B^* and for each vertex v of I^d not in the hyperplane determined by B^*, we form the simplex $S = \text{conv}(B^* \cup \{v\})$ and record: (1) the numbers of the base classes represented by facets of S, (2) The signs of the barycentric coordinates of c relative to the vertices of S, (3) the side indicators of S for its facets relative to our canonical representatives of the base classes, (4) the volume of S, (5) the exterior-facet tuple of S, and (6) the base and vertex used to construct S. We keep all this information only if it differs in at least some of the items (1)–(5) from that of all previously considered simplices. Item (6) is included in order that a simplex yielding (1)–(5) can easily be reconstructed.

Also a program was written to produce triangulations by the method of convex enlargements. This was a straightforward implementation of the description in §2.

5. Results

Dimension 5

Results concerning $S(5)$

The basic linear programming problem for $S(5)$, from §3, has 237 variables and 15 equality constraints. Our computer generated optimal objective value is 65.00 which implies $S(5) \geq 65$. By partitioning the decompositions into classes and using various additional constraints, we obtain the better bound $S(5) \geq 67$. This improves the best lower bound of 61 given previously [12]. Since decompositions, actually mostly triangulations, of I^5 with 67 simplices have been given [2, 3, 14, 16, 18], we conclude $S(5) = 67$.

Case 1: $\mathcal{D}$ is a decomposition which contains a simplex having c in its interior. Then no other simplex in $\mathcal{D}$ can contain c. Whether or not c is in the interior of a simplex, or even in the relative interior of a k-face, can be determined by the signs of the barycentric coordinates of c with respect to the vertices of the simplex. For $n = 2, \ldots, 6$, let J_n be the set of indices $i \in \{1, \ldots, 237\}$ such that for any simplex S represented by the problem variable x_i none of the barycentric coordinates of c relative to the vertices of S is negative and exactly n of them are positive. One can easily show that $J_3 = \emptyset$. The constraints which reflect the conditions on $\mathcal{D}$ are

$$\sum_{i \in J_6} x_i = 1 \quad \text{and} \quad \sum_{i \in J_2 \cup J_4 \cup J_5} x_i = 0.$$

Adjoining these constraints to the basic problem for $S(5)$ yields a problem for which the computed objective value is approximately 66.7619, and we conclude $|\mathcal{D}| \geq 67$.

Case 2: $\mathcal{D}$ is a decomposition which has two simplices each of which contains c in the relative interior of one of its facets. Clearly no decomposition can contain more that two such simplices. The appropriate additional constraints are

$$\sum_{i \in J_5} x_i = 2 \quad \text{and} \quad \sum_{i \in J_2 \cup J_4 \cup J_6} x_i = 0.$$

The resulting problem gives a computed integer-valued optimal solution whose objective value is 67. Hence $|\mathcal{D}| \geq 67$.

Four configuration classes play a central role in our additional constraints for the rest of the cases and we introduce them now. They are the classes represented by the simplices in turn represented by the matrices

$$\underset{\substack{\text{vol } 4/5! \\ YR1-214}}{\begin{bmatrix} 1&1&0&1&0 \\ 1&1&1&0&0 \\ 0&0&1&1&0 \\ 0&1&0&0&1 \\ 1&0&0&0&0 \\ 1&1&1&1&1 \end{bmatrix}} \quad \underset{\substack{\text{vol } 4/5! \\ YR2-67}}{\begin{bmatrix} 1&1&0&1&0 \\ 0&1&1&0&0 \\ 0&0&1&1&0 \\ 0&1&0&0&1 \\ 1&0&0&0&0 \\ 1&1&1&1&1 \end{bmatrix}} \quad \underset{\substack{\text{vol } 3/5! \\ XEF1-197}}{\begin{bmatrix} 1&1&0&1&0 \\ 1&1&1&0&0 \\ 0&0&1&1&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 1&1&1&1&1 \end{bmatrix}} \quad \underset{\substack{\text{vol } 3/5! \\ XEF4-148}}{\begin{bmatrix} 0&0&1&1&1 \\ 0&1&0&1&1 \\ 0&1&1&0&1 \\ 0&1&1&1&0 \\ 0&0&0&0&0 \\ 1&1&1&1&1 \end{bmatrix}}.$$

The names are from an earlier paper, [11], and the numbers come from the appendix of the current paper. For $i \in \{1, \ldots, 5\}$ and $r \in \{0, 1\}$, we let F_{ir} be the facet $\{x \in I^d : x_i = r\}$. Any $YR1$- or $YR2$-simplex has a ridge (3-face) containing c and any $XEF1$- or $XEF4$-simplex has an exterior facet.

The classes $YR1$ and $YR2$ are similar in many respects. Let S with vertices $v_1, \ldots, v_6$ be a simplex from either class. Then c is a convex combination, with all coefficients $1/4$, of four of these vertices, say v_3, v_4, v_5, v_6. These are divided equally among two opposite facets of I^5, say v_3 and v_4 on F_{i0} and v_5 and v_6 on F_{i1}, in such a way that the centers of F_{i0} and F_{i1}, $\text{cen}(F_{i0})$ and $\text{cen}(F_{i1})$, are the midpoints of the edges $[v_3, v_4]$ and $[v_5, v_6]$, respectively. In particular the line segment from $\text{cen}(F_{i0})$ to $\text{cen}(F_{i1})$ is contained in S. The facets of S opposite v_1 and v_2 are orthogonal. They contain c and no other facets of S contain c. There are distinct coordinates j and k, different from i, and numbers p and q in $\{0, 1\}$ such that the open line segments $\{\lambda c + (1-\lambda)\text{cen}(F_{jp}) : 1/3 < \lambda < 1\}$ and $\{\lambda c + (1-\lambda)\text{cen}(F_{kq}) : 1/3 < \lambda < 1\}$ are in the interior of S. (For the simplices we used to define the classes $YR1$ and $YR2$, we have $i = 1$, $j = 2$, $p = 1$, $k = 5$, and $q = 0$.)

Next let T be an $XEF1$- or an $XEF4$-simplex. Then T has an exterior facet F contained in some facet F_{mr} of I^5. The facet F is the type of 4-simplex described by the variable X_{17} in §2. It contains the center of F_{mr} in its relative interior. Moreover, the open line segment $\{\lambda c + (1-\lambda)\text{cen}(F_{mr}) : 0 < \lambda < 1/2\}$ is in the interior of T.

The key point is that if a decomposition contains the simplices S and T just described, then the facet F_{mr} is not in $\{F_{i0}, F_{i1}, F_{jp}, F_{kq}\}$. Moreover if S_1 and S_2 are simplices in $YR1 \cup YR2$ and in a decomposition $\mathcal{D}$ with S_1 blocking facets F_{i0}, F_{i1}, F_{jp}, F_{kq} and

S_2 blocking facets F_{u0}, F_{u1}, F_{vy}, F_{wz} as just described, then neither F_{vy} nor F_{wz} is in $\{F_{i0}, F_{i1}, F_{jp}, F_{kq}\}$ and at least 6 facets of I^5 are blocked from containing the exterior facet of any $XEF1$- or $XEF4$-simplex in $\mathcal{D}$. Similarly three simplices from $YR1 \cup YR2$ in a decomposition block at least 8 facets, four such simplices block all 10 facets of I^5, and, by the above mentioned orthogonality, no decomposition has more than four such simplices. We use this information to formulate additional constraints in the remaining cases.

Case 3: $\mathcal{D}$ is a decomposition which has exactly one simplex that contains c in the relative interior of one of its facets and $\mathcal{D}$ also contains two simplices in $YR1 \cup YR2$. Three or more simplices in $YR1 \cup YR2$ are ruled out by the previously mentioned orthogonality. Also, by this orthogonality, one can show that no other simplices from $\mathcal{D}$ can contain c. The appropriate constraints are

$$\sum_{i \in J_5} x_i = 1, \qquad \sum_{i \in J_2 \cup J_4 \cup J_6} x_i = 2, \quad \text{and} \quad x_{YR1} + x_{YR2} = 2$$

where x_{YR1} and x_{YR2} are the problem variables representing the classes $YR1$ and $YR2$. The enlarged problem then has a computed optimal objective value of 66.90 and again $|\mathcal{D}| \geq 67$.

Case 4: $\mathcal{D}$ is a decomposition of I^5 which contains exactly one simplex containing c in the relative interior of one of its facets and $\mathcal{D}$ also contains at most one simplex in $YR1 \cup YR2$. We adjoin the constraints

$$\sum_{i \in J_5} x_i = 1, \quad \sum_{i \in J_6} x_i = 0, \quad \text{and} \quad x_{YR1} + x_{YR2} \leq 1$$

and obtain a problem with a computed optimal objective value of approximately 66.1111. Again we conclude $|\mathcal{D}| \geq 67$.

Case 5: $\mathcal{D}$ is a decomposition with exactly four simplices from $yr1 \cup yr2$ and no simplices that contain c in the relative interior of a facet. We include the constraints

$$x_{YR1} + x_{YR2} = 4, \qquad \sum_{i \in J_5 \cup J_6} x_i = 0, \quad \text{and} \quad x_{XEF1} + x_{XEF4} = 0$$

where x_{XEF1} and x_{XEF4} are the problem variables representing the classes $XEF1$ and $XEF4$. This problem yields a computed optimal objective value of 66.80 and $|\mathcal{D}| \geq 67$.

Case 6: $\mathcal{D}$ has exactly three simplices from $YR1 \cup YR2$ and no simplices that contain c in the relative interior of a facet. We include the constraints

$$x_{YR1} + x_{YR2} = 3, \qquad \sum_{i \in J_5 \cup J_6} x_i = 0, \quad \text{and} \quad x_{XEF1} + x_{XEF4} \leq 2.$$

Our computed objective value is approximately 66.5810.

Case 7: $\mathcal{D}$ has exactly two simplices from $YR1 \cup YR2$ and no simplices that contain c in the relative interior of a facet. We adjoin

$$x_{YR1} + x_{YR2} = 2, \quad \sum_{i \in J_5 \cup J_6} x_i = 0, \quad \text{and} \quad x_{XEF1} + x_{XEF4} \leq 4$$

and get a computed objective value of approximately 66.3810.

Case 8: $\mathcal{D}$ has at most one simplex from $YR1 \cup YR2$, no simplex that contains c in its interior, and no simplices that contain c in the relative interior of a facet. We adjoin

$$x_{YR1} + x_{YR2} \leq 1 \quad \text{and} \quad \sum_{i \in J_5 \cup J_6} x_i = 0$$

and get a computed optimal objective value of approximately 66.3333.

Every decomposition of I^5 meets the conditions of one of our cases and thus no decomposition can have fewer than 67 simplices.

We summarize what we have learned about $S(5)$.

Result 5.1. *The minimum of the cardinalities of all decompositions of I^5 is 67, i.e. $S(5) = 67$.*

Results concerning $T(5)$

Our basic linear programming for $T(5)$, from §3, has 237 variables, 44 equality constraints, and yields a computed integer-valued optimal solution with objective value 67. Since triangulations with 67 simplices exist, this confirms that $T(5) = 67$.

Here, mainly for the sake of completeness, we characterize the inventories of configuration classes (i.e. which classes and how many simplices from each class) that correspond to minimum-cardinality triangulations of I^5 and describe all such triangulations. In a less specific framework, Böhm has described the three main classes of these triangulations and has found the numbers of the various types of simplices used [2]. Unexpectedly, the descriptions of all inventories of configuration classes that correspond to minimum-cardinality triangulations of I^d become progressively simpler in dimensions 5, 6, and 7.

Table 5.1 lists the matrices representing all the configuration classes which have simplices in minimum-cardinality triangulations of I^5. We have labeled these with both the letters A–N, which we will use in this section, and the corresponding numbers from the appendix. The volumes and exterior-facet tuples are given. For each vertex of each representative simplex, we indicate whether the opposite facet (1) is an exterior facet, (2) determines a central selfcomplementing base, or (3) determines a noncentral base. In this last case, the number of the base class from the appendix and the half-space indicator (+ if the simplex is in the long half-space determined by facet) are given. None of the bases of simplices in the classes of Table 5.1 is central and non-selfcomplementing.

We first describe all the integer-valued optimal solutions and then discuss the corresponding triangulations. We let $x_A, x_B, \ldots, x_N$ be the variables in our linear programming problem corresponding to the classes $A, B, \ldots, N$. After some experimentation with additional bounds on some of the variables, we see the integer-valued optimal solutions split into three main types.

To learn about the first type, we solve the basic problem for $T(5)$ with all 237 variables and with the additional constraint $x_A = 1$. We compute the optimal solution

$$x_A = 1,\ x_B = 5,\ x_C = 15,\ x_D = 15,\ x_E = 15,\ x_F = 16. \tag{5.1}$$

By solving five additional problems where in each one we attempt to lower the number of simplices from one of the classes by one or more, we learn that (5.1) is the unique integer-valued optimal solution to our problem for $T(5)$ with the additional constraint $x_A = 1$. Changing the additional constraint to $x_A \geq 2$ gives an infeasible problem.

For the second type we adjoin the two constraints $x_A = 0$ and $x_G = 1$. Some more experimentation leads us to the conjecture that any integer-valued optimal solution to our problem with these constraints satisfies

$$\begin{gathered} x_D = 5,\ x_E = 5,\ x_F - x_M = 5,\ x_G = 1,\ x_H = 5, \\ x_I = 5,\ x_J = 20,\ x_K + x_F + x_M = 25,\ x_L = 1. \end{gathered}$$

This conjecture is verified by solving several problems—each is obtained by adjoining one of the last nine equations, changing it to an inequality, and appropriately perturbing its right-hand side by ± 1, and thereby producing a new problem which is either infeasible or has an optimal objective value larger than 67.

Hence every integer-valued optimal solution to our problem for $T(5)$ with the additional constraints $x_A = 0$ and $x_G = 1$ satisfies

$$\begin{gathered} x_D = 5,\ x_E = 5,\ x_F = 5 + s,\ x_G = 1,\ x_H = 5,\ x_I = 5, \\ x_J = 20,\ x_K = 20 - 2s,\ x_L = 1,\ x_M = s \end{gathered} \tag{5.2}$$

$$\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\\0&0&0&0&0\end{bmatrix}\begin{matrix}101+\\101+\\101+\\101+\\101+\\93+\end{matrix}$$

vol $4/5!$

eft $(0,0,0)$

$A - 168$

$$\begin{bmatrix}1&1&0&0&0\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\\0&0&0&0&0\end{bmatrix}\begin{matrix}101-\\\text{csc}\\96+\\96+\\96+\\\text{ef}\end{matrix}$$

vol $2/5!$

eft $(1,1,0)$

$B - 174$

$$\begin{bmatrix}1&1&0&0&0\\1&0&1&1&1\\1&0&1&0&0\\1&1&1&0&1\\1&1&1&1&0\\0&0&0&0&0\end{bmatrix}\begin{matrix}\text{csc}\\94+\\96-\\95+\\95+\\\text{ef}\end{matrix}$$

vol $2/5!$

eft $(1,1,0)$

$C - 133$

$$\begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\1&0&1&0&0\\1&1&1&0&1\\1&1&1&1&0\\0&0&0&0&0\end{bmatrix}\begin{matrix}94-\\94-\\94-\\\text{ef}\\\text{ef}\\93+\end{matrix}$$

vol $2/5!$

eft $(2,2,0)$

$D - 116$

$$\begin{bmatrix}1&1&0&0&0\\1&0&1&1&1\\1&0&1&0&0\\1&1&1&0&1\\1&0&0&0&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}94+\\\text{ef}\\95-\\94+\\95-\\\text{ef}\end{matrix}$$

vol $2/5!$

eft $(2,2,0)$

$E - 123$

$$\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}\begin{matrix}\text{ef}\\\text{ef}\\\text{ef}\\\text{ef}\\\text{ef}\\93-\end{matrix}$$

vol $1/5!$

eft $(5,20,60)$

$F - 110$

$$\begin{bmatrix}1&1&1&0&0\\1&1&0&1&0\\1&0&1&0&1\\0&1&0&1&1\\0&0&1&1&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}104+\\104+\\104+\\104+\\104+\\97+\end{matrix}$$

vol $3/5!$

eft $(0,0,0)$

$G - 204$

$$\begin{bmatrix}1&0&0&1&1\\1&1&0&1&0\\1&0&1&0&1\\0&1&0&1&1\\0&0&1&1&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}104-\\98+\\98+\\103+\\103+\\95+\end{matrix}$$

vol $3/5!$

eft $(0,0,0)$

$H - 211$

$$\begin{bmatrix}1&0&0&1&1\\1&1&0&1&0\\1&0&1&0&1\\1&0&1&1&0\\0&0&1&1&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}103-\\\text{ef}\\98+\\103-\\98+\\94+\end{matrix}$$

vol $3/5!$

eft $(1,0,0)$

$I - 197$

$$\begin{bmatrix}1&0&0&1&1\\0&0&0&0&1\\1&0&1&0&1\\0&1&0&1&1\\0&0&1&1&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}\text{csc}\\98-\\77+\\\text{ef}\\\text{csc}\\\text{ef}\end{matrix}$$

vol $2/5!$

eft $(2,2,0)$

$J - 147$

$$\begin{bmatrix}1&0&0&1&1\\0&0&0&0&1\\0&0&0&1&1\\0&1&0&1&1\\0&0&1&1&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}\text{ef}\\\text{csc}\\77-\\\text{ef}\\\text{ef}\\\text{ef}\end{matrix}$$

vol $1/5!$

eft $(4,13,33)$

$K - 27$

$$\begin{bmatrix}1&1&1&0&0\\1&1&0&1&0\\1&0&1&0&1\\0&1&0&1&1\\0&0&1&1&1\\1&1&1&1&1\end{bmatrix}\begin{matrix}95+\\95+\\95+\\95+\\95+\\97-\end{matrix}$$

vol $2/5!$

eft $(0,0,0)$

$L - 137$

$$\begin{bmatrix}1&0&0&1&1\\0&0&0&0&1\\0&0&0&1&0\\0&1&0&1&1\\0&0&1&1&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}\text{ef}\\77-\\77-\\\text{ef}\\\text{ef}\\93+\end{matrix}$$

vol $1/5!$

eft $(3,6,6)$

$M - 109$

$$\begin{bmatrix}0&1&0&1&1\\1&0&1&0&1\\1&1&0&1&0\\1&1&1&0&0\\1&0&1&1&0\\0&0&0&0&0\end{bmatrix}\begin{matrix}98+\\103+\\\text{csc}\\104+\\104+\\96+\end{matrix}$$

vol $3/5!$

eft $(0,0,0)$

$N - 199$

Table 5.1. The Configuration Classes for Minimum-Cardinality Triangulations of I^5

for some $s \in \{0, 1, \ldots, 10\}$. Using information displayed in Table 5.1, one can check that any inventory conforming to (5.2) is an optimal solution to the problem for $T(5)$ with the additional constraints $x_A = 0$ and $x_G = 1$. Changing the additional constraints to $x_A = 0$ and $x_G \geq 2$ gives an infeasible problem.

To learn about the third type of optimal solution, we adjoin the constraints $x_A = 0$, $x_G = 0$, and $x_N = 2$. (Replacing $x_N = 2$ by $x_N \leq 1$ gives an optimal objective of 67.5 and replacing it with $x_N \geq 3$ produces infeasibility.) Carrying on as for the second type, we learn that any integer-valued optimal solution to our problem with the additional constraints $x_A = 0,\ x_G = 0,\ x_N = 2$ satisfies

$$\begin{aligned} x_C = 2,\ x_D = 5,\ x_E = 4,\ x_F = 5 + s,\ x_H = 4,\ x_I = 5, \\ x_J = 20,\ x_K = 20 - 2s,\ x_M = s,\ x_N = 2 \end{aligned} \tag{5.3}$$

for some $s \in \{0, 1, \ldots, 10\}$ and the converse also holds.

Thus an integer-valued inventory of configuration classes is an optimal solution to the linear programming problem for $T(5)$ if and only if it satisfies (5.1), (5.2), or (5.3). We have justified the first part of the following result.

Result 5.2 *Any triangulation of I^5 with 67 simplices conforms to an inventory of configuration classes satisfying (5.1), (5.2), or (5.3). Conversely, to any such inventory, there corresponds a triangulation conforming to this inventory.*

We turn to the justification of the last part and, moreover, will sketch a description of all minimum-cardinality triangulations of I^5.

We first consider triangulations corresponding to (5.1). Each has has a single class-A simplex, and without loss of generality, the only such triangulations we consider are those which contain the class-A simplex represented by the first matrix from Table 5.1. Each of these contains 16 corners and contains the corner at the vertex 11111 and thus contains the corners at all 16 vertices with odd coordinate sums. Let E and O be the sets of vertices of I^5 with even and odd coordinate sums, respectively. Let $Q_5 = \text{conv}(E)$. With the help of Table 5.1, the serious reader can show that, for any triangulation $\mathcal{T}$ under consideration, the triangulation of Q_5 provided by the noncorners of $\mathcal{T}$ can be obtained by coning off to 0 a triangulation of the complex $\mathcal{O}$ of facets of Q_5 opposite 0. We pursue this point of view to describe all triangulations corresponding to (5.1).

For $i \in \{1, 2, \ldots, 5\}$ and $j \in \{0, 1\}$, let $G_{ij} = \text{conv}\{\, v \in E : v_i = j \,\}$. For $v \in O$, let F_v be the interior facet of the corner of I^5 at v. The complex $\mathcal{C}$ of all facets of Q_5 consists of the 10 G_{ij}'s and the 16 F_v's, and $\mathcal{O}$ consists of the 5 G_{i1}'s and the 11 F_v's for v's with coordinate sums in $\{3,5\}$.

Each G_{ij} is a regular 4-crosspolytope. For definiteness we discuss G_{11}.

$$G_{11} = \text{conv}\{11110, 11101, 11011, 10111, 11000, 10100, 10010, 10001\}.$$

Two vertices of G_{11} are *opposite* if the midpoint of the line segment joining them is the center of G_{11}, (1, 0.5, 0.5, 0.5, 0.5). We leave it to the reader to show that, without extra vertices, every triangulation of G_{11} consists of 8 simplices. Moreover a collection of 8 sets of the form conv(V), where each V is a set of 5 vertices of G_{11}, is a triangulation of G_{11} if and only if each of the 8 sets of vertices contains a common pair of opposite vertices and contains no other pair of opposite vertices.

Let S be the simplex conv$\{o_1, o_2, v_1, v_2, v_3, 0\}$, with $\{o_1, o_2, v_1, v_2, v_3\}$ contained in G_{i1}, where o_1 and o_2 are a pair of opposite vertices of G_{i1} and v_1, v_2, v_3 are selected, one each, from the other 3 pairs of opposite vertices. Then S is a class-B simplex, (respectively, C, E, D) if the coordinate sums of v_1, v_2, and v_3 are all 4's (respectively, two 4's and a 2, a 4 and two 2's, all 2's). Thus any triangulation of G_{i1} coned off to 0, contains one B-simplex, three simplices from each of the classes C and E, and one D-simplex.

Let S_A be the class-A simplex described by the first matrix of Table 5.1. Any triangulation $\mathcal{T}$ under consideration is such that across each of the five facets of S_A containing 0 is a class-B simplex of $\mathcal{T}$ in some G_{i1}. For any facet F of S_A containing 0, there are 4 possible choices for the class-B simplex of $\mathcal{T}$ adjacent to S_A across F. For example, the set of vertices v of I^5, such that replacing the first vertex of S_A by v creates a class-B simplex, is $\{11000, 10100, 10010, 10001\}$. If S_B^i is a class-B simplex sharing the facet of S_A obtained by omitting the i^{th} vertex, then the facet of S_B^i opposite 0 is in G_{i1}.

Using our earlier description of triangulations of the G_{ij}'s, one can easily check that any choice of triangulations for all the G_{i1}'s gives a triangulation of $\mathcal{O}$ (the other 11 polytopes of $\mathcal{O}$ are simplices), and thus, by coning off to 0 and adjoining corners, gives a unique triangulation of I^5. A triangulation of G_{i1} is uniquely determined by selecting a vertex v of G_{i1} with coordinate sum 2 which is to be one member of the pair of opposite vertices common to all simplices in the triangulation of G_{i1}. (In this case, conv$(G_i \cup \{v\})$, where G_i is the facet of S_A opposite the i^{th} vertex, is a class-B simplex adjacent to S_A.) Thus a 5×5 $\{0,1\}$-matrix with 1's on the diagonal and common row sums 2 uniquely determines triangulations of G_{11}–G_{51} and hence uniquely determines a triangulation of I^5. Let $\mathcal{M}$ be the set of 1024 such matrices. We want to describe all essentially different triangulations of I^5 which arise by coning off triangulations of $\mathcal{O}$ to 0. We call two such triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ equivalent if there is a permutation matrix P such that any 6×5 matrix S represents a simplex in $\mathcal{T}_1$ if and only if SP represents a simplex in $\mathcal{T}_2$. With some effort the reader can show that the 5×5 matrices M and N generate equivalent triangulations if and only if there is a permutation matrix P such that $P^T M P = N$, and we note that this is a equivalence relation in $\mathcal{M}$. Then, with some more effort, the reader can check that Table 5.2 lists matrices which generate canonical representatives of the 13 equivalence classes in $\mathcal{M}$.

$$\underset{M_1}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\1&0&0&0&1\end{bmatrix}} \quad \underset{M_2}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}} \quad \underset{M_3}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\0&0&1&0&1\end{bmatrix}} \quad \underset{M_4}{\begin{bmatrix}1&0&1&0&0\\1&1&0&0&0\\0&1&1&0&0\\1&0&0&1&0\\1&0&0&0&1\end{bmatrix}} \quad \underset{M_5}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&1\end{bmatrix}}$$

$$\underset{M_6}{\begin{bmatrix}1&1&0&0&0\\0&1&0&1&0\\1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}} \quad \underset{M_7}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\0&0&1&1&0\\0&0&0&1&1\end{bmatrix}} \quad \underset{M_8}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&1&0&1\end{bmatrix}} \quad \underset{M_9}{\begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\1&0&0&1&0\\1&0&0&0&1\end{bmatrix}} \quad \underset{M_{10}}{\begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\1&0&1&0&0\\1&0&0&1&0\\0&0&0&1&1\end{bmatrix}}$$

$$\underset{M_{11}}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\0&0&1&1&0\\0&0&1&0&1\end{bmatrix}} \quad \underset{M_{12}}{\begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\1&0&0&0&1\end{bmatrix}} \quad \underset{M_{13}}{\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&1&0&1\end{bmatrix}}$$

Table 5.2. Matrices Generating Triangulations Corresponding to (5.1)

We record our progress.

Result. *The matrices of Table 5.2 represent simplices which generate the 13 equivalence classes of triangulations of I^5 conforming to the inventory (5.1).*

The method of convex enlargements, described in §2, can be used to produce some of these triangulations. As mentioned in §3, any simplex in Q_5 has volume $m/5$ for some even integer m and also from §2 we have $m \in \{2, 4\}$. From earlier work [2, 5, 11] or our computer generated information about dimension-5 configuration classes, among all configuration classes whose simplices have volume larger than $2/5!$, only class A has simplices in Q_5. And any triangulation can have at most one of these since any class-A simplex has c in its interior. Hence the method of convex enlargements applied to the volume-$4/5!$ simplex S_A in the volume-$104/5!$ polytope Q_5 with the vertices E in any order yields a triangulation of Q_5 with 51 simplices and thus extends to a triangulation of I^5 with 67 simplices. One easily sees that the triangulations which arise in this manner are those determined by M_i for $i \in \{1, 2, 3, 5, 7, 8, 11\}$.

The triangulations corresponding to the solution (5.2) are easier to describe. Without loss of generality, the only such triangulations we consider are those whose single class-G simplex is S_G, the one represented by the appropriate matrix from Table 5.1. Then for $s = 0$ (resp. $s = 10$), with Table 5.1 and some effort, one can show that there is a unique triangulation $\mathcal{T}_0$ ($\mathcal{T}_{10}$) of I^5 with inventory (5.2).

In $\mathcal{T}_{10}$ there are 10 pairs of adjacent simplices, (S^i_M, S^i_F), from classes M and F. As described by Böhm in a different framework [2], for each pair (S^i_M, S^i_F), there is a corresponding pair of adjacent class-K simplices from $\mathcal{T}_0$, $(S^i_K, \overline{S^i_K})$, such that $S^i_M \cup S^i_F = S^i_K \cup \overline{S^i_K}$. Any inventory (5.2) with $s \notin \{0, 10\}$ corresponds to triangulations which use s of the (S^i_M, S^i_F)-pairs and $10 - s$ of the $(S^i_K, \overline{S^i_K})$-pairs, and only to such triangulations.

Briefly postponing additional comments on these triangulations, we turn to triangulations corresponding to (5.3). Without loss of generality, the only such triangulations we consider contain the class-N simplex, S_N, which is represented by the appropriate matrix from Table 5.1. From this table, one can see that any triangulation under consideration has its other class-N simplex adjacent to S_N across a central hyperplane. There are two possibilities: S^1_N and S^2_N obtained by replacing the vertex 11010 of S_N by 00111 and 01110 respectively. The complex $\{S_N, S^1_N\}$ is not the image of $\{S_N, S^2_N\}$ under any sequence of mappings of the form (2.3).

The situation here is similar to that just described. For $s = 0$ (resp. $s = 10$) there is a unique triangulation $\mathcal{R}_0$ (resp. $\mathcal{R}_{10}$) of I^5 with inventory (5.3) which contains the complex $\{S_N, S^1_N\}$ and similarly there are $\mathcal{S}_0$ and $\mathcal{S}_{10}$ containing $\{S_N, S^2_N\}$. As above, the inventories (5.3) with $s \notin \{0, 10\}$ also correspond to triangulations.

Each of $\mathcal{R}_0$, $\mathcal{S}_0$, and $\mathcal{T}_0$ is a middle-cut triangulation for the cut $\sum x_i = 3$.

Using a program which carries out the process of convex enlargements, one easily shows that this method can produce $\mathcal{R}_{10}$, $\mathcal{S}_{10}$, and $\mathcal{T}_{10}$. For $\mathcal{R}_{10}$ and $\mathcal{S}_{10}$, the starting simplex is S_N and for $\mathcal{T}_{10}$ it is S_G. In all three cases, one uses the ordering of the vertices such that the coordinate sums appear in the order 0, 3, 5, 1, 4, 2, and vertices with the same sum are in lexicographical order except that for $\mathcal{R}_{10}$ (resp. $\mathcal{S}_{10}$) the first sum-3 vertex should be 00111 (resp. 01110).

Dimension 5 is the smallest d for which some minimum-cardinality triangulation of I^d fails to induce minimum-cardinality triangulations on all the facets of I^d. All the triangulations corresponding to (5.2) and (5.3) are examples.

Results concerning $T^c(5)$

Our problem for $T^c(5)$ has 10 variables, 11 constraints, and has the unique optimal solution (5.1) with an objective value of 67. We have just discussed the corresponding triangulations. This confirms $T^c(5) = 67$ [5, 11].

Dimension 6

Results concerning $S(6)$

The problem for $S(6)$ has 9890 variables and 60 constraints. We obtained a computed optimal objective value of approximately 269.24 and we conclude $S(6) \geq 270$. This improves the best lower bound of 259 given previously [12].

Results concerning $T(6)$

Our linear programming problem for $T(6)$ has 1154 constraints in 9890 variables. Using the software package CPLEX[2] with the dual simplex method, this problem was solved using 221 iterations and gave an integer-valued optimal solution with objective value of 308. Since a triangulation of I^6 with 308 simplices exists [13], we have achieved one of our major objectives.

Result 5.3. *The minimum of the cardinalities of all triangulations of I^6 is 308, i.e. $T(6) = 308$.*

We will characterize the inventories of configuration classes (i.e. which classes and how many simplices of each class) for all triangulations of I^6 with 308 simplices. We learn which configuration classes are involved by experimenting with additional constraints which modify inventories from previous optimal solutions. In addition to the configuration classes A–L previously defined [13], we need the configuration classes M, N, and O represented by the simplices which are in turn represented by the following matrices:

[2] CPLEX is a registered trademark of CPLEX Optimization, Inc.

$$\begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&0&1&0&1&1\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&1&1&0&1&0\\ 0&0&1&0&1&1\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&0&0&1&1&0\\ 1&1&1&1&1&0\\ 1&1&0&1&0&0\\ 1&1&0&0&1&0\\ 1&1&0&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix}.$$

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M N O

We let $x_A, x_B, \ldots, x_O$ be the variables in our problem corresponding to the classes $A, B, \ldots, O$.

One can check that each of these variables represents only one configuration class. This can be carried out for a particular variable x as follows. For the simplex S used to define a configuration class corresponding to x, find a base B of S which is not central and selfcomplementing. For each vertex v of I^d such that the simplex $T = \operatorname{conv}(B \cup \{v\})$ generates a configuration class represented by x, check that S and T are equivalent. Here S and T are represented by the same variable if they have the same volume, the same exterior-facet tuple, and, relative to some orderings of the vertices of S and T, the same sequence of base classes (except for possible discrepancies in central selfcomplementing base classes and base classes of bases contained in facets of I^d), the same sequence of side indicators, and the same sequence of signs for the barycentric coordinates of c relative to the vertices.

We claim that any integer-valued optimal solution to our problem for $T(6)$ satisfies

$$x_A = 2,\ x_B = 12,\ x_C + x_D = 30,\ x_D - x_E = 0,\ x_D - x_F = 0,$$
$$x_D - x_G = 0,\ x_H = 48,\ x_I = 24,\ x_J = 60,\ x_K - x_L = 0,$$
$$x_D + x_M = 24,\ 2x_D + x_N = 48,\ 2x_K + x_O = 60.$$

These equations were conjectured by experimenting with additional constraints. They are justified by solving several problems—each is obtained by adjoining one of these constraints, changing it to an inequality, and perturbing its right-hand side by ± 1, giving a new problem which is either infeasible or has an optimal objective value larger than 308. Hence every all-integer optimal solution to our problem satisfies

$$\begin{aligned} &x_A = 2,\ x_B = 12,\ x_C = 30 - t,\ x_D = x_E = x_F = x_G = t,\ x_H = 48,\ x_I = 24, \\ &x_J = 60,\ x_K = x_L = s,\ x_M = 24 - t,\ x_N = 48 - 2t,\ x_O = 60 - 2s \end{aligned} \tag{5.4}$$

for some $s \in \{0,1,2,\ldots,30\}$ and some $t \in \{0,1,2,\ldots,24\}$. Conversely, using information about the configuration classes A–O, not all of which has been displayed in this paper, one can check that any such inventory gives an optimal solution. It turns out that not all these optimal solutions correspond to triangulations; t must be further restricted.

Result. *Any triangulation of I^6 with 308 simplices yields an inventory of configuration classes satisfying (5.4) for some $s \in \{0,1,2,\ldots,30\}$ and some $t \in \{0,2,4,\ldots,24\}$. Conversely, to any such inventory, there corresponds a triangulation of I^6 which conforms to this inventory.*

Justification. The inventory of configuration classes for the previously published triangulation of I^6 into 308 simplices [13] satisfies (5.4) with $s = 30$ and $t = 24$. We use this triangulation as a starting point and discuss the possibility of modifying it to yield triangulations with other inventories.

The parameter s, which is the number of corners, is treated first. The union of a class-L simplex and an adjacent class-K simplex is a polytope which is also the union of two adjacent class-O simplices. Thus for s_1 and s_2 in $\{0,1,\ldots,30\}$, any triangulation satisfying (5.4) with $s = s_1$ and some t can be modified, two simplices at a time, to be a triangulation satisfying (5.4) with $s = s_2$ and the same t.

The treatment of the parameter t is more complicated. We will first show that no triangulation satisfies (5.4) with t odd. Let B_1 and B_2 be the bases represented by the following matrices:

$$\underset{M_1}{\begin{bmatrix} 1&1&1&0&0&0\\ 1&1&0&1&0&0\\ 1&1&0&0&1&0\\ 1&0&1&1&0&0\\ 1&0&1&0&0&1\\ 0&1&0&1&0&1 \end{bmatrix}} \quad \text{and} \quad \underset{M_2}{\begin{bmatrix} 1&1&1&1&0&0\\ 1&1&0&0&1&0\\ 1&1&0&0&0&1\\ 1&0&1&0&1&0\\ 1&0&0&1&0&1\\ 0&1&1&0&0&1 \end{bmatrix}}.$$

The bases B_1 and B_2 are central and are contained in the hyperplanes $\sum_{i=1}^{6} x_i = 3$ and $x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 = 4$, respectively. Of the configuration classes A–O, only D and E have simplices with bases in $[B_1] \cup [B_2]$. Each of the base classes $[B_1]$ and $[B_2]$ is represented exactly once in each simplex from class D or E. Moreover, for each of the two base classes, the side indicator of a class-D simplex for the facet generated by a base in this class is opposite that of a class-E simplex. Suppose S_D^1 is a class-D simplex. Then

S_D^1 is adjacent to some class-E simplex S_E^1 across a class-$[B_1]$ base and one can check that S_E^1 is uniquely determined. Similarly S_E^1 uniquely determines a class-D simplex S_D^2 which is adjacent across a base from class $[B_2]$. Then S_D^2 in turn uniquely determines a class-E simplex S_E^2 across a base from class $[B_1]$. It turns out that S_D^1 and S_E^2 are adjacent across a base of class $[B_2]$. It follows that if $\mathcal{T}$ is a triangulation of I^6 with 308 simplices which contains at least one of S_D^1, S_D^2, S_E^1, S_E^2, then $\mathcal{T}$ contains all four of these. We conclude that no inventory for a triangulation of I^6 with 308 simplices can be described by (5.4) with t odd.

The remaining goal is to show that any inventory of configuration classes described by (5.4) with t even corresponds to a triangulation. Suppose $\mathcal{T}$ is a minimum-cardinality triangulation with parameters s and t with $t \geq 2$. We will describe 18 simplices to be removed from $\mathcal{T}$ and 18 to be inserted so that $\mathcal{T}$ remains a triangulation and has parameters s and $t-2$. This will be sufficient to accomplish our goal.

The triangulation $\mathcal{T}$ must contain a quadruple S_D^1, S_D^2, S_E^1, S_E^2, as described in the penultimate paragraph. Each of S_D^1, S_D^2, S_E^1, S_E^2 has exactly one facet with fundamental normal 111222. Across these four facets are the $\mathcal{T}$-simplices S_F^1, S_F^2, S_G^1, S_G^2, respectively, with S_F^1 and S_F^2 in class F and S_G^1 and S_G^2 in class G. Each of S_D^1, S_D^2, S_E^1, S_E^2, S_F^1, S_F^2 has exactly one facet with fundamental normal 111112 and each of S_G^1 and S_G^2 has two such facets. Across these 10 facets are the $\mathcal{T}$-simplices S_I^1, S_I^2, S_H^1, S_H^2, S_H^3, S_H^4, S_J^1, S_J^2, S_J^3, S_J^4, respectively, in the configuration classes suggested by this labeling. We have identified the 18 simplices to be removed and we turn to the problem of defining the simplices to replace them.

We first let S_M^a and S_M^b be the class-M simplices adjacent to the same class-B simplices as are S_D^1 and S_D^2. Each of S_M^a and S_M^b has two facets with fundamental normal 111222. Across each of these facets there is exactly one adjacent simplex of volume 3/6! and it's a class-N simplex. This distinguishes four simplices and we denote them by S_N^a, S_N^b, S_N^c, S_N^d. Each of these has one facet not shared with S_M^a or S_M^b which has fundamental normal 111222. Across each of these facets is exactly one volume-3/6! simplex. This distinguishes only two simplices, S_C^a and S_C^b, and these are in class C. We may assume S_C^a is adjacent to S_N^a and S_N^b while S_C^b is adjacent to S_N^c and S_N^d. Now each of S_M^a, S_M^b, S_N^a, S_N^b, S_N^c, S_N^d has exactly one facet with fundamental normal 111112 while each of S_C^a and S_C^B has two such facets. For each of these facets, there is exactly one simplex of volume 2/6! which is adjacent to the corresponding simplex across this facet. We obtain S_I^a, S_I^b, S_H^a, S_H^b,

S^c_H, S^d_H, S^a_J, S^b_J, S^c_J, S^d_J, respectively, in the classes suggested by this notation. These 18 simplices fill the cavity exactly with a facet to facet match across the part of the boundary in the interior of the cube.

□

The minimum-cardinality triangulations of I^6 are middle-cut triangulations. Let $\mathcal{T}$ be a triangulation corresponding to (5.4) with parameters s and t. Then $\mathcal{T}$ contains the corners at s of the 30 vertices of I^6 with coordinate sums 2 and 4. Using notation from §2, suppose the corners of $A(6,3)$, which are in $\mathcal{T}$, are removed and the resulting set is closed to form $\bar{A}(6,3)$. Let $\bar{B}(6,3)$ be defined similarly. Then $\mathcal{T}$ induces a triangulation on the complex

$$\bar{\mathcal{C}} = \{\, F : F \text{ is a facet of } \bar{A}(6,3) \text{ opposite } 0 \,\} \cup \{\, F : F \text{ is a facet of } \bar{B}(6,3) \text{ opposite } e \,\}.$$

If $s = 0$ (respectively 30), $\mathcal{T}$ is similar to the dimension-6 instances of the middle-cut creations of Sallee (Böhm) with 324 simplices—16 more than ours. A key difference is that $\mathcal{T}$ induces a triangulation of $I^d \cap H(6,3)$ with 58 simplices while the analogous number for Sallee and Böhm is 66. Doubling this difference for the two sides of $H(6,3)$ gives the difference of 16.

Triangulations of I^6 with 308 simplices can also be created using the method of convex enlargements. For example, a triangulation corresponding to the inventory (5.4) with $s = 30$ and $t = 0$ is generated by starting with the simplex S_A and using the vertices of I^6 in the following order: the vertices of S_A, 111111, the complements of the first 6 vertices of S_A, 011010, 010101, 101100, 100011, 010011, 011100, 101010, 100101, and then vertices with coordinate sums 5, 1, 4, 2 respectively, using the lexicographical order within each group with constant sum.

Results concerning $T^c(6)$

The problem for $T^c(6)$ has 126 variables and 52 constraints. We obtained a unique computed integer-valued optimal solution with an objective value of 324. This confirms $T^c(6) \geq 324$ [11] and, as previously announced [11], we have $T^c(6) = 324$. The optimal solution agrees with the unique inventory of configuration classes for minimum-cardinality corner-slicing triangulations of I^6 given earlier [11]. As noted before [11], the method of convex enlargements can be used to produce such triangulations.

Dimension 7

Results concerning $T(7)$

Our linear programming problem for $T(7)$ is rather large; it has 103706 constraints in 1456318 variables. We were not able to solve this problem. We will describe a smaller problem, specific to dimension 7, which is good enough for our purposes. As for the larger problem, and for the same reasons, the optimal objective value of the smaller problem provides a lower bound on the cardinalities of triangulations of I^7. Among its optimal solutions are some yielding inventories of configuration classes corresponding to triangulations of I^7 with 1493 simplices. This will allow us to conclude that $T(7) = 1493$. Again we will characterize the inventories of configuration classes for all triangulations of I^7 with 1493 simplices. From this we will see that no minimum-cardinality triangulation of I^7 induces minimum-cardinality triangulations on all the facets.

This new smaller problem has constraints corresponding to a smaller family of base classes. Using intuition based on experience with other problems and experimentation with the current problem, we select those base classes which correspond to constraints we think might affect the optimal solution. Among all base classes $[B]$ where B is neither central and selfcomplementing nor on a facet of I^7, we let $\mathcal{B}_7$ be the set of base classes containing bases from any of the following simplices:

(a) simplices of volume at least $10/7!$,

(b) simplices with exterior-facet tuple $(1,0,0,0,0)$ and volume $k/7!$ for $k \in \{6,7,8,9\}$,

(c) simplices with exterior-facet tuple $(2,2,0,0,0)$ and volume $k/7!$ for $k \in \{4,5\}$,

(d) simplices with exterior-facet tuple $(3,6,6,0,0)$ and volume $3/7!$,

(e) simplices with exterior-facet tuple $(2,12,24,24,0)$ and volume $2/7!$,

(f) simplices with at least 5 exterior facets and volume $1/7!$.

There are 9234 base classes in $\mathcal{B}_7$ and our new linear programming problem has one constraint for each of these. As before, we will have 6 constraints based on volume considerations.

If the same variables were used as in our basic problem for $T(7)$, then the majority would not enter into the constraints corresponding to base classes, and, within this majority, variables with common volume-exterior-facet-tuple pair would be involved in the problem in the same way. We take advantage of these facts to significantly reduce the number of variables. For the new smaller problem, each variable is either a variable of the basic dimension 7 problem or a sum of such variables. More explicitly, if a simplex S

in I^7 has a facet which determines a base in $\mathcal{B}_7$, then the variable of the larger problem representing the configuration class $[S]$ is a variable of the smaller problem. In this way we get 421487 variables. The additional variables of the smaller problem arise by lumping together variables of the larger problem (other than the 421487 already used) with common volume-exterior-facet-tuple pairs. There are 1378 volume-exterior-facet-tuple pairs. For each of the 1329 of these not included in the 49 pairs described implicitly in (a)–(f) above, we have a variable which represents all the corresponding configuration classes. Thus we have 422816 variables.

The objective function to be minimized is the sum of the 422816 nonnegative variables. There are 9234 equality constraints in 421487 variables corresponding to the base classes in $\mathcal{B}_7$ and there are six constraints, in all 422816 variables, based on volume considerations.

Again using CPLEX's dual simplex implementation we obtained an integer-valued optimal solution to this problem using 3876 iterations. The optimal objective value is 1493 which means that no triangulation of I^7 has fewer than 1493 simplices. In order to make sense of the values of the variables in the optimal solutions we introduce the relevant configuration classes in Table 5.3. In each case we give a matrix representing a simplex in the class and we append the volume and exterior-facet tuple. For the opposite facet of each vertex of each representative simplex, we give one of the following:

(1) an indication that the facet is an exterior facet,
(2) the number of the base class and half-space indicator in case the base is noncentral,
(3) the number of the base class in parentheses in case the base is cental and selfcomplementing.

For these classes, none of the bases is central and non-selfcomplementing.

$$\begin{bmatrix} 1&1&1&0&0&0&0\\ 1&0&0&1&1&0&0\\ 1&0&0&0&0&1&1\\ 0&1&0&1&0&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 1+\\ 1+\\ 1+\\ 1+\\ 1+\\ 1+\\ 1+\\ 1+ \end{matrix}$$

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eft (0, 0, 0, 0, 0)

A

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 1&0&0&1&1&0&0\\ 1&0&0&0&0&1&1\\ 0&1&0&1&0&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 1-\\ 2+\\ 2+\\ 2+\\ 2+\\ 2+\\ 2+\\ 2+ \end{matrix}$$

vol 24/7!

eft (0, 0, 0, 0, 0)

B

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&0&0&0&1&1\\ 0&1&0&1&0&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 2-\\ 2-\\ 3+\\ 3+\\ 3+\\ 3+\\ 3+\\ 3+ \end{matrix}$$

vol 16/7!

eft (0, 0, 0, 0, 0)

C

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&1&0&1&0&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} (12)\\ (12)\\ 3-\\ 4+\\ 4+\\ 4+\\ 4+\\ \mathrm{ef} \end{matrix}$$

vol $8/7!$

eft $(1,0,0,0,0)$

D

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 5+\\ 5+\\ 4-\\ 4-\\ 6+\\ 6+\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $4/7!$

eft $(2,2,0,0,0)$

E

$$\begin{bmatrix} 0&1&1&0&1&0&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 5-\\ 7+\\ 5-\\ 5-\\ \mathrm{ef}\\ 8+\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $3/7!$

eft $(3,6,6,0,0)$

F

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&0&1&1&0&1&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} \mathrm{ef}\\ 8+\\ 6-\\ 6-\\ 6-\\ \mathrm{ef}\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $2/7!$

eft $(4,12,24,24,0)$

G

$$\begin{bmatrix} 0&1&1&0&1&0&1\\ 0&0&1&1&1&0&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 7-\\ 7-\\ 7-\\ 7-\\ \mathrm{ef}\\ \mathrm{ef}\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $2/7!$

eft $(4,12,24,24,0)$

H

$$\begin{bmatrix} 0&1&1&0&1&0&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&0&0&1&0&1\\ 0&1&1&0&1&1&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} (13)\\ \mathrm{ef}\\ (13)\\ (13)\\ \mathrm{ef}\\ 8-\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $1/7!$

eft $(4,15,51,150,360)$

I

$$\begin{bmatrix} 0&1&1&0&1&0&1\\ 0&1&0&0&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&0&0&1&0&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 8-\\ 8-\\ \mathrm{ef}\\ (14)\\ (14)\\ 9+\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $1/7!$

eft $(3,8,18,30,30)$

J

$$\begin{bmatrix} 0&1&1&0&1&0&1\\ 0&1&0&0&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&1&0&1&1&1\\ 0&0&1&0&1&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} \mathrm{ef}\\ \mathrm{ef}\\ \mathrm{ef}\\ (15)\\ (15)\\ 9-\\ \mathrm{ef}\\ \mathrm{ef} \end{matrix}$$

vol $1/7!$

eft $(5,22,84,270,690)$

K

$$\begin{bmatrix} 0&0&1&0&1&1&1\\ 0&1&1&0&1&1&0\\ 0&0&1&0&1&1&0\\ 0&1&1&0&1&0&1\\ 0&1&0&0&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 9-\\ 9-\\ 10+\\ \mathrm{ef}\\ \mathrm{ef}\\ \mathrm{ef}\\ (16)\\ \mathrm{ef} \end{matrix}$$

vol $1/7!$

eft $(4,13,33,60,60)$

L

$$\begin{bmatrix} 0&0&1&0&1&1&1\\ 0&1&1&0&1&1&0\\ 0&1&1&0&1&1&1\\ 0&1&1&0&1&0&1\\ 0&1&0&0&1&1&1\\ 0&1&1&0&0&1&1\\ 0&1&1&1&1&1&1\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} \text{ef}\\ \text{ef}\\ 10-\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ (17)\\ \text{ef} \end{matrix}$$

vol $1/7!$

eft $(6, 31, 135, 480, 1320)$

M

$$\begin{bmatrix} 0&1&1&1&1&1&1\\ 1&1&1&0&1&1&1\\ 0&0&1&0&1&1&1\\ 0&1&1&0&1&1&0\\ 0&1&1&0&1&0&1\\ 0&1&0&0&1&1&1\\ 0&1&1&0&0&1&1\\ 1&1&1&1&1&1&1 \end{bmatrix} \begin{matrix} 10-\\ 10-\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ 11+ \end{matrix}$$

vol $1/7!$

eft $(5, 20, 60, 120, 120)$

N

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef}\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ \text{ef}\\ 11- \end{matrix}$$

vol $1/7!$

eft $(7, 42, 210, 840, 2520)$

O

Table 5.3 The Configuration Classes for Minimum-Cardinality Triangulations of I^7

Let $x_A, \ldots, x_O$ be the variables in our problem coresponding to the classes A–O. One can check that each of these variables represents only one configuration class. As in the case of dimensions 5 and 6, by solving many linear programming problems, we learn that every integer-valued optimal solution to our 422816-variable problem satisfies

$$\begin{gathered} x_A = 1,\ x_B = 8,\ x_C = 28,\ x_D = 168,\ x_E = 336,\ x_F = 224, \\ x_G = 224,\ x_H = 56,\ x_I = 448 - 2r,\ x_J = r,\ x_K = r - 2s, \\ x_L = s,\ x_M = s - 2t,\ x_N = t,\ x_O = t \end{gathered} \tag{5.5}$$

for some integers r, s, and t with $r \in \{0, 1, 2, \ldots, 224\}$, $s \in \{0, 1, 2, \ldots, \lfloor r/2 \rfloor\}$, and $t \in \{0, 1, 2, \ldots, \lfloor s/2 \rfloor\}$. Conversely, using information about the configuration classes $I, J, \ldots, O$, one easily checks that any such inventory yields an optimal solution.

It turns out that not all these optimal solutions correspond to triangulations; r and s must be further restricted

Result 5.4. *Any triangulation of I^7 with 1493 simplices yields an inventory of configuration classes satisfying* (5.5) *with $r \in \{0, 4, 8, \ldots, 224\}$, $s \in \{0, 2, 4, \ldots, r/2\}$, and $t \in \{0, 1, 2, \ldots, s/2\}$. Conversely, to any such inventory, there corresponds a triangulation of I^7 conforming to this inventory.*

Justification. We first indicate how to construct a triangulation of I^7 which corresponds to the inventory (5.5) with $r = s = t = 0$. (This triangulation is relatively simple and can be formed and checked with only moderate computer involvement. The main tool would be a program which finds the fundamental normals of bases.) Start with any class-A simplex. Adjacent to it are 8 uniquely determined class-B simplices. Adjacent to these, across all facets not shared with the class-A simplices, are 28 uniquely determined class-C simplices. (Each of the 28 is adjacent to two class-B simplices.) Each class-C simplices has 6 facets generated by a base from base class 3—a class-3 facet, for short. Across each of the class-3 facets of our 28 class-C simplices is a unique class-D simplex and altogether this yields 168 class-D simplices. Each class-D simplex has four class-4 facets and across each of these is exactly one class-E simplex. Our class-D simplices give us 336 class-E simplices in this manner. Each class-E simplex has two class-5 facets and two class-6 facets. Across each class-5 facet is exactly one class-F simplex and across each class-6 facet is a unique class-G simplex. Our class-E simplices yield 224 class-F simplices and 224 class-G simplices in this manner. Each class-F simplex has one class-7 facet and across this facet is exactly one class-H simplex. We obtain 56 class-H simplices from our class-F simplices. Each class-F simplex and each class-G simplex has a class-8 facet and across this facet is a unique class-I simplex. We obtain a total of 448 class-I simplices from our simplices of classes F and G.

One can use Theorem 2.3 of [11] to verify that the collection $\mathcal{T}_0$ of 1493 simplices we have described is a triangulation. To carry this out, a few simple computer programs would be needed. We also note that $\mathcal{T}_0$ is uniquely determined by the class-A simplex with which we started.

We have several comments before we consider triangulations conforming to inventories (5.5) with some nonzero parameter values.

The following two facts can be established by routine checking. For the first, the reader probably needs a program which finds base class numbers of bases.

(1) If S_1 is a simplex from class I (K, M, respectively) and F is a facet of S_1 whose vertices give a base of class 13 (15, 17), then there is a unique simplex S_2 from class I (K, M) such that S_1 and S_2 are adjacent with common facet F.

(2) The union of two adjacent simplices from class I (K, M, respectively) can be written uniquely as the union of a simplex from class J and a simplex from class K (simplices from classes L and M, simplices from classes N and O).

In the case of class-I simplices in (1), we give a rule for finding S_2 from S_1. Suppose V is a matrix whose rows are the vertices of S_1 and row i gives the vertex opposite the class-13 facet, F. Let p be the column of V such that the entry in position (i,p) appears in exactly one other row of column p, say row r. Then let q be the column such that the entry in the (r,q) position does not appear elsewhere in column q. Then the j^{th} coordinate of the vertex of S_2 opposite F is the same as the majority of the entries in column j of V

with the exception that for column q we use the minority instead of the majority.

We are in a position to see why, for an inventory (5.5) which corresponds to a triangulation $\mathcal{T}$, the parameter r must be an integer multiple of 4. Let S_1 be a class-I simplex in $\mathcal{T}$. From information displayed in Table 5.3, S_1 must be adjacent to three other class-I simplices in $\mathcal{T}$. Using the above pivot rule, one can easily find these three simplices and then continue as long as possible finding new class-I simplices in $\mathcal{T}$ adjacent to those already found. This process terminates upon obtaining 8 simplices. If we form the graph with these 8 simplices as nodes and having $\{ S_i, S_j \}$ as an arc if S_i and S_j are adjacent, then the graph is bipartite and each node has degree 3. From the uniqueness mentioned in (1) above and the definition of a configuration class, any starting class-I simplex in $\mathcal{T}$ yields the same structure. Hence the number of class-I simplices in $\mathcal{T}$ is an integer multiple of 8 and r is a multiple of 4.

The situation is somewhat similar for the K-simplices of a minimum-cardinality triangulation $\mathcal{T}$. From Table 5.3, each class-K simplex of $\mathcal{T}$ is adjacent to exactly two others. Starting with a K-simplex of $\mathcal{T}$ leads to a complex of four class-K simplices in $\mathcal{T}$ which, in graph theoretic language, is a cycle. Thus the parameter s for the inventory corresponding to $\mathcal{T}$ is even.

Let $\mathcal{C}_1$ be a complex of 8 class-I simplices as just described. We let the nodes be $S_I^1, S_I^2, \ldots, S_I^8$ and assume the adjacent pairs of simplices are $\{S_I^1, S_I^2\}$, $\{S_I^1, S_I^4\}$, $\{S_I^1, S_I^6\}$, $\{S_I^3, S_I^2\}$, $\{S_I^3, S_I^4\}$, $\{S_I^3, S_I^8\}$, $\{S_I^5, S_I^2\}$, $\{S_I^5, S_I^6\}$, $\{S_I^5, S_I^8\}$, $\{S_I^7, S_I^4\}$, $\{S_I^7, S_I^6\}$, $\{S_I^7, S_I^8\}$. We will describe how $\mathcal{C}_1$ can be modified successively to yield three other complexes which could replace $\mathcal{C}_1$ in a triangulation so that the result continues to be a triangulation.

The first of these is the complex $\mathcal{C}_2$ formed by replacing, as in (2), the four pairs of adjacent simplices $\{S_I^1, S_I^2\}$, ..., $\{S_I^7, S_I^8\}$ with the pairs $\{S_J^1, S_K^1\}, \ldots, \{S_J^4, S_K^4\}$ of adjacent simplices in the configuration classes suggested by the notation. The adjacent pairs of simplices for $\mathcal{C}_2$ are $\{S_J^1, S_J^2\}$, $\{S_J^1, S_J^3\}$, $\{S_J^2, S_J^4\}$, $\{S_J^3, S_J^4\}$, $\{S_K^1, S_K^2\}$, $\{S_K^1, S_K^3\}$, $\{S_K^2, S_K^4\}$, $\{S_K^3, S_K^4\}$, $\{S_J^1, S_K^1\}$, $\{S_J^2, S_K^2\}$, $\{S_J^3, S_K^3\}$, $\{S_J^4, S_K^4\}$.

Then the complex $\mathcal{C}_3$ is formed from $\mathcal{C}_2$ by replacing the pairs $\{S_K^1, S_K^2\}$ and $\{S_K^3, S_K^4\}$ with $\{S_L^1, S_M^1\}$ and $\{S_L^2, S_M^2\}$ so that neither replacement affects the union. The adjacent pairs of simplices for $\mathcal{C}_3$ are $\{S_J^1, S_J^2\}$, $\{S_J^1, S_J^3\}$, $\{S_J^2, S_J^4\}$, $\{S_J^3, S_J^4\}$, $\{S_J^1, S_L^1\}$, $\{S_J^2, S_L^1\}$, $\{S_J^3, S_L^2\}$, $\{S_J^4, S_L^2\}$, $\{S_L^1, S_L^2\}$, $\{S_L^1, S_M^1\}$, $\{S_L^2, S_M^2\}$, $\{S_M^1, S_M^2\}$.

If in $\mathcal{C}_3$ the pair $\{S_M^1, S_M^2\}$ is replaced by $\{S_N^1, S_O^1\}$ with the same union, we obtain the complex $\mathcal{C}_4$ whose adjacent pairs of simplices are $\{S_J^1, S_J^2\}$, $\{S_J^1, S_J^3\}$, $\{S_J^2, S_J^4\}$, $\{S_J^3, S_J^4\}$,

$\{S_J^1, S_L^1\}$, $\{S_J^2, S_L^1\}$, $\{S_J^3, S_L^2\}$, $\{S_J^4, S_L^2\}$, $\{S_L^1, S_L^2\}$, $\{S_L^1, S_N^1\}$, $\{S_L^2, S_N^1\}$, $\{S_N^1, S_O^1\}$.

Finally we consider triangulations whose inventories (5.5) have some nonzero parameter values. Let $r \in \{0, 4, 8, \ldots, 224\}$, $s \in \{0, 2, 4, \ldots, r/2\}$, and $t \in \{1, 2, \ldots, s/2\}$. The triangulation $\mathcal{T}_0$ has 56 complexes of class-I simplices with the same adjacency structure as $\mathcal{C}_1$. Suppose we modify $\mathcal{T}_0$ by changing $r/4$ of these complexes to obtain: $(r-2s)/4$ complexes with the same structure as $\mathcal{C}_2$, $(s-2t)/2$ complexes similar to $\mathcal{C}_3$, and t complexes similar to $\mathcal{C}_4$. The result is a triangulation whose inventory satisfies (5.5) with the prescribed parameter values.

□

In view of this result, we have achieved another major objective.

Result 5.5. *The minimum of the cardinalities of all triangulations of I^7 is* 1493, *i.e,* $T(7) = 1493$.

We were not able to generate any minimum-cardinality triangulation of I^7 using any of the general methods of §2.

One might hope that a minimum-cardinality triangulation of I^d could somehow be formed from minimum-cardinality triangulations of all the facets of I^d. This is true for $d \leq 6$ But every minimum-cardinality triangulation of I^7 has $4424 = (14)(316)$ exterior facets while each minimum-cardinality triangulation of I^6 has 308 simplices. Thus we have the following result.

Remark *No minimum-cardinality triangulation of I^7 induces minimum-cardinality triangulations on all the facets.*

Results concerning $T^c(7)$

Our problem for $T^c(7)$ is relatively small; it has 1653 constraints in 8303 variables. Our computed optimal objective value is 1820.00 which implies $T^c(7) \geq 1820$. We produced several integer-valued optimal solutions but none with fewer than 42 variables taking nonzero values. By learning from a few failures, we found one of our computed optimal solutions for which we were able to construct a corresponding corner-slicing triangulation of I^7 with 1820 simplices. We summarize our claims.

Result 5.6. *The minimum of the cardinalities of all corner-slicing triangulations of I^7 is 1820, i.e.* $T^c(7) = 1820$.

Justification. Here we give a concise description of the formation of a corner-slicing triangulation, $\mathcal{T}$, with 1820 simplices. The construction of such a triangulation by the reader would require a computer and significant effort.

The triangulation contains the corners at all the 64 vertices of I^7 with odd coordinate sums. The remainder of the simplices have only vertices with even coordinate sums and come from 42 configuration classes. Table 5.4 gives matrices representing simplices in these classes. For each class the volume of the simplices and the number of simplices in $\mathcal{T}$ are given. For each vertex of each representative simplex, we indicate whether the opposite facet (1) is an exterior facet, (2) determines a central selfcomplementing base, or (3) determines either a noncentral base or a central non-selfcomplementing base. In this last case, we give a number for the base class and the appropriate half-space or side indicator.

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 1&1&0&1&1&0&0 \\ 1&0&1&1&1&0&0 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 1- \\ 1- \\ 2+ \\ 1- \\ 2+ \\ 1- \\ 2+ \\ 2+ \end{matrix}$$

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num. req. 4

A

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&0&1&1&1&0&0 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 1+ \\ 4- \\ 6+ \\ 3- \\ 7+ \\ 1+ \\ 2+ \\ 5+ \end{matrix}$$

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num. req. 8

B

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&0&1&1&1&0&0 \\ 0&1&1&1&1&0&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 4+ \\ 3+ \\ 7+ \\ 3+ \\ 8+ \\ 4+ \\ 6+ \\ 9+ \end{matrix}$$

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num. req. 4

C

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 14+ \\ 10+ \\ 2- \\ 12+ \\ 12+ \\ 13- \\ 11+ \\ 15+ \end{matrix}$$

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num. req. 24

D

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 14- \\ 16+ \\ 6- \\ 17+ \\ 17+ \\ 14- \\ 15+ \\ \text{ef} \end{matrix}$$

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num. req. 12

E

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&0&1&1&1&0&0 \\ 1&0&0&0&0&0&1 \\ 1&0&0&1&1&1&0 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 13+ \\ 21+ \\ 21+ \\ 18+ \\ 7- \\ 13+ \\ 11+ \\ 20+ \end{matrix}$$

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num. req. 12

F

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 1&1&0&1&1&0&0\\ 1&0&1&1&1&0&0\\ 0&1&1&1&1&0&0\\ 1&1&1&1&1&1&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 24-\\ 22+\\ 22+\\ 22+\\ 8-\\ 24-\\ \text{ef}\\ 23+ \end{matrix}$$

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num. req. 4

G

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 1&1&0&1&1&0&0\\ 1&0&1&1&1&0&0\\ 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&1&1&0&0 \end{bmatrix} \begin{matrix} 25+\\ 24+\\ \text{ef}\\ 26-\\ 20+\\ 25+\\ 25+\\ 5- \end{matrix}$$

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num. req. 8

H

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 1&1&0&1&1&0&0\\ 1&0&1&1&1&0&0\\ 0&1&1&1&1&0&0\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&1&1&0&0 \end{bmatrix} \begin{matrix} 28+\\ 26+\\ 20+\\ 26+\\ 27+\\ 28+\\ 28+\\ 9- \end{matrix}$$

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num. req. 4

I

$$\begin{bmatrix} 1&1&1&0&0&1&0\\ 1&1&0&1&1&0&0\\ 1&1&0&0&0&1&1\\ 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 1&1&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef}\\ 31+\\ 30-\\ 29+\\ 29+\\ 32+\\ 32+\\ 15- \end{matrix}$$

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num. req. 36

J

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 1&1&0&1&1&0&0\\ 1&1&0&0&0&1&1\\ 0&1&0&1&1&1&0\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 33+\\ 35+\\ 21-\\ 17-\\ 36+\\ 33+\\ 34+\\ \text{ef} \end{matrix}$$

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num. req. 24

K

$$\begin{bmatrix} 1&1&1&0&0&1&0\\ 1&1&0&1&1&0&0\\ 1&1&0&0&0&1&1\\ 0&1&0&1&1&1&0\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 33-\\ 37+\\ 12-\\ 12-\\ 37+\\ 33-\\ 34+\\ 34+ \end{matrix}$$

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num. req. 24

L

$$\begin{bmatrix} 1&1&1&0&0&1&0\\ 1&1&0&1&1&0&0\\ 1&1&0&0&0&1&1\\ 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&1\\ 1&1&0&0&1&1&0\\ 0&0&0&0&1&1&0\\ 1&1&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef}\\ \text{ef}\\ \text{csc}\\ 40+\\ 40+\\ 32-\\ 39+\\ 38- \end{matrix}$$

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num. req. 72

M

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 1&1&0&1&1&0&0\\ 1&0&1&1&1&0&0\\ 1&0&0&0&0&0&1\\ 1&0&0&1&1&1&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&1&1&0&0 \end{bmatrix} \begin{matrix} 42+\\ \text{ef}\\ \text{ef}\\ 41+\\ 20-\\ 42+\\ 42+\\ 20- \end{matrix}$$

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num. req. 12

N

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&1&0&0&0&1&1\\ 0&1&0&1&1&1&0\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 44-\\ 35-\\ 35-\\ 44-\\ \text{ef}\\ 44-\\ 43+\\ 23+ \end{matrix}$$

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num. req. 12

O

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&0&1&1&1&0&0 \\ 1&1&1&1&1&1&0 \\ 1&0&0&1&1&1&0 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 46- \\ 45+ \\ 45+ \\ 18- \\ 22- \\ 46- \\ \text{ef} \\ 27+ \end{matrix}$$

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num. req. 12

P

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 1&1&0&1&0&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 38+ \\ \text{ef} \\ 30+ \\ \text{csc} \\ \text{csc} \\ 48+ \\ 11- \\ 38+ \end{matrix}$$

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num. req. 36

Q

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 46- \\ 10- \\ 25- \\ 47+ \\ 47+ \\ 46+ \\ \text{ef} \\ 48+ \end{matrix}$$

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num. req. 24

R

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 0&1&0&1&1&1&0 \\ 1&0&0&1&1&1&0 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 50+ \\ \text{ef} \\ 49- \\ 36- \\ 36- \\ 50+ \\ 31+ \\ \text{ef} \end{matrix}$$

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num. req. 12

S

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 46+ \\ 16- \\ 28- \\ 51+ \\ 51+ \\ 46+ \\ 48+ \\ \text{ef} \end{matrix}$$

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num. req. 12

T

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&1&0&0&1&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 53- \\ 44+ \\ 51- \\ 51- \\ \text{ef} \\ 53- \\ 52+ \\ 27+ \end{matrix}$$

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num. req. 12

U

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 0&1&0&0&0&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef} \\ 54+ \\ \text{csc} \\ 29- \\ 56+ \\ 55+ \\ 55+ \\ 57- \end{matrix}$$

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num. req. 72

V

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&0&1&1&1&0&0 \\ 0&1&1&1&1&0&0 \\ 1&1&1&1&1&1&0 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&1&1&0&0 \end{bmatrix} \begin{matrix} \text{ef} \\ 58+ \\ 58+ \\ 58+ \\ 27- \\ \text{ef} \\ \text{ef} \\ 23- \end{matrix}$$

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num. req. 16

W

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 59+ \\ \text{csc} \\ \text{csc} \\ 57+ \\ 48+ \\ 59+ \\ 34- \\ \text{ef} \end{matrix}$$

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num. req. 72

X

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 1&1&0&1&1&0&0 \\ 1&1&1&1&1&1&0 \\ 0&1&0&1&1&1&0 \\ 1&0&0&1&1&1&0 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 60+ \\ \text{ef} \\ 49+ \\ 45- \\ 45- \\ 60+ \\ \text{ef} \\ 41+ \end{matrix}$$

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num. req. 12

Y

$$\begin{bmatrix} 1&1&0&1&0&1&0 \\ 1&1&0&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 59- \\ 39+ \\ 40+ \\ 40+ \\ 39+ \\ \text{ef} \\ 59- \\ \text{ef} \end{matrix}$$

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num. req. 72

Z

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&1&0&0&1&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 53+ \\ 50+ \\ 47- \\ 47- \\ \text{ef} \\ 60- \\ \text{ef} \\ 52+ \end{matrix}$$

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num. req. 24

AA

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 0&1&1&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 44+ \\ 37- \\ 37- \\ 50- \\ \text{ef} \\ 50- \\ 31+ \\ 43+ \end{matrix}$$

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num. req. 24

BB

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} 54+ \\ \text{csc} \\ \text{csc} \\ 54+ \\ \text{ef} \\ 54+ \\ 43- \\ 27+ \end{matrix}$$

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num. req. 36

CC

$$\begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&1&0&0&1&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&1&1&0&1&1 \end{bmatrix} \begin{matrix} \text{ef} \\ 54- \\ 39+ \\ 39+ \\ \text{ef} \\ \text{ef} \\ 52- \\ 41+ \end{matrix}$$

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num. req. 36

DD

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 0&1&0&0&0&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef} \\ 54- \\ 63+ \\ 61- \\ 62+ \\ \text{ef} \\ \text{ef} \\ 39- \end{matrix}$$

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num. req. 144

EE

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0 \\ 0&1&0&0&0&1&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef} \\ 39- \\ 63- \\ 64- \\ 65+ \\ \text{ef} \\ \text{ef} \\ \text{csc} \end{matrix}$$

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num. req. 144

FF

$$\begin{bmatrix} 1&1&1&0&0&1&0 \\ 0&0&0&0&0&1&1 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0 \\ 1&1&0&0&0&0&0 \end{bmatrix} \begin{matrix} \text{ef} \\ 31- \\ 66+ \\ 61+ \\ 61+ \\ \text{ef} \\ \text{ef} \\ 48- \end{matrix}$$

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num. req. 72

GG

$$\begin{bmatrix}1&1&1&0&0&1&0\\0&0&0&0&0&1&1\\0&0&0&0&0&0&0\\1&0&0&0&0&0&1\\0&1&0&0&0&0&1\\0&0&0&1&0&1&0\\0&0&0&0&1&1&0\\1&1&0&0&0&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\48-\\66-\\64+\\64+\\\text{ef}\\\text{ef}\\42-\end{array}$$

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num. req. 72

HH

$$\begin{bmatrix}1&1&1&0&0&1&0\\0&0&0&0&0&1&1\\1&1&0&0&0&1&1\\0&1&0&0&0&1&0\\1&0&0&0&0&1&0\\0&0&0&1&0&1&0\\0&0&0&0&1&1&0\\1&1&0&0&0&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\41+\\41+\\62-\\62-\\\text{ef}\\\text{ef}\\\text{ef}\end{array}$$

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num. req. 72

II

$$\begin{bmatrix}1&1&1&0&0&1&0\\1&1&0&1&1&0&0\\1&1&0&0&0&1&1\\0&1&0&0&0&1&0\\0&1&0&0&0&0&1\\1&1&0&0&1&1&0\\0&0&0&0&1&1&0\\1&1&0&0&0&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\\text{ef}\\68+\\40-\\67+\\55-\\\text{ef}\\40-\end{array}$$

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JJ

$$\begin{bmatrix}1&0&0&1&0&1&1\\1&0&1&1&0&1&1\\1&0&1&1&1&0&0\\1&0&0&0&0&0&1\\1&0&1&1&0&0&1\\0&0&0&1&0&1&0\\1&0&0&1&0&0&0\\1&1&1&1&0&1&1\end{bmatrix}\begin{array}{c}68-\\68-\\\text{ef}\\\text{ef}\\68-\\\text{ef}\\68-\\\text{ef}\end{array}$$

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num. req. 36

KK

$$\begin{bmatrix}0&0&0&1&1&1&1\\0&0&1&1&1&1&0\\1&0&1&1&1&0&0\\0&1&1&1&1&0&0\\1&1&1&1&1&1&0\\0&0&0&1&0&1&0\\0&0&0&0&1&1&0\\0&0&0&1&1&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\58-\\65+\\65+\\41-\\\text{ef}\\\text{ef}\\27-\end{array}$$

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num. req. 48

LL

$$\begin{bmatrix}1&1&1&0&0&1&0\\1&1&0&1&1&0&0\\1&1&0&0&0&1&1\\0&1&0&0&0&1&0\\1&0&0&0&0&1&0\\0&0&0&1&0&1&0\\0&0&0&0&1&1&0\\1&1&0&0&0&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\41-\\\text{ef}\\56-\\56-\\69+\\69+\\42+\end{array}$$

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num. req. 36

MM

$$\begin{bmatrix}1&1&1&0&0&1&0\\1&1&0&1&1&0&0\\1&1&0&0&0&1&1\\0&1&0&0&0&1&0\\1&0&0&0&0&1&0\\1&1&0&0&1&1&0\\0&0&0&0&1&1&0\\1&1&0&0&0&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\\text{ef}\\\text{ef}\\67-\\67-\\69-\\70+\\\text{csc}\end{array}$$

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num. req. 72

NN

$$\begin{bmatrix}0&0&0&1&1&1&1\\0&0&1&1&1&1&0\\0&1&0&1&1&1&0\\0&1&1&1&1&0&0\\1&1&1&1&1&1&0\\0&0&0&1&0&1&0\\0&0&0&0&1&1&0\\0&0&0&1&1&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\65-\\65-\\70+\\\text{ef}\\\text{ef}\\\text{ef}\\41-\end{array}$$

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num. req. 120

OO

$$\begin{bmatrix}1&1&1&0&0&1&0\\1&1&0&1&1&0&0\\1&1&0&0&0&1&1\\0&1&0&0&0&1&0\\1&0&0&0&0&1&0\\1&1&0&0&1&1&0\\1&1&0&1&0&1&0\\1&1&0&0&0&0&0\end{bmatrix}\begin{array}{c}\text{ef}\\71+\\\text{ef}\\\text{ef}\\\text{ef}\\70-\\70-\\70-\end{array}$$

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num. req. 64

PP

Table 5.4 The Configuration Classes for the Corner-Slicing triangulation of I^7

We turn to an explanation of how $\mathcal{T}$ can be formed. First we describe the formation of a core complex of 4 class-A simplices, 8 class-B simplices, and 4 class-C simplices. One easily shows that, for any class-A simplex, across any facet generated by a class-1 base (a class-1 facet, for short), there is a unique adjacent class-B simplex and vice versa. Similarly for any class-B simplex, across any class-3 or -4 facet, there is a unique class-C simplex and vice versa. We start with $\mathcal{C} = \{S_A\}$ where S_A is the class-A simplex represented by the first matrix of Table 5.4. Then we enlarge $\mathcal{C}$ by adjoining adjacent simplices from classes A, B, and C as long as possible. The final $\mathcal{C}$ is the desired core complex.

In order to save space in the description to follow, we introduce some notation. For example, $A\ (2) \Longrightarrow 16\ D$ means that across every class-2 facet of each class-A simplex there is exactly one adjacent class-D simplex and using the class-A simplices in $\mathcal{T}$ we obtain 16 class-D simplices for $\mathcal{T}$ in this manner. In some cases (e.g. the creation of the class-L simplices from the class-D simplices) the same simplex is generated by more than one facet.

The simplices in classes D–L are created as follows:

$A\ (2) \Longrightarrow 16\ D$	$B\ (2) \Longrightarrow 8\ D$	$B\ (6) \Longrightarrow 8\ E$	$C\ (6) \Longrightarrow 4\ E$
$B\ (7) \Longrightarrow 8\ F$	$C\ (7) \Longrightarrow 4\ F$	$C\ (8) \Longrightarrow 4\ G$	$B\ (5) \Longrightarrow 8\ H$
$C\ (9) \Longrightarrow 4\ I$	$D\ (15) \Longrightarrow 24\ J$	$E\ (15) \Longrightarrow 12\ J$	$E\ (17) \Longrightarrow 24\ K$
$D\ (12) \Longrightarrow 24\ L$	$J\ (32) \Longrightarrow 72\ M$		

In the last case, easy identification of the class-M simplices may not be clear, and we state a pivot rule which the reader can verify. (The reader constructing $\mathcal{T}$ would likely invent similar rules for most of the rest of the classes to be constructed.)

Suppose a matrix M_J reprsents a class-J simplex S_J and for S_J a class-32 facet, F, is opposite the vertex represented by the i^{th} row of J. Let r be the row vector obtained from the i^{th} row of M_J as follows:

1) For any column where the sum for M_J is not 1, 4, or 7, the entry for r is the complement of the entry for the i^{th} row of M_J.
2) For any column where the sum for M_J is 1 (respectively 7), use 0 (1) as the entry for r.
3) For any column where the sum for M_J is 4, let r have the same entry as in the i^{th} row of M_J.

Then replacing the i^{th} row of M_J by r yields a matrix M_M which represents the class-M simplex adjacent to S_J across F.

We continue with the definitions of the simplices in $\mathcal{T}$.

$F\ (20) \Longrightarrow 12\ N$	$K\ (35) \Longrightarrow 12\ O$	$F\ (18) \Longrightarrow 12\ P$	$D\ (11) \Longrightarrow 24\ Q$
$F\ (11) \Longrightarrow 12\ Q$	$D\ (10) \Longrightarrow 24\ R$	$K\ (36) \Longrightarrow 12\ S$	$E\ (16) \Longrightarrow 12\ T$
$T\ (51) \Longrightarrow 12\ U$	$J\ (29) \Longrightarrow 72\ V$	$I\ (27) \Longrightarrow 4\ W$	$U\ (27) \Longrightarrow 12\ W$

K (34)$\Longrightarrow$ 24 X $\quad$ L (34)$\Longrightarrow$ 48 X $\quad$ S (49)$\Longrightarrow$ 12 Y $\quad$ X (59)$\Longrightarrow$ 72 Z
U (53)$\Longrightarrow$ 24 AA $\quad$ L (37)$\Longrightarrow$ 24 BB $\quad$ O (43)$\Longrightarrow$ 12 CC $\quad$ BB (43)$\Longrightarrow$ 24 CC
U (52)$\Longrightarrow$ 12 DD $\quad$ AA (52)$\Longrightarrow$ 24 DD

The formation of the class-EE simplices for $\mathcal{T}$ is more complicated. The 36 class-CC simplices will generate 72 class-EE simplices and 72 more come from the 72 class-V simplices. For each class-CC simplex in $\mathcal{T}$, across each of the two class-54 facets not shared with a class-DD simplex of $\mathcal{T}$, there are two class-EE simplices and we need to select one of these for $\mathcal{T}$. With some effort these 72 selections can be made systematically so that the resulting set $\mathcal{S}$ of 72 class-EE simplices has the following two properties:

1) There is a one-to-one correspondence between $\mathcal{S}$ and the set of class-Z simplices in $\mathcal{T}$ such that corresponding simplices are adjacent.
2) Each simplex in $\mathcal{S}$ uniquely determines an adjacent class-GG simplex across a class-61 facet and each of these class-GG simplices is adjacent to two simplices in $\mathcal{S}$.

There are four ways to accomplish this. We arbitrarily take the one which produces the simplices represented by the matrices

$$\begin{bmatrix} 0&1&1&1&0&1&0\\ 0&1&1&0&0&1&1\\ 1&1&0&0&0&1&1\\ 0&1&0&1&1&1&0\\ 0&1&0&0&0&0&1\\ 0&0&0&1&0&1&0\\ 0&1&0&1&0&1&1\\ 1&1&1&1&0&1&1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0&0&0&1&0&0&1\\ 1&0&0&1&0&0&0\\ 1&0&1&1&1&0&0\\ 1&0&0&0&0&0&1\\ 1&0&1&1&0&0&1\\ 1&1&1&1&1&0&1\\ 1&1&1&1&0&0&0\\ 1&1&1&1&0&1&1 \end{bmatrix}.$$

(The other three joint selections may also work but we only checked out our selection.) Similarly there are two class-EE simplices across the single class-54 facet of each of our 72 class-V simplices. Exactly one will be adjacent to one of our class-DD simplices and this is the one we take for $\mathcal{T}$. We get 72 class-EE simplices in this way for a total of 144 class-EE simplices.

We continue as before.

EE (63)$\Longrightarrow$ 144 FF $\quad$ EE (62)$\Longrightarrow$ 72 II $\quad$ EE (61)$\Longrightarrow$ 72 GG $\quad$ GG (66)$\Longrightarrow$ 72 HH
V (56)$\Longrightarrow$ 36 MM $\quad$ V (55)$\Longrightarrow$ 144 JJ $\quad$ JJ (68)$\Longrightarrow$ 36 KK $\quad$ W (58)$\Longrightarrow$ 48 LL
FF (65)$\Longrightarrow$ 72 OO $\quad$ LL (65)$\Longrightarrow$ 48 OO $\quad$ MM (69)$\Longrightarrow$ 72 NN $\quad$ OO (70)$\Longrightarrow$ 64 PP

In this last case, each of 28 of the 64 class-PP simplices is produced three times and each of the remainder is produced twice.

Finally the 64 class-PP simplices, across their class-71 facets, generate the corners at the 64 vertices of I^7 with odd coordinate sums.

One can use Theorem 2.3 of [11] along with Table 5.4 to verify the collection of 1820 simplices we have described is indeed a triangulation.

$\square$

We were not able to generate $\mathcal{T}$ with any of the general methods described in §2. The triangulation $\mathcal{T}$ induces triangulations of cardinality 339 on 12 facets of I^7 and triangulations of cardinality 334 on the other two facets. The value $d = 7$ is the smallest for which some triangulation of minimum cardinality among all corner-slicing triangulations of I^d fails to induce triangulations with the same property on all the facets of I^d.

Higher dimensions

From the main theorem of Haiman's paper [9], the values of $T(d)$ for $d \leq 7$ yield upper bounds on $T(d)$ for $d > 7$, e.g. $T(8) \leq 11944$.

6. Remarks

For a triangulation of I^d with $T(d)$ simplices, the number $\rho = (T(d)/d!)^{1/d}$ has been proposed by Todd [20] as a measure of the efficiency of the triangulation for simplicial algorithms. Haiman [9] showed that if a particular value of ρ is achievable in dimension d, then it is also achievable in dimension kd for all positive integers k. The minimum cardinality triangulations of I^7 yield $(1493/7!)^{1/7} \approx 0.840463$. This is the smallest value of ρ for any triangulations of the cube published to date.

A nice feature of our linear programming approach is that the constraints are strong enough that we obtain many integer-valued optimal solutions. This is essential in order that the optimal solutions help in creating minimum-cardinality triangulations. Without our notion of non-selfcomplementing bases and the corresponding constraints, this would not have been the case, e.g., the optimal objective value for our problem for $T(5)$ would have been approximately 66.6667.

Our guess is that the minimum of the cardinalities of all decompositions of I^6 is far closer to $T(6) = 308$ than to our lower bound of 270.

Without drastic modification, our techniques have no chance of establishing good lower bounds on $T(d)$ for $d \geq 8$. However we think it is reasonable that our linear programming approach can help find minimum-cardinality triangulations of some polytopes such as

$H(6,2) \cap I^6$, $H(7,3) \cap I^7$, and, perhaps, $H(8,4) \cap I^8$. Then, using the middle-cut ideas, this would lead to decompositions and better upper bounds on $S(d)$ for $d \geq 8$. Perhaps some technique could be devised to create triangulations instead of just decompositions.

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Helpful assistance, related to the solution of large linear programming problems, is acknowledged from Dick Curtis at the Idaho National Engineering Laboratory, Irv Lustig, and Robert Bixby. Also thanks are given to Victor Klee for suggesting this area of research and for continued help and encouragement.

APPENDIX

DATA FOR DIMENSION 5

Canonical representatives of base classes in dimension 5

To conserve space, vertices of I^5, viewed as binary numbers, have been converted to octal.

Base classes with representatives on $x_5 = 1$

Base	Vertices in octal	Base	Vertices in octal	Base	Vertices in octal
1	37 35 33 27 17	2	37 35 33 27 15	3	37 35 33 27 11
4	37 35 33 27 01	5	37 35 33 25 03	6	37 35 33 25 13
7	37 35 33 25 11	8	37 35 33 21 07	9	37 35 33 21 05
10	37 35 33 21 11	11	37 35 33 21 01	12	37 35 23 13 07
13	37 35 23 13 05	14	37 35 23 13 01	15	37 35 23 11 03
16	37 31 25 23 15	17	37 31 25 15 03		

Base classes with representatives on $x_4 + x_5 = 1$

Base	Vertices in octal	Base	Vertices in octal	Base	Vertices in octal
18	36 35 32 26 16	19	36 35 32 26 15	20	36 35 32 26 12
21	36 35 32 26 11	22	36 35 32 26 02	23	36 35 32 26 01
24	36 35 32 25 12	25	36 35 32 25 11	26	36 35 32 25 02
27	36 35 32 22 12	28	36 35 32 22 11	29	36 35 32 22 06
30	36 35 32 22 05	31	36 35 32 22 02	32	36 35 32 22 01
33	36 35 32 21 11	34	36 35 32 21 06	35	36 35 32 21 05
36	36 35 32 21 02	37	36 35 32 21 01	38	36 35 22 12 06
39	36 35 22 12 05	40	36 35 22 12 02	41	36 35 22 12 01
42	36 35 22 11 02	43	36 32 26 21 16	44	36 32 26 21 15
45	36 32 26 21 12	46	36 32 26 21 11	47	36 32 26 21 02
48	36 32 26 21 01	49	36 32 26 16 01	50	36 32 26 15 02
51	36 32 26 12 05	52	36 32 26 11 02	53	36 32 25 15 06
54	36 32 25 15 02	55	36 32 25 15 01	56	36 32 25 11 06
57	36 32 25 11 05	58	36 32 25 06 01	59	36 31 25 22 15
60	36 31 25 15 02	61	36 31 25 15 01		

Base classes with representatives on $x_3 + x_4 + x_5 = 1$

Base	Vertices in octal	Base	Vertices in octal	Base	Vertices in octal
62	34 32 31 24 14	63	34 32 31 24 12	64	34 32 31 24 04
65	34 32 31 24 02	66	34 32 24 21 14	67	34 32 24 21 12
68	34 32 24 21 11	69	34 32 24 14 01	70	34 32 24 12 01
71	34 32 24 11 04	72	34 32 24 11 02	73	34 32 24 04 01
74	34 32 24 02 01	75	34 32 21 11 04	76	34 32 21 11 01

Base classes with representatives on $x_2 + x_3 + x_4 + x_5 = 1$

Base	Vertices in octal	Base	Vertices in octal
77	30 24 22 21 10	78	30 24 22 10 01

Base classes with representatives on $x_2 + x_3 + x_4 + x_5 = 2$

Base	Vertices in octal	Base	Vertices in octal	Base	Vertices in octal
79	34 32 31 26 14	80	34 32 31 26 11	81	34 32 31 26 05
82	34 32 31 14 06	83	34 32 31 14 03	84	34 32 25 14 11
85	34 32 25 12 11	86	34 32 25 11 06	87	34 32 25 11 03

Base classes with representatives on $x_2 + x_3 + x_4 + 2x_5 = 2$

Base	Vertices in octal	Base	Vertices in octal	Base	Vertices in octal
88	34 32 26 21 14	89	34 32 26 21 01	90	34 32 26 14 01
91	34 32 21 14 06	92	34 32 21 06 01		

Base classes with representatives on $\sum_{i=1}^{5} x_i = 1$

Base	Vertices in octal
93	20 10 04 02 01

Base classes with representatives on $\sum_{i=1}^{5} x_i = 2$

Base	Vertices in octal	Base	Vertices in octal	Base	Vertices in octal
94	30 24 22 21 14	95	30 24 22 14 11	96	30 24 22 14 03
97	30 24 12 05 03				

Base classes with representatives on $\sum_{i=1}^{4} x_i + 2x_5 = 2$

Base	Vertices in octal
98	30 24 22 14 01

Base classes with representatives on $\sum_{i=1}^{4} x_i + 2x_5 = 3$

Base	Vertices in octal	Base	Vertices in octal
99	34 32 26 21 16	100	34 32 26 21 11

Base classes with representatives on $\sum_{i=1}^{4} x_i + 3x_5 = 3$

Base	Vertices in octal
101	34 32 26 16 01

Base classes with representatives on $x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 2$

Base	Vertices in octal
102	30 24 14 02 01

Base classes with representatives on $x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$

Base	Vertices in octal	Base	Vertices in octal
103	34 22 21 12 06	104	34 22 21 12 05

Base classes with representatives on $x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 3$

Base	Vertices in octal
105	34 22 12 06 01

Base classes with representatives on $x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 = 4$

Base	Vertices in octal
106	34 32 21 11 06

Linear programming variables in dimension 5

Each row of the following table splits into 7 groups:

(a) The variable number.

(b) The base classes for any representative simplex.

(c) The barycentric coordinates of c relative to the vertices of a representative simplex.

(d) The side indicators.

(e) The exterior-facet tuple.

(f) The volume times 5!.

(g) A vertex of I^5 which, with the canonical base from the largest numbered base class, gives the vertices of a representative simplex.

var	base classes	1/2-space ind.	side ind.	eft	vol	vert
1	6 6 24 24 31 31	1 1 0 0 0 0	0 0 0 0 0 0	2 4 8	1	11100
2	2 2 2 2 62 62	1 1 1 1 -1 -1	0 0 0 0 0 0	4 12 28	1	11000
3	2 2 6 20 20 63	1 1 1 0 0 -1	0 0 0 0 0 0	3 8 18	1	11000
4	11 24 24 40 48 64	1 0 0 0 0 1	0 0 0 0 0 0	1 1 1	1	11011
5	2 6 6 19 27 64	1 1 1 0 0 -1	0 0 0 0 0 0	3 7 15	1	11000
6	2 5 27 45 65 66	1 1 0 0 -1 1	0 0 0 0 0 0	2 4 9	1	11110
7	6 7 7 63 66 66	1 1 1 1 -1 -1	0 0 0 0 0 0	3 6 10	1	10000
8	6 7 20 29 64 67	1 1 0 0 1 -1	0 0 0 0 0 0	2 4 7	1	11000
9	5 24 37 48 51 67	1 0 0 0 0 1	0 0 0 0 0 0	1 1 2	1	10101
10	7 11 19 45 63 67	1 1 0 0 1 -1	0 0 0 0 0 0	2 3 4	1	10000
11	6 32 32 51 51 70	1 0 0 0 0 1	0 0 0 0 0 0	1 2 4	1	11110
12	26 26 57 57 70 70	0 0 0 0 1 1	0 0 0 0 0 0	0 0 0	1	00101
13	15 25 25 67 67 70	1 0 0 1 1 -1	0 0 0 0 0 0	1 0 0	1	00000
14	6 45 45 70 71 71	1 0 0 -1 1 1	0 0 0 0 0 0	1 2 4	1	11110
15	5 6 20 25 66 71	1 1 0 0 1 -1	0 0 0 0 0 0	2 3 6	1	11000
16	11 25 29 65 70 71	1 0 0 1 -1 1	0 0 0 0 0 0	1 1 1	1	01101
17	5 5 68 68 71 71	1 1 1 1 -1 -1	0 0 0 0 0 0	2 2 4	1	00000
18	6 26 30 31 51 73	1 0 0 0 0 1	0 0 0 0 0 0	1 2 4	1	11110
19	24 32 42 57 70 73	0 0 0 0 1 1	0 0 0 0 0 0	0 0 0	1	11011
20	5 6 28 45 67 73	1 1 0 0 1 -1	0 0 0 0 0 0	2 3 6	1	11000
21	30 37 48 57 71 74	0 0 0 0 1 1	0 0 0 0 0 0	0 0 0	1	00011
22	15 19 28 66 67 74	1 0 0 1 1 -1	0 0 0 0 0 0	1 0 0	1	00000
23	48 48 48 48 76 76	0 0 0 0 1 1	0 0 0 0 0 0	0 0 0	1	11110
24	11 11 19 19 62 76	1 1 0 0 1 -1	0 0 0 0 0 0	2 2 2	1	11000
25	5 28 45 68 74 76	1 0 0 1 -1 1	0 0 0 0 0 0	1 1 2	1	10101
26	10 11 47 63 63 77	1 1 0 -1 -1 1	0 0 0 0 0 0	2 3 3	1	10110

27	1	2	2	2	18	77	1	1	1	1	0	-1	0	0	0	0	0	0	4	13	33	1	10000
28	5	7	47	66	68	78	1	1	0	-1	-1	1	0	0	0	0	0	0	2	3	5	1	11100
29	11	47	67	67	76	78	1	0	-1	-1	1	1	0	0	0	0	0	0	1	1	1	1	10110
30	15	63	68	68	78	78	1	-1	-1	-1	1	1	0	0	0	0	0	0	1	0	0	1	10101
31	2	7	7	18	62	78	1	1	1	0	1	-1	0	0	0	0	0	0	3	7	13	1	10000
32	5	5	33	64	76	78	1	1	0	-1	-1	1	0	0	0	0	0	0	2	2	4	1	01001
33	15	33	65	67	68	78	1	0	-1	-1	1	1	0	0	0	0	0	0	1	0	0	1	00101
34	7	7	10	62	77	78	1	1	1	-1	1	-1	0	0	0	0	0	0	3	6	8	1	00000
35	2	3	7	20	66	79	1	1	1	0	-1	0	0	0	0	0	0	0	3	7	14	1	11000
36	7	9	20	62	68	79	1	1	0	1	-1	0	0	0	0	0	0	0	2	3	4	1	10100
37	9	11	25	63	71	79	1	1	0	1	-1	0	0	0	0	0	0	0	2	2	2	1	10010
38	3	11	24	24	43	80	1	1	0	0	0	0	0	0	0	0	0	1	2	3	5	1	11110
39	2	3	22	31	31	80	1	1	0	0	0	0	0	0	0	0	0	-1	2	5	11	1	11000
40	9	24	37	52	63	80	1	0	0	0	1	0	0	0	0	0	0	-1	1	1	1	1	10100
41	16	61	77	80	80	80	1	0	1	0	0	0	0	0	0	-1	-1	-1	1	1	0	2	10000
42	3	32	43	52	71	81	1	0	0	0	1	0	0	0	0	0	0	0	1	2	4	1	11000
43	7	7	22	22	82	82	1	1	0	0	0	0	0	0	0	0	1	1	2	4	6	1	11110
44	11	26	46	52	67	82	1	0	0	0	1	0	0	0	0	0	0	1	1	1	1	1	11011
45	2	5	21	31	43	82	1	1	0	0	0	0	0	0	0	0	0	-1	2	4	9	1	11000
46	15	36	36	63	82	82	1	0	0	1	0	0	0	0	0	0	-1	-1	1	0	0	1	10010
47	7	9	21	31	80	82	1	1	0	0	0	0	0	0	0	0	1	1	2	3	4	1	01110
48	9	22	36	66	81	82	1	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	01101
49	5	31	35	46	64	82	1	0	0	0	1	0	0	0	0	0	0	-1	1	1	2	1	00100
50	11	22	26	26	69	83	1	0	0	0	1	0	0	0	0	0	0	1	1	1	1	1	11011
51	2	43	51	51	69	83	1	0	0	0	1	0	0	0	0	0	0	-1	1	3	7	1	11000
52	9	32	52	72	81	83	1	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	01110
53	35	37	57	65	72	83	0	0	0	1	1	0	0	0	0	0	0	-1	0	0	0	1	00010
54	5	20	66	71	72	84	1	0	1	1	-1	0	0	0	0	0	0	0	1	1	2	1	10100
55	15	25	68	68	72	84	1	0	1	1	-1	0	0	0	0	0	0	0	1	0	0	1	10001
56	5	7	29	65	69	84	1	1	0	1	-1	0	0	0	0	0	0	0	2	3	5	1	01000
57	5	31	32	36	71	85	1	0	0	0	1	0	0	0	0	0	0	1	1	1	2	1	11110
58	15	21	24	37	66	85	1	0	0	0	1	0	0	0	0	0	0	1	1	0	0	1	11101
59	7	31	52	66	82	85	1	0	0	1	0	0	0	0	0	0	1	-1	1	2	3	1	11000

60	24	46	57	67	71	85	0	0	0	1	1	0	0	0	0	0	0	-1	0	0	0	1	10100
61	37	37	68	68	85	85	0	0	1	1	0	0	0	0	0	0	-1	-1	0	0	0	1	10001
62	15	26	35	65	82	85	1	0	0	1	0	0	0	0	0	0	-1	1	1	0	0	1	01101
63	5	30	46	51	74	85	1	0	0	0	1	0	0	0	0	0	0	1	1	1	2	1	01011
64	7	21	40	51	65	85	1	0	0	0	1	0	0	0	0	0	0	-1	1	2	3	1	01000
65	26	36	42	67	74	85	0	0	0	1	1	0	0	0	0	0	0	-1	0	0	0	1	00001
66	59	75	78	83	85	85	0	1	1	0	0	0	0	0	0	-1	-1	-1	0	0	0	2	00000
67	75	75	75	75	86	86	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	4	00000
68	21	32	66	72	85	87	0	0	1	1	0	0	0	0	0	0	-1	0	0	0	0	1	11000
69	36	36	72	72	87	87	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	10010
70	21	42	46	69	74	87	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	00001
71	2	4	45	45	69	88	1	1	0	0	-1	1	0	0	0	0	0	0	2	4	10	1	11110
72	13	62	66	66	75	88	1	1	1	1	-1	1	0	0	0	0	0	0	1	0	0	2	11101
73	14	25	63	72	79	88	1	0	1	-1	0	1	0	0	0	0	0	0	1	0	0	1	11001
74	7	8	20	66	84	88	1	1	0	1	0	-1	0	0	0	0	0	0	2	3	4	1	11000
75	14	37	37	80	83	88	1	0	0	0	0	1	0	0	0	1	1	0	1	0	0	1	10011
76	8	11	67	67	69	88	1	1	1	1	-1	-1	0	0	0	0	0	0	2	2	2	1	10010
77	12	12	63	63	88	88	1	1	1	1	-1	-1	0	0	0	0	0	0	2	2	0	2	10000
78	4	48	48	48	49	89	1	0	0	0	0	1	0	0	0	0	0	0	1	1	3	1	11110
79	8	19	50	67	67	89	1	0	0	1	1	-1	0	0	0	0	0	0	1	1	1	1	11000
80	12	38	64	64	64	89	1	0	1	1	1	-1	0	0	0	0	0	0	1	1	0	2	10000
81	2	23	49	51	51	90	1	0	0	0	0	1	0	0	0	0	0	0	1	3	7	1	11110
82	56	56	63	70	90	90	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11011
83	11	44	50	70	70	90	1	0	0	1	1	-1	0	0	0	0	0	0	1	1	1	1	10010
84	39	54	64	73	73	90	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	01101
85	26	55	58	67	82	90	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	1	01001
86	13	53	67	67	89	90	1	0	1	1	1	-1	0	0	0	0	0	0	1	0	0	2	00000
87	7	45	50	69	90	91	1	0	0	1	-1	1	0	0	0	0	0	0	1	2	3	1	11110
88	54	54	67	67	91	91	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11011
89	26	58	63	82	83	91	0	0	1	0	0	1	0	0	0	-1	-1	0	0	0	0	1	11001
90	5	20	34	69	73	91	1	0	0	1	1	-1	0	0	0	0	0	0	1	1	2	1	11000
91	37	57	58	72	85	91	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	1	10011
92	15	34	67	74	84	91	1	0	1	1	0	-1	0	0	0	0	0	0	1	0	0	1	10010

93	13	53	63	67	88	91	1	0	1	1	1	-1	0	0	0	0	0	0	1	0	0	2	10000
94	7	23	51	52	83	91	1	0	0	0	0	1	0	0	0	0	1	0	1	2	3	1	01110
95	56	66	69	71	86	91	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	2	01101
96	36	41	65	82	87	91	0	0	1	0	0	1	0	0	0	-1	0	0	0	0	0	1	00101
97	5	44	45	71	72	91	1	0	0	1	1	-1	0	0	0	0	0	0	1	1	2	1	00100
98	15	70	72	72	91	91	1	1	1	1	-1	-1	0	0	0	0	0	0	1	0	0	1	00010
99	23	48	57	57	69	92	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	11110
100	56	56	67	70	91	92	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11101
101	32	32	41	64	83	92	0	0	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	11001
102	19	44	69	70	70	92	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	11000
103	25	67	72	84	91	92	0	1	1	0	1	-1	0	0	0	0	0	0	0	0	0	1	10100
104	55	57	57	83	90	92	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	1	01110
105	54	54	70	70	92	92	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	00111
106	32	46	58	71	87	92	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	00101
107	25	34	70	72	74	92	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	00100
108	53	71	71	75	76	92	0	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	2	00000
109	10	10	10	77	77	93	1	1	1	-1	-1	1	0	0	0	0	0	0	3	6	6	1	11000
110	1	1	1	1	1	93	1	1	1	1	1	-1	0	0	0	0	0	0	5	20	60	1	00000
111	3	3	10	64	64	94	1	1	1	-1	-1	1	0	0	0	0	0	0	3	6	10	1	11100
112	9	10	29	65	79	94	1	1	0	-1	0	1	0	0	0	0	0	0	2	3	3	1	11010
113	10	40	40	80	80	94	1	0	0	0	0	1	0	0	0	-1	-1	0	1	2	2	1	10011
114	1	3	3	27	27	94	1	1	1	0	0	-1	0	0	0	0	0	0	3	8	20	1	10000
115	9	9	33	64	77	94	1	1	0	-1	1	-1	0	0	0	0	0	0	2	2	2	1	01000
116	16	16	93	94	94	94	1	1	1	-1	-1	-1	0	0	0	0	0	0	2	2	0	2	00000
117	3	3	29	29	73	95	1	1	0	0	-1	1	0	0	0	0	0	0	2	4	8	1	11100
118	9	29	64	74	79	95	1	0	1	-1	0	1	0	0	0	0	0	0	1	1	1	1	11010
119	9	30	35	40	80	95	1	0	0	0	0	1	0	0	0	0	1	0	1	1	1	1	10110
120	35	42	65	80	81	95	0	0	1	0	0	1	0	0	0	-1	0	0	0	0	0	1	10011
121	3	9	27	28	64	95	1	1	0	0	1	-1	0	0	0	0	0	0	2	3	5	1	10000
122	9	9	65	65	73	95	1	1	1	1	-1	-1	0	0	0	0	0	0	2	2	2	1	00100
123	16	16	94	94	95	95	1	1	1	1	-1	-1	0	0	0	0	0	0	2	2	0	2	00000
124	3	30	30	40	49	96	1	0	0	0	0	1	0	0	0	0	0	0	1	2	4	1	11100
125	39	39	95	95	96	96	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11011

126	9	29	34	73	90	96	1	0	0	1	-1	1	0	0	0	0	0	0	1	1	1	1	11010
127	30	30	55	64	80	96	0	0	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	10011
128	3	27	50	73	73	96	1	0	0	1	1	-1	0	0	0	0	0	0	1	2	4	1	10000
129	41	42	42	80	90	96	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	1	01101
130	35	42	55	74	81	96	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	01011
131	9	28	50	65	74	96	1	0	0	1	1	-1	0	0	0	0	0	0	1	1	1	1	01000
132	14	28	28	64	89	96	1	0	0	1	1	-1	0	0	0	0	0	0	1	0	0	1	00010
133	16	38	94	95	95	96	1	0	1	1	1	-1	0	0	0	0	0	0	1	1	0	2	00000
134	39	39	96	96	97	97	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11110
135	30	30	42	42	73	97	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	11100
136	28	28	73	74	74	97	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	10000
137	95	95	95	95	95	97	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	2	00000
138	3	4	47	71	71	98	1	1	0	-1	-1	1	0	0	0	0	0	0	2	3	7	1	11100
139	9	47	71	72	88	98	1	0	1	-1	-1	1	0	0	0	0	0	0	1	1	1	1	11010
140	13	75	78	79	84	98	1	-1	1	0	0	1	0	0	0	0	0	0	1	0	0	2	11001
141	14	65	65	77	94	98	1	-1	-1	1	-1	1	0	0	0	0	0	0	1	0	0	1	10001
142	3	8	18	66	66	98	1	1	0	1	1	-1	0	0	0	0	0	0	2	3	5	1	10000
143	13	59	80	85	85	98	1	0	0	0	0	1	0	0	1	1	1	0	1	0	0	2	01101
144	14	33	65	74	95	98	1	0	1	-1	-1	1	0	0	0	0	0	0	1	0	0	1	01001
145	8	9	66	68	78	98	1	1	-1	1	1	-1	0	0	0	0	0	0	2	2	2	1	01000
146	33	74	74	89	96	98	0	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	1	00011
147	12	16	77	79	79	98	1	1	1	0	0	-1	0	0	0	0	0	0	2	2	0	2	00000
148	17	93	99	99	99	99	1	1	0	0	0	0	0	0	1	1	1	1	1	0	0	3	11111
149	1	4	43	43	43	99	1	1	0	0	0	0	0	0	0	0	0	1	2	5	15	1	11110
150	13	59	77	82	82	99	1	0	1	0	0	0	0	0	0	-1	-1	1	1	0	0	2	11101
151	14	35	35	80	94	99	1	0	0	0	1	0	0	0	0	1	0	1	1	0	0	1	11001
152	8	10	22	82	82	99	1	1	0	0	0	0	0	0	0	-1	-1	-1	2	3	3	1	11000
153	12	16	80	80	80	99	1	1	0	0	0	0	0	0	1	1	1	-1	2	2	0	2	10000
154	10	52	52	83	88	99	1	0	0	0	1	0	0	0	0	-1	0	-1	1	2	2	1	01100
155	16	61	81	81	98	99	1	0	0	0	1	0	0	0	0	0	0	-1	1	1	0	2	01000
156	4	43	46	46	76	100	1	0	0	0	1	0	0	0	0	0	0	1	1	1	3	1	11110
157	13	59	78	82	82	100	1	0	1	0	0	0	0	0	0	-1	1	1	1	0	0	2	11101
158	14	21	21	62	80	100	1	0	0	1	0	0	0	0	0	0	-1	1	1	0	0	1	11001

159	8	22	62	82	82	100	1	0	1	0	0	0	0	0	0	-1	-1	-1	1	1	1	1	11000
160	13	75	83	85	85	100	1	1	0	0	0	0	0	0	1	1	1	1	1	0	0	2	10111
161	14	36	68	81	82	100	1	0	1	0	0	0	0	0	0	0	1	1	1	0	0	1	10101
162	8	21	52	68	85	100	1	0	0	1	0	0	0	0	0	0	1	-1	1	1	1	1	10100
163	12	61	78	82	82	100	1	0	1	0	0	0	0	0	0	1	1	-1	1	1	0	2	10000
164	36	68	82	87	88	100	0	1	0	0	1	0	0	0	1	0	0	-1	0	0	0	1	01100
165	78	80	85	85	98	100	1	0	0	0	1	0	0	-1	-1	-1	0	-1	0	0	0	2	01000
166	59	75	81	87	98	100	0	1	0	0	1	0	0	0	0	0	0	-1	0	0	0	2	00100
167	46	46	55	76	89	100	0	0	0	1	1	0	0	0	0	0	0	-1	0	0	0	1	00001
168	93	101	101	101	101	101	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	4	11111
169	1	49	49	49	49	101	1	0	0	0	0	1	0	0	0	0	0	0	1	4	12	1	11110
170	60	77	90	90	90	101	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	3	11101
171	39	39	94	96	96	101	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11001
172	10	50	50	90	90	101	1	0	0	1	1	-1	0	0	0	0	0	0	1	2	2	1	11000
173	55	55	55	89	99	101	0	0	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	10001
174	16	38	96	96	96	101	1	0	1	1	1	-1	0	0	0	0	0	0	1	1	0	2	10000
175	17	89	89	89	89	101	1	1	1	1	1	-1	0	0	0	0	0	0	1	0	0	3	00000
176	17	99	100	100	100	102	1	0	0	0	0	1	0	-1	1	1	1	0	1	0	0	3	11110
177	4	4	76	76	76	102	1	1	-1	-1	-1	1	0	0	0	0	0	0	2	2	6	1	11100
178	13	68	68	75	88	102	1	1	1	-1	-1	1	0	0	0	0	0	0	1	0	0	2	11010
179	100	100	100	100	102	102	0	0	0	0	1	1	-1	-1	-1	-1	0	0	0	0	0	3	10011
180	14	68	68	78	98	102	1	-1	-1	1	-1	1	0	0	0	0	0	0	1	0	0	1	10010
181	8	8	62	78	78	102	1	1	-1	1	1	-1	0	0	0	0	0	0	2	2	2	1	10000
182	76	76	76	89	89	102	1	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	2	00011
183	12	12	62	62	62	102	1	1	1	1	1	-1	0	0	0	0	0	0	2	2	0	2	00000
184	12	38	65	65	90	103	1	0	1	1	-1	1	0	0	0	0	0	0	1	1	0	2	11110
185	39	54	74	74	89	103	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11101
186	8	29	66	84	91	103	1	0	1	0	-1	1	0	0	0	0	0	0	1	1	1	1	11010
187	35	58	68	85	100	103	0	0	1	0	0	1	0	0	0	-1	1	0	0	0	0	1	11001
188	14	34	65	79	95	103	1	0	1	0	1	-1	0	0	0	0	0	0	1	0	0	1	11000
189	64	71	71	88	92	103	1	1	1	1	-1	1	0	0	0	0	0	0	0	0	0	2	10011
190	13	53	64	66	66	103	1	0	1	1	1	-1	0	0	0	0	0	0	1	0	0	2	10000
191	8	23	40	85	85	103	1	0	0	0	0	1	0	0	0	1	1	0	1	1	1	1	01110

192	14	74	74	90	96	103	1	1	1	-1	1	-1	0	0	0	0	0	0	1	0	0	1	01100
193	56	65	72	86	88	103	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	2	01011
194	13	65	68	75	91	103	1	1	1	1	-1	-1	0	0	0	0	0	0	1	0	0	2	01000
195	41	81	81	94	99	103	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	1	00011
196	4	27	44	71	71	103	1	0	0	1	1	-1	0	0	0	0	0	0	1	1	3	1	00010
197	17	94	98	98	103	103	1	1	1	1	-1	-1	0	0	0	0	0	0	1	0	0	3	00000
198	39	54	65	74	90	104	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	11110
199	60	96	98	103	104	104	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	3	11011
200	23	30	58	66	85	104	0	0	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	11010
201	34	65	79	90	96	104	0	1	0	1	1	-1	0	0	0	0	0	0	0	0	0	1	11000
202	56	56	71	71	73	104	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	10011
203	66	66	73	91	91	104	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	2	10000
204	97	104	104	104	104	104	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	3	01111
205	42	58	85	87	91	104	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	1	01110
206	34	34	74	74	97	104	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	01100
207	56	72	74	86	92	104	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	2	01011
208	53	68	74	75	91	104	0	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	2	01000
209	41	41	81	81	95	104	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	00011
210	28	44	71	72	92	104	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	00010
211	95	98	98	103	103	104	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	3	00000
212	12	61	83	83	83	105	1	0	0	0	0	1	0	0	1	1	1	0	1	1	0	2	11110
213	60	89	92	92	92	105	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	3	11101
214	86	86	98	98	105	105	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	4	11011
215	8	47	69	91	91	105	1	0	1	-1	-1	1	0	0	0	0	0	0	1	1	1	1	11010
216	59	75	87	87	100	105	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	2	11001
217	14	72	72	88	98	105	1	1	1	-1	1	-1	0	0	0	0	0	0	1	0	0	1	11000
218	33	72	72	92	103	105	0	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	1	10001
219	13	75	78	84	84	105	1	1	1	0	0	-1	0	0	0	0	0	0	1	0	0	2	10000
220	77	83	83	83	99	105	1	0	0	0	0	1	0	-1	-1	-1	1	0	0	0	0	2	00011
221	4	18	69	69	69	105	1	0	1	1	1	-1	0	0	0	0	0	0	1	1	3	1	00010
222	17	77	88	88	88	105	1	1	1	1	1	-1	0	0	0	0	0	0	1	0	0	3	00000
223	102	105	105	106	106	106	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	5	11111
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225	60	78	90	91	91	106	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	3	11101
226	23	23	62	83	83	106	0	0	1	0	0	1	0	0	0	-1	-1	0	0	0	0	1	11001
227	62	69	69	90	90	106	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	2	11000
228	75	75	104	104	106	106	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	4	10111
229	58	58	83	88	100	106	0	0	0	1	0	1	0	0	1	0	-1	0	0	0	0	1	10110
230	56	68	72	86	91	106	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	2	10101
231	34	68	84	91	104	106	0	1	0	1	1	-1	0	0	0	0	0	0	0	0	0	1	10100
232	78	88	91	91	105	106	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	3	10000
233	92	92	92	92	106	106	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	3	00111
234	41	87	87	100	103	106	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	1	00101
235	53	72	72	75	92	106	0	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	2	00100
236	44	44	76	92	92	106	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	00001
237	75	75	102	103	103	106	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	4	00000

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