On the Number of “Magic Squares”

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Semi–magic square: square matrix whose entries are nonnegative integers and whose row and column sums are equal

Magic square: semi–magic square whose main diagonals add up to the row–column sum

Symmetric magic square: magic square which is symmetric

Pandiagonal magic square: semi–magic square whose pandiagonals add up to the row–column sum.

$H_n(t) :=$ # semi–magic squares

$M_n(t) :=$ # magic squares

$S_n(t) :=$ # symmetric magic squares

$P_n(t) :=$ # pandiagonal magic squares

for $n \times n$ squares with row–column–(pan)diagonal sum t

Example:

$$H_2(t) = t + 1$$

$$M_2(t) = S_2(t) = P_2(t) = \begin{cases} 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

Theorem (Macmahon, 1915)

$$H_3(t) = \tfrac{1}{8}t^4 + \tfrac{3}{4}t^3 + \tfrac{15}{8}t^2 + \tfrac{9}{4}t + 1$$

$$M_3(t) = \begin{cases} \tfrac{2}{9}t^2 + \tfrac{2}{3}t + 1 & \text{if } 3|t \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Stein–Stein, 1970) $H_n(t)$ is a polynomial in t of degree $(n-1)^2$.

Theorem (Ehrhart, Stanley, 1973)

$$H_n(-n-t) = (-1)^{\deg(H_n)} H_n(t)$$

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$$

Conditions for magic 3×3 squares:

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ &\vdots \\ x_9 &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= t \\ x_4 + x_5 + x_6 &= t \\ x_7 + x_8 + x_9 &= t \\ x_1 + x_4 + x_7 &= t \\ x_2 + x_5 + x_8 &= t \\ x_3 + x_6 + x_9 &= t \\ x_1 + x_5 + x_9 &= t \\ x_3 + x_5 + x_7 &= t \end{aligned}$$

This system describes a polytope in $\mathbb{R}^9$. We are interested in counting integer points in this polytope. Geometrically, the "magic variable" t is a dilation parameter.

Dilate the d –dimensional rational polytope $\mathcal{P}$ by a positive integer t :

$$t\mathcal{P} := \{tx : x \in \mathcal{P}\}$$

and count the number of integer points ("lattice points") in $t\mathcal{P}$:

$$L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$$

Theorem (Ehrhart, 1960's) $L_{\mathcal{P}}(t)$ is a quasi-polynomial in t whose degree is the dimension of $\mathcal{P}$. The period of this quasipolynomial divides any common multiple of the denominators of the vertices of $\mathcal{P}$.

A quasipolynomial is an expression

$$c_d(t)\, t^d + \cdots + c_1(t)\, t + c_0(t)\ ,$$

where $c_0, \ldots, c_n$ are periodic functions in t

Example:

$$M_3(t) = \begin{cases} 0 & \text{for } t = 1, 4, 7, 10, \ldots \\ 0 & \text{for } t = 2, 5, 8, 11, \ldots \\ \frac{2}{9}t^2 + \frac{2}{3}t + 1 & \text{for } t = 3, 6, 9, 12, \ldots \end{cases}$$

Theorem $M_n(t), S_n(t), P_n(t)$ are quasipolynomials in t with degrees

$$\deg(M_n) = n^2 - 2n - 1$$
$$\deg(S_n) = \tfrac{1}{2}n^2 - \tfrac{1}{2}n - 2$$
$$\deg(P_n) = n^2 - 3n + 2$$

Idea of proof: The conditions of the 3×3 magic square yield the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \\ t \\ t \\ t \\ t \end{pmatrix}$$

Denote by $M_n^\star(t), S_n^\star(t), P_n^\star(t)$ the counting functions for magic squares, symmetric, and pandiagonal magic squares, respectively, as before, but now with the restriction that the entries are positive integers.

For a d –dimensional rational polytope $\mathcal{P}$, define

$$L_{\mathcal{P}}^\star(t) := \#\left(t\mathcal{P}^{\mathrm{int}} \cap \mathbb{Z}^d\right)$$

Theorem (Ehrhart–Macdonald reciprocity law, 1971) If $\mathcal{P}$ is a d –dimensional rational polytope homeomorphic to a d –sphere then

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}}^\star(t) \; .$$

$$\Downarrow$$

Proposition

$$
\begin{aligned}
M_n(-t) &= (-1)^{n^2-2n-1} M_n^\star(t) \\
S_n(-t) &= (-1)^{\frac{1}{2}n^2-\frac{1}{2}n-2} S_n^\star(t) \\
P_n(-t) &= (-1)^{n^2-3n+2} P_n^\star(t)
\end{aligned}
$$

$$
\Downarrow \qquad \begin{gathered} M_n^\star(t) = M_n(t-n) \\ M_n^\star(1) = \cdots = M_n^\star(n-1) = 0 \end{gathered} \qquad \text{etc.}
$$

Theorem

$$
\begin{aligned}
M_n(-t) &= (-1)^{n^2-2n-1} M_n(t-n) \\
S_n(-t) &= (-1)^{\frac{1}{2}n^2-\frac{1}{2}n-2} S_n(t-n) \\
P_n(-t) &= (-1)^{n^2-3n+2} P_n(t-n)
\end{aligned}
$$

$$
\begin{aligned}
M_n(-1) = \cdots = M_n(-n+1) = 0 \\
S_n(-1) = \cdots = S_n(-n+1) = 0 \\
P_n(-1) = \cdots = P_n(-n+1) = 0
\end{aligned}
$$

Find vertices by setting some $x_k = 0$.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Vertices:

$$\left(\tfrac{2}{3}, 0, \tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, 0\right), \quad \left(\tfrac{1}{3}, \tfrac{2}{3}, 0, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0, \tfrac{1}{3}\right),$$
$$\left(0, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0, \tfrac{1}{3}, 0, \tfrac{2}{3}\right), \quad \left(\tfrac{1}{3}, 0, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0, 0, \tfrac{2}{3}, \tfrac{1}{3}\right)$$

To interpolate a quasipolynomial of degree d and period p , we need to compute $p(d+1)$ values.

Example: $M_3(t)$ has degree 2 and period 3

$$
\begin{aligned}
M_3(1) &= 0 \\
M_3(2) &= 0 \\
M_3(3) &= 5 \\
M_3(4) &= 0 \\
M_3(5) &= 0 \\
M_3(6) &= 13 \\
M_3(7) &= 0 \\
M_3(8) &= 0 \\
M_3(9) &= 25
\end{aligned}
$$

$$
\Longrightarrow M_3(t) = \begin{cases} 0 & \text{for } t = 1, 4, 7, 10, \ldots \\ 0 & \text{for } t = 2, 5, 8, 11, \ldots \\ \frac{2}{9}t^2 + \frac{2}{3}t + 1 & \text{for } t = 3, 6, 9, 12, \ldots \end{cases}
$$

$$M_3(t) = \begin{cases} \frac{2}{9}t^2 + \frac{2}{3}t + 1 & \text{if } 3|t \\ 0 & \text{otherwise} \end{cases}$$

$$S_3(t) = \begin{cases} \frac{2}{3}t + 1 & \text{if } 3|t \\ 0 & \text{otherwise} \end{cases}$$

$$P_3(t) = \frac{1}{2}t^2 + \frac{2}{3} + 1$$

$$M_4(t) = \begin{cases} \frac{1}{480}t^7 + \frac{7}{240}t^6 + \frac{89}{480}t^5 + \frac{11}{16}t^4 \\ \quad + \frac{49}{30}t^3 + \frac{38}{15}t^2 + \frac{71}{30}t + 1 & \text{if } t \text{ is even} \\ \frac{1}{480}t^7 + \frac{7}{240}t^6 + \frac{89}{480}t^5 + \frac{11}{16}t^4 \\ \quad + \frac{779}{480}t^3 + \frac{593}{240}t^2 + \frac{1051}{480}t - \frac{3}{16} & \text{if } t \text{ is odd} \end{cases}$$

$$S_4(t) = \begin{cases} \frac{5}{128}t^4 + \frac{5}{16}t^3 + t^2 + \frac{3}{2}t + 1 & \text{if } t \equiv 0 \ (4) \\ \frac{5}{128}t^4 + \frac{5}{16}t^3 + t^2 + \frac{3}{2}t + \frac{7}{8} & \text{if } t \equiv 2 \ (4) \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

$$P_4(t) = \begin{cases} \frac{7}{1440}t^6 + \frac{7}{120}t^5 + \frac{23}{72}t^4 + t^3 \\ \quad + \frac{341}{180}t^2 + \frac{31}{15}t + 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

What about the vertices/periods?

Conjecture For $n \geq 4$, M_n, S_n, and P_n are *not* polynomials.

Application/next step: integer solutions to transportation problem/polytope

$H_n^d(t) :=$ # semi–magic hypercubes of dimension d , size n , and axial sums t

$H_n^d(t)$ is a quasipolynomial in t .

Stein–Stein: $\deg\left(H_n^2\right) = (n-1)^2$

Theorem $\deg\left(H_n^d\right) = (n-1)^d$

$M_n^d(t) :=$ # magic hypercubes of dimension d , size n , and axial/diagonal sums t

$M_n^d(t)$ is a quasipolynomial in t .

Earlier Theorem:
$\deg\left(M_n^2\right) = n^2 - 2n - 1 = (n-1)^2 - 2$

Conjecture $\deg\left(M_n^d\right) = (n-1)^d - 2^{d-1}$

Conjecture For $n, d \geq 3$, H_n^d and M_n^d are *not* polynomials.