A TRIANGULATION OF THE 6-CUBE WITH 308 SIMPLICES

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ABSTRACT

We give a triangulation of the 6-cube into 308 simplices; this is the smallest number in any triangulation of I^6 presented so far.

The problem of finding the minimum number of simplices required to triangulate the d-dimensional cube, $I^d = [0,1]^d$, has received considerable attention [1]–[6]. Results in this area are of interest to people developing simplicial algorithms for finding fixed points. We consider triangulations into simplices whose vertices are vertices of the cube itself. For d=2, 3, 4, 5 this minimum is known to be 2, 5, 16, 67 [4]. For $d = 6$ the smallest known triangulations have 324 simplices [1], [5]. It is shown in [4] that among all triangulations of I^6 which slice off corners at alternate vertices, the minimum cardinality is 324. Here we describe a triangulation of I^6 into 308 simplices. This makes 6 the lowest dimension for which no corner slicing triangulation has minimum cardinality.

We will first introduce 12 equivalence classes of simplices which contain all the simplices in our triangulation and will then turn to a description of the triangulation. We call two simplices in I^6 equivalent if the vertices of one can be obtained from the vertices of the other by successive use of the following operations:

1. For some coordinate, for each vertex, if the entry is 0 change it to 1 and vice versa.
2. For some pair of coordinates, for each vertex, interchange the entries.

Equivalent simplices are congruent.

The 12 equivalence classes are the classes represented by the following matrices.

$$A \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 1&1&0&1&0&0\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 1&0&1&0&0&1\\ 0&0&0&0&0&0 \end{bmatrix} \qquad B \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 1&1&0&1&0&0\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix} \qquad C \quad \begin{bmatrix} 1&0&0&1&1&0\\ 0&0&0&1&1&1\\ 1&1&0&1&0&0\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix} \qquad D \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix}$$

$$E \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 0&0&0&0&0&0 \end{bmatrix} \qquad F \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&1&0&0&1&1\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix} \qquad G \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&1&0&0&1&1\\ 0&1&0&1&1&0\\ 0&0&0&0&0&0 \end{bmatrix} \qquad H \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 0&1&1&1&1&1\\ 0&1&0&0&1&1\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix}$$

$$I \quad \begin{bmatrix} 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 0&1&1&1&1&1\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix} \qquad J \quad \begin{bmatrix} 0&0&0&0&1&0\\ 0&0&0&1&1&1\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&1&0&0&1&1\\ 0&1&0&1&1&0\\ 0&0&0&0&0&0 \end{bmatrix} \qquad K \quad \begin{bmatrix} 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&1&0&0&1&1\\ 0&1&0&1&1&0\\ 0&0&0&0&0&0 \end{bmatrix} \qquad L \quad \begin{bmatrix} 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&1&1&0&1&0\\ 1&1&0&0&1&0\\ 0&1&0&0&1&1\\ 0&1&0&1&1&0\\ 0&1&0&0&1&0 \end{bmatrix}$$

In our triangulation some pair of antipodal vertices plays a distinguished role and we have selected 000000 and 111111 for the realization we now describe. We start with the convex hull of

$$\{011001, 000111, 110100, 110010, 001110, 101001, 000000, 111111\}$$

which is triangulated by 2 simplices from class A. Then on each of the 12 facets of this polytope we erect a uniquely determined simplex from class B. Each of these has exactly one facet that has a normal which can become 111222 by taking the absolute values of its coordinates and permuting them. (We say 111222 is the fundamental normal of the facet.) There are 6 uniquely determined simplices from class C which are adjacent to our class B simplices across these 12 facets. (We call two simplices adjacent if they have a common facet.)

In addition each of our class B simplices has exactly two facets with fundamental normal 111224. On these facets we erect 24 simplices of class D which become uniquely determined by agreeing that the new vertices are among 101100, 100011, 011010, 010101.

Each of these class D simplices has exactly one of 000000 and 111111 as a vertex. Replacing 000000 by 111111 and vice versa we obtain 24 class E simplices adjacent to our 24 class D simplices.

Adjacent to each class D simplex across the facet with fundamental normal 111222 is exactly one simplex in class F and we include these 24 simplices in our triangulation.

Swapping 000000 for 111111 and vice versa in the simplices from class F yields 24 simplices in class G for our triangulation.

Each simplex in class E or F has exactly one facet with fundamental normal 111112 and across this facet is exactly one simplex in class H. This gives us 48 simplices in class H.

Similarly we obtain 24 class I simplices from the 24 class D simplices across facets with fundamental normal 111112.

Each simplex from class C or class G has exactly two facets with fundamental normal 111112 from which we similarly obtain 60 uniquely determined simplices in class J.

We take the 30 class L simplices to be the corners at the vertices $(x_1, x_2, \ldots, x_6)$ of I^6 with $\sum x_i \in \{2, 4\}$. These uniquely determine 30 adjacent class K simplices and this completes the definition of the simplices in the triangulation. The adjacency is more fully described by the graph in Figure 1. Here the nodes are the classes of simplices and there is an arc between two nodes if in the triangulation there exist simplices in the corresponding classes which are adjacent. Beside each arc we have put the fundamental normal of the shared facet.

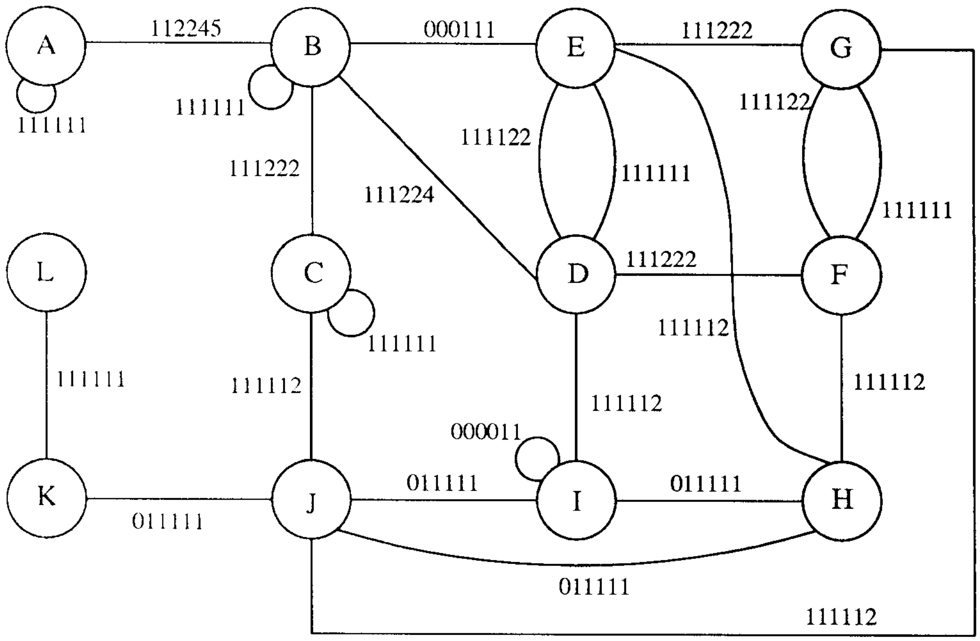

Figure 1 Simplex Adjacency Graph

One can use Theorem 2.3 of [4] to verify that the collection of simplices we have described is a triangulation. In this connection, see Table 1.

Class	A	B	C	D	E	F	G	H	I	J	K	L
Volume	9/6!	6/6!	3/6!	3/6!	3/6!	3/6!	3/6!	2/6!	2/6!	2/6!	1/6!	1/6!
Num. used	2	12	6	24	24	24	24	48	24	60	30	30

Table 1 Data on Classes

We were led to the specific classes A–L and the information in the third row of Table 1 by computer solutions of linear programming problems we developed to find the minimum cardinality of triangulations of I^6. As in [4] any triangulation yields a feasible solution to the linear programming problem and the optimal objective value is a lower bound for the minimum cardinality. We identified 1149 equivalence classes of facets of simplices in I^6. Based on these we identified 9890 classes of simplices. There is a decision variable for each class of simplices. Along with 5 constraints based on volume considerations as in [4], for each of the 1149 facet classes we have an equation balancing the simplices on the two sides of all the facets of the given class. We have a preliminary computer aided proof that no triangulation of I^6 with fewer than 308 simplices is possible. We hope to report on this at some future date.

For a triangulation of I^d with $T(d)$ simplices the number $\rho = (T(d)/d!)^{1/d}$ has been proposed by Todd in [6] as a measure of the efficiency of the triangulation for simplicial algorithms. In [3] Haiman shows that if a particular value of ρ is achievable in dimension d, it is also achievable in dimension kd for all positive integers k. Our triangulation yields $\rho \approx 0.868033$. This improves the smallest value of ρ obtainable from triangulations previously published, $\rho = (13136/8!)^{1/8} \approx 0.869196$ from Böhm's triangulation of I^8 [1].

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