Mathematical Programming manuscript No.
(will be inserted by the editor)

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Gröbner bases of acyclic directed graphs, minimum cost flow and hypergeometric systems

the date of receipt and acceptance should be inserted later

Abstract: In recent years, relations between Gröbner bases of graphs and integer programs on graphs have been studied through Conti-Traverso algorithm via toric ideals. However, Gröbner bases of directed graphs have not well studied. In this paper, we study Gröbner bases for toric ideals of acyclic directed graphs. These toric ideals can be homogenized by changing the positive grading, or by adding one more variable. Some Gröbner bases are characterized in terms of graphs, and gives the bounds for the degrees and the cardinalities of the Gröbner bases of acyclic tournament graphs. Homogenization by changing the positive grading enables to apply Gröbner bases of acyclic directed graphs to the minimum cost flow problems for any cost vector using Conti-Traverso algorithm. Another homogenization gives the alternative proof for the result by Gelfand-Graev-Postnikov about the number of independent solutions of hypergeometric

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systems on the group of unipotent matrices via the relations of Gröbner bases and regular triangulations.

1. Introduction

Conti and Traverso [5] showed the Gröbner bases methods for integer programs using the discreteness of toric ideals. Applying to some integer programs on graphs, properties of Gröbner bases of graphs may give some insights into properties of integer programs, or conversely, properties of integer programs may give some insights into properties of Gröbner bases of graphs. Especially, Gröbner bases of undirected graphs and their applications to integer programs have been studied [6,7,13] in recent years. The toric ideals of these graphs are "homogeneous", which is well-studied property in commutative algebra, for the standard grading. On the other hand, Gröbner bases of directed graphs have not well studied since the toric ideals of these graphs are not homogeneous for the standard grading. ← Is this the reason?

In this paper, we study Gröbner bases for toric ideals of *acyclic* directed graphs and their applications to minimum cost flow problems and to some hypergeometric systems. We focus especially on the degrees and the cardinalities of reduced Gröbner bases. The cardinality of reduced Gröbner basis is related to the complexity of the algorithms for minimum cost flow problems using the Conti-Traverso algorithm.

The toric ideal of an acyclic directed graph is not homogeneous for the standard grading, but can be homogenized by changing the grading of each variable (we call the *graphical grading*) or by adding one more variable. In addition, each element in Gröbner bases corresponds to a circuit of graphs. Three Gröbner bases are characterized in terms of circuits. Two are the set of incidence vectors for all of the circuits of length 3 and some of the circuits of length 4, and the other is the set of incidence vectors for all of the fundamental circuits for some spanning tree.

The fact that toric ideals of acyclic directed graphs are homogeneous for the graphical grading enables to use the Conti-Traverso algorithm in the minimum cost flow for any cost vector. On the other hand, minimum cost flow problems are one of the most fundamental and easiest network problems. So Gröbner bases of acyclic directed graphs may have several good properties.

Moreover, the vertex-arc incidence matrices of acyclic tournament graphs relate with the hypergeometric system on the group of unipotent matrices are hypergeometric systems. These hypergeometric systems have been studied recently by Gelfand, Graev and Postnikov [8]. By adding one more variable, toric ideals of acyclic tournament graphs can be homogenized for the standard grading. Then we can relate the Gröbner bases with the regular triangulations [16]. Two Gröbner bases for homogenized toric ideals are obtained from the Gröbner bases characterized above, and these bases implies the alternative proof for the result of Gelfand, Graev and Postnikov [8].

This paper is organized as follows. In Section 2, we give some basic definitions of toric ideals and Gröbner bases. In Section 3, we study the Gröbner bases for acyclic directed graphs. We give the positive grading which homogenize the toric ideals, and characterize three reduced Gröbner bases in terms of graphs. In Section 4, we analyze the bounds for the degrees and the cardinalities of the reduced Gröbner bases with respect to various cost vectors. In Section 5, we describe the Conti-Traverso algorithm, and apply the reduced Gröbner bases to the minimum cost flow problems on acyclic directed graphs. Finally in Section 6, we summarize [12] which shows that some results in [8] can be obtained by analyzing the Gröbner bases of acyclic tournament graphs.

2. Preliminaries

In this section, we give basic definitions of toric ideals and Gröbner bases. We refer to [4] for Gröbner bases, and [16] for toric ideals. We denote the set of non-negative integers by $\mathbb{N}_0$ and the set of non-negative real numbers by $\mathbb{R}_0$.

We consider the integer programming problem $IP_{A,\mathbf{c}}(\mathbf{b}) := minimize\{\mathbf{c} \cdot \mathbf{x} \colon A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{N}_0^n\}$, where $A \in \mathbb{Z}^{d \times n}$ is the coefficient matrix, $\mathbf{b} \in \mathbb{Z}^d$ is the right hand side vector, and $\mathbf{c} \in \mathbb{R}_0^n$ is the cost vector. We denote $P_\mathbf{b} := \mathrm{conv}\{\mathbf{x} \in \mathbb{N}_0^n \colon A\mathbf{x} = \mathbf{b}\}$, where "conv" means the convex hull.

Let k be a field and $k[x_1, \dots, x_n]$ be the polynomial ring in n variables. For an exponent vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}_0^n$, we denote $\mathbf{x}^\mathbf{a} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in k[x_1, \dots, x_n]$. A total order on monomials in $k[x_1, \dots, x_n]$ is a *term order* if 1 is the unique minimal element, and $\mathbf{x}^\mathbf{u} \succ \mathbf{x}^\mathbf{v}$ implies $\mathbf{x}^{\mathbf{u}+\mathbf{w}} \succ \mathbf{x}^{\mathbf{v}+\mathbf{w}}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{N}_0^n$.

For a fixed term order $\succ$, the *refinement* cost vector $\succ_\mathbf{c}$ is a term order such that $\mathbf{x}^\mathbf{u} \succ \mathbf{x}^\mathbf{v}$ if either $\mathbf{c} \cdot \mathbf{u} > \mathbf{c} \cdot \mathbf{v}$ or "$\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{v}$ and $\mathbf{x}^\mathbf{u} \succ \mathbf{x}^\mathbf{v}$". Let $IP_{A,\succ_\mathbf{c}}(\mathbf{b})$

be the problem to find the unique minimal element in $P_{\mathbf{b}} \cap \mathbb{N}_0^n$ with respect to $\succ_{\mathbf{c}}$. Then the solution $\mathbf{u}$ of $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$ is one of the optimal solutions of $IP_{A,\mathbf{c}}(\mathbf{b})$.

The *toric ideal* I_A of A is defined as $I_A := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \colon A\mathbf{u} = A\mathbf{v},\ \mathbf{u}, \mathbf{v} \in \mathbb{N}_0^n \rangle$. For any $f \in I_A$, the *initial term* $in_{\succ_{\mathbf{c}}}(f)$ of f is the largest term in f with respect to $\succ_{\mathbf{c}}$. Then we define the *initial ideal* $in_{\succ_{\mathbf{c}}}(I_A)$ of I_A as $in_{\succ_{\mathbf{c}}}(I_A) := \langle in_{\succ_{\mathbf{c}}}(f) \colon f \in I_A \rangle$.

Definition 1. *The* Gröbner basis *for I_A with respect to $\succ_{\mathbf{c}}$ is a finite subset $\mathcal{G}_{\succ_{\mathbf{c}}} = \{g_1, \dots, g_s\} \subseteq I_A$ such that $in_{\succ_{\mathbf{c}}}(I_A) = \langle in_{\succ_{\mathbf{c}}}(g_1), \dots, in_{\succ_{\mathbf{c}}}(g_s) \rangle$. In addition, Gröbner basis $\mathcal{G}_{\succ_{\mathbf{c}}}$ is* reduced *if $\mathcal{G}_{\succ_{\mathbf{c}}}$ satisfies the following:*

1. *For any i, the coefficient of $in_{\succ_{\mathbf{c}}}(g_i)$ is equal to 1.*
2. *For any i, any term of g_i is not divisible by $in_{\succ_{\mathbf{c}}}(g_j)$ $(i \neq j)$.*

The reduced Gröbner basis $\mathcal{G}_{\succ_{\mathbf{c}}}$ is uniquely defined for any $\succ$ and $\mathbf{c}$, and calculated by Buchberger algorithm (see [4]). The reduced Gröbner basis $\mathcal{G}_{\succ_{\mathbf{c}}}$ is a test set for the family $\{IP_{A,\succ_{\mathbf{c}}}(\mathbf{b}) \colon \mathbf{b} = A\mathbf{p}$ for some $\mathbf{p} \in \mathbb{N}_0^n\}$ for fixed A and $\mathbf{c}$. For more details, see [17, 18].

The *support* $\mathrm{supp}(\mathbf{u})$ of a vector $\mathbf{u}$ is the index set $\{i \colon u_i \neq 0\}$. Any $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}_0^n$ and have disjoint support. Then $G_{\succ_{\mathbf{c}}}$ can be written as $G_{\succ_{\mathbf{c}}} = \{\mathbf{x}^{\mathbf{u}_1^+} - \mathbf{x}^{\mathbf{u}_1^-}, \dots, \mathbf{x}^{\mathbf{u}_p^+} - \mathbf{x}^{\mathbf{u}_p^-}\}$ for some $\mathbf{u}_1, \dots, \mathbf{u}_p \in \ker(A)$ [16].

Proposition 1 ([4]). *Let $G_{\succ_{\mathbf{c}}} = \{g_1, \dots, g_s\}$ be the reduced Gröbner basis for I_A with respect to $\succ_{\mathbf{c}}$. Then every $f \in k[x_1, \dots, x_n]$ can be written as $f = a_1 g_1 + \cdots + a_s g_s + r$, $(a_i, r \in k[x_1, \dots, x_n])$, where either $r = 0$ or no term of r*

is divisible by any of $in_{\succ_{\mathbf{c}}}(g_1), \ldots, in_{\succ_{\mathbf{c}}}(g_s)$*. In addition,* r *is unique for any* $\succ$ *and* $\mathbf{c}$*, and called the* normal form *of* f *by* $G_{\succ_{\mathbf{c}}}$.

Proposition 2 ([4]). *Any Gröbner basis for* I_A *is a basis of* I_A.

An ideal $I \subseteq k[x_1, \ldots, x_n]$ is called *homogeneous* with respect to the positive grading $\deg(x_i) = d_i > 0$ $(i = 1, \ldots, n)$ if for any $f = f_1 + f_2 + \cdots + f_m \in I$ (f_i is the homogeneous component of degree i in f), $f_i \in I$ for any i. Then I is homogeneous if and only if I is generated by homogeneous polynomials [4].

Proposition 3 ([16]). *If* I_A *is a homogeneous with respect to some positive grading* $\deg(x_i) = d_i > 0$*, then reduced Gröbner basis* $\mathcal{G}_{\succ_{\mathbf{c}}}$ *exists for any* $\mathbf{c} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

3. Gröbner Bases for Acyclic Tournament Graphs

Let D_d be the acyclic tournament graph with d vertices which have labels $1, 2, \ldots, d$ such that each arc (i, j) $(i < j)$ is directed from i to j. Let $n = \binom{d}{2}$ be the number of arcs in D_d. We associate each arc (i, j) with the variable x_{ij} in the polynomial ring $k[\mathbf{x}] := k[x_{ij} \colon 1 \leq i < j \leq d]$. Let A_d be the vertex-arc incidence matrix of D_d.

3.1. Toric Ideals of Acyclic Tournament Graphs

A *walk* in D_d is a sequence $(v_1, v_2, \ldots, v_p)$ of vertices such that (v_i, v_{i+1}) or (v_{i+1}, v_i) is an arc of D_d for each $1 \leq i < p$. A *cycle* is a walk $(v_1, v_2, \ldots, v_p, v_1)$. A *circuit* is a cycle $(v_1, v_2, \ldots, v_p, v_1)$ such that $v_i \neq v_j$ for any $i \neq j$.

Definition 2. *Let C be a cycle in D_d and fix a direction of C. If C passes the arc (i,j) u_{ij}^+ times forwardly and u_{ij}^- times backwardly, then we define $\mathbf{u}_C^+ = (u_{ij}^+)_{1\le i<j\le d}, \mathbf{u}_C^- = (u_{ij}^-)_{1\le i<j\le d} \in \mathbb{R}^n$. The vector $\mathbf{u}_C := \mathbf{u}_C^+ - \mathbf{u}_C^-$ is called the* incidence vector *of C.*

For any vector $\mathbf{u}$, $\mathbf{u} \in \ker(A_d) \cap \mathbb{Z}^n$ if and only if there exists some cycle C of D_d such that $\mathbf{u}$ is the incidence vector of C. Thus we can identify a cycle C of D_d with the binomial $f_C := \mathbf{x}^{\mathbf{u}_C^+} - \mathbf{x}^{\mathbf{u}_C^-} \in I_{A_d}$.

We denote $\mathcal{U}_{A_d}$ the union of all reduced Gröbner bases for I_{A_d} with respect to all $\succ_{\mathbf{c}}$. $\mathcal{U}_{A_d}$ is called the *universal Gröbner basis* for I_{A_d}. The universal Gröbner basis $\mathcal{U}_{A_d}$ is a universal test set for the family $\{IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})\colon \mathbf{b} = A\mathbf{p}$ for some $\mathbf{p} \in \mathbb{N}_0^n$, $\mathbf{c} \in \mathbb{R}^n\}$ for fixed A [17,18].

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Proposition 4. $\mathcal{U}_{A_d} = \{f_C\colon C$ *is a circuit of* $D_d\}$. *Especially, the number of elements in $\mathcal{U}_{A_d}$ is of exponential order with respect to d.*

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Proof. Let $f_C = \mathbf{x}^{\mathbf{u}_C^+} - \mathbf{x}^{\mathbf{u}_C^-}$ be a binomial corresponding to a circuit C. We define the cost vector $\mathbf{c} = (c_{ij})_{1\le i<j\le d}$ as $c_{ij} = 0$ if $(i,j) \in C$ and $c_{ij} = 1$ otherwise, and a term order $\succ$ such that $\mathbf{x}^{\mathbf{u}_C^+} \succ_{\mathbf{c}} \mathbf{x}^{\mathbf{u}_C^-}$. Then $x^{\mathbf{u}_C^+} \in in_{\succ_{\mathbf{c}}}(I_A)$, and there exists a binomial $g := \mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-} \in G_{\succ_{\mathbf{c}}}$ such that $\mathbf{x}^{\mathbf{u}_i^+}$ divides $\mathbf{x}^{\mathbf{u}_C^+}$. The choice of $\mathbf{c}$ and the inclusion $\mathrm{supp}(\mathbf{u}_i^+) \subseteq \mathrm{supp}(\mathbf{u}_C^+) \cup \mathrm{supp}(\mathbf{u}_C^-)$ implies $\mathrm{supp}(\mathbf{u}_i^-) \subseteq \mathrm{supp}(\mathbf{u}_C^+) \cup \mathrm{supp}(\mathbf{u}_C^-)$, hence $\mathrm{supp}(\mathbf{u}_i^+) \cup \mathrm{supp}(\mathbf{u}_i^-) \subseteq \mathrm{supp}(\mathbf{u}_C^+) \cup \mathrm{supp}(\mathbf{u}_C^-)$. Since C is a circuit and g corresponds a cycle in D_n, g corresponds to C, i.e. $f_C = g$.

Conversely, suppose that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in G_{\succ_{\mathbf{c}}}$ corresponds to a cycle C which is not a circuit. Then C contains a circuit C'. For a suitable direction, $\mathrm{supp}(\mathbf{u}_{C'}^+) \subseteq$

$\mathrm{supp}(\mathbf{u}_C^+)$ and $\mathrm{supp}(\mathbf{u}_{C'}^-) \subsetneq \mathrm{supp}(\mathbf{u}_C^-)$. Since C' is a circuit, each element in $\mathbf{u}_{C'}^+$ and $\mathbf{u}_{C'}^-$ is either 0 or 1, and thus $\mathbf{x}^{\mathbf{u}_{C'}^+}$ divides $\mathbf{x}^{\mathbf{u}_C^+}$ and $\mathbf{x}^{\mathbf{u}_{C'}^-}$ divides $\mathbf{x}^{\mathbf{u}_C^-}$. This is contradiction for the definition of $G_{\succ_\mathbf{c}}$. □

Theorem 1. *I_{A_d} is not homogeneous for the grading $\deg(x_{ij}) = 1$, but is homogeneous for the grading $\deg(x_{ij}) = j - i$.*

We call the grading $\deg(x_{ij}) = 1$ the *standard grading*, and the grading $\deg(x_{ij}) = j - i$ the *graphical grading.*

Proof. For any d, $x_{12}x_{23} - x_{13} \in I_{A_d}$ and $x_{12}x_{23} \notin I_{A_d}$. This implies that I_{A_d} is not homogeneous for the standard grading.

Consider the graphical grading. Let $C = (v_1, v_2, \dots, v_p, v_1)$ be a circuit in D_d, $C^+ := \{k\colon v_k < v_{k+1}\}$ and $C^- := \{k\colon v_k > v_{k+1}\}$ (we set $v_{p+1} := v_1$). The binomial f_C corresponding to C is $f_C = \prod_{k\in C^+} x_{v_k v_{k+1}} - \prod_{k\in C^-} x_{v_{k+1} v_k}$. Then f_C is homogeneous because

$$\begin{aligned}\deg\left(\prod_{k\in C^+} x_{v_k v_{k+1}}\right) - \deg\left(\prod_{k\in C^-} x_{v_{k+1} v_k}\right) &= \sum_{k\in C^+}(v_{k+1} - v_k) - \sum_{k\in C^-}(v_k - v_{k+1}) \\ &= \sum_{k=1}^{p}(v_{k+1} - v_k) \ = \ 0.\end{aligned}$$

□

Thus reduced Gröbner basis exists for any $\mathbf{c} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ by Proposition 3.

3.2. Gröbner Bases for Acyclic Directed Graphs

For the case of incidence matrices of acyclic directed graphs or (undirected) bipartite graphs, Gröbner bases can be obtained from those of acyclic tournament graphs automatically using the following *elimination theorem.*

Theorem 2 (Elimination Theorem [4]). *Fix an integer $1 \leq l \leq n$ and let $\succ$ be a term order on $k[x_1, x_2, \ldots, x_n]$ such that any monomial involving at least one of $x_1, \ldots, x_l$ is greater than all monomials in $k[x_{l+1}, \ldots, x_n]$. Let $\succ'$ be a term order which is the restriction of $\succ$ to $k[x_{l+1}, \ldots, x_n]$. If I is an ideal in $k[x_1, x_2, \ldots, x_n]$ and $\mathcal{G}$ is a Gröbner basis of I with respect to $\succ$, then $\mathcal{G} \cap k[x_{l+1}, \ldots, x_n]$ is a Gröbner basis for $I \cap k[x_{l+1}, \ldots, x_n]$ with respect to $\succ'$.*

Let B_d be the vertex-arc incidence matrix of an acyclic directed graph G_d with d vertices. We consider G_d as a subgraph of D_d, and let $E(D_d)$ (resp. $E(G_d)$) be the arc set of D_d (resp. G_d).

Proposition 5. $I_{B_d} = I_{A_d} \cap k[x_{ij} \colon (i,j) \in E(G_d)]$.

Proof. If $f = \mathbf{x}^{\mathbf{a}}(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}) \in I_{B_d}$ (where $\mathbf{a}$, $\mathbf{u}^+$, $\mathbf{u}^- \in \mathbb{N}_0^n$, $\mathbf{x}^{\mathbf{a}} \in k[x_{ij} \colon (i,j) \in E(G_d)]$ and $\operatorname{supp}(\mathbf{u}^+) \cap \operatorname{supp}(\mathbf{u}^-) = \emptyset$), then there exists a cycle C in G_d such that for a suitable orientation of C, $\mathbf{u} := \mathbf{u}^+ - \mathbf{u}^-$ is the incidence vector of C. Then C is also a cycle in D_d, which implies that $f \in I_{A_d} \cap k[x_{ij} \colon (i,j) \in E(G_d)]$.

Conversely, let $f = \mathbf{x}^{\mathbf{a}}(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}) \in I_{A_d} \cap k[x_{ij} \colon (i,j) \in E(G_d)]$ (where $\mathbf{a}$, $\mathbf{u}^+$, $\mathbf{u}^- \in \mathbb{N}_0^n$, $\mathbf{x}^{\mathbf{a}} \in k[x_{ij} \colon (i,j) \in E(G_d)]$ and $\operatorname{supp}(\mathbf{u}^+) \cap \operatorname{supp}(\mathbf{u}^-) = \emptyset$). Since $f \in I_{A_d}$, there exists a cycle C in D_d such that for a suitable orientation, $\mathbf{u} := \mathbf{u}^+ - \mathbf{u}^-$ is the incidence vector of C. Furthermore, since $f \in k[x_{ij} \colon (i,j) \in$

$E(G_d)]$, all arcs in C are contained in $E(G_d)$. Then C is also a cycle in G_d, which implies that $f \in I_{B_d}$. □

Let B_{d_1,d_2} be the (undirected) bipartite graph with the vertex sets V, W such that $|V| = d_1$, $|W| = d_2$, and C_{d_1,d_2} the vertex-edge incidence matrix of B_{d_1,d_2}. We define the acyclic directed graph G_{d_1,d_2} from B_{d_1,d_2} by orienting each arc of B_{d_1,d_2} from the vertex in V to that in W. Then we can consider G_{d_1,d_2} as a subgraph of $D_{d_1+d_2}$. Let $\{1, \ldots, d_1\}$ and $\{d_1+1, \ldots, d_1+d_2\}$ be the vertex set of V and W, C'_{d_1,d_2} the vertex-arc incidence matrix of G_{d_1,d_2}, and $E(G_{d_1,d_2})$ the arc set of G_{d_1,d_2}.

Proposition 6. $I_{C'_{d_1,d_2}} = I_{C_{d_1,d_2}} = I_{A_{d_1+d_2}} \cap k[x_{ij} : (i,j) \in E(G_{d_1,d_2})]$.

Proof. The i-th row of C'_{d_1,d_2} is same as the i-th row of C_{d_1,d_2} for $1 \le i \le d_1$ and as (-1) times the i-th row of C_{d_1,d_2} for $d_1+1 \le i \le d_1+d_2$. Thus $I_{C'_{d_1,d_2}} = I_{C_{d_1,d_2}}$ since $\ker(C'_{d_1,d_2}) = \ker(C_{d_1,d_2})$.

The proof of the second equality is similar to that of Proposition 5. □

Thus we may consider Gröbner bases of acyclic tournament graphs when we consider those of acyclic directed graphs or bipartite graphs.

3.3. Some Gröbner bases for I_{A_d}

In this section, we show that the elements in reduced Gröbner bases with respect to some specific term orders can be given in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order.

Theorem 3. *Let* $\mathbf{c} = (c_{ij})_{1 \le i < j \le d}$ *be the cost vector such that* $c_{ij} = j - i$, *and* $\succ$ *be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \iff i < k \text{ or } (i = k \text{ and } j < l).$$

Then the reduced Gröbner basis for I_{A_d} *with respect to* $\succ_{\mathbf{c}}$ *is*

$$\{g_{ijk} := x_{ij}x_{jk} - x_{ik} : i < j < k\} \cup \{g_{ijkl} := x_{ik}x_{jl} - x_{il}x_{jk} : i < j < k < l\}. \tag{1}$$

In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{3} + \binom{d}{4}$.

The set $\{g_{ijk} : i < j < k\}$ corresponds to all of the circuits of length three, and $\{g_{ijkl} : i < j < k < l\}$ corresponds to some of the circuits of length four uniquely determined for each four vertices i, j, k, l (Figure 1).

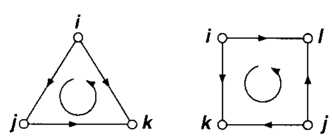

Fig. 1. Circuit corresponding to g_{ijk} (left) and circuit corresponding to g_{ijkl} (right).

Proof. By Proposition 4, it suffices to show that any binomial which corresponds to a circuit in D_d is either contained in (1) or that its initial term is divisible by the initial term of some element in (1).

Any binomial corresponding to a circuit of length 3 is contained in $\{g_{ijk}\}$.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i, j, k, l, i)$, $C_2 := (i, j, l, k, i)$, $C_3 := (i, k, j, l, i)$ and their opposites. The binomial which corresponds to C_1 or its opposite is $x_{ij}x_{jk}x_{kl} - x_{il}$, whose initial term $x_{ij}x_{jk}x_{kl}$ is divisible by $in_{\succ_c}(g_{ijk})$. Similarly, the initial term of binomial which corresponds to C_2 or its opposite is divisible by $in_{\succ_c}(g_{ijl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

Let C be a circuit of length more than five. Let v_1 be the vertex whose label is minimum in C, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Let f_C be the binomial corresponding to C, then $in_{\succ_c}(f_C)$ is the product of all variables whose associated arcs have the same direction as (v_1, v_2) on C. If $v_2 < v_3$, then (v_1, v_2) and (v_2, v_3) have the same direction on C. Thus both $x_{v_1v_2}$ and $x_{v_2v_3}$ appear in $in_{\succ_c}(f_C)$, and $in_{\succ_c}(f_C)$ is divisible by $in_{\succ_c}(g_{v_1v_2v_3})$ (Figure 2 left). If $v_2 > v_3$, then since $v_3 < v_2 < v_p$, there exists k $(3 \le k \le p-1)$ such that $v_1 < v_k < v_2 < v_{k+1}$. Then both $x_{v_1v_2}$ and $x_{v_kv_{k+1}}$ appear in $in_{\succ_c}(f_C)$, and $in_{\succ_c}(f_C)$ is divisible by $in_{\succ_c}(g_{v_1v_kv_2v_{k+1}})$ (Figure 2 right). □

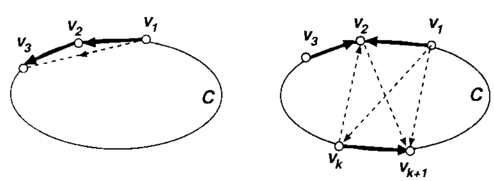

Fig. 2. $x_{v_1v_2}$ and $x_{v_2v_3}$ (left) or $x_{v_1v_2}$ and $x_{v_kv_{k+1}}$ (right) appear in $in_{\succ_c}(f_C)$.

Corollary 1. *Let $\succ$ be any term order and $\mathbf{c} = (c_{12}, \ldots, c_{1d}, c_{23}, \ldots, c_{d-1,d}) \in \mathbb{R}^n$ satisfy $c_{ij} + c_{jk} > c_{ik}$ for any $i < j < k$ and $c_{ik} + c_{jl} > c_{il} + c_{jk}$ for any*

$i < j < k < l$. Then the reduced Gröbner basis for I_{A_d} with respect to $\succ_{\mathbf{c}}$ is same as the basis (1) *in Theorem 3.*

Proof. Let $\succ'$ be the term order defined in Theorem 3. Then $in_{\succ_{\mathbf{c}}}(g_{ijk}) = x_{ij}x_{jk} = in_{\succ'}(g_{ijk})$ since $c_{ij} + c_{jk} > c_{ik}$, and $in_{\succ_{\mathbf{c}}}(g_{ijkl}) = x_{ik}x_{jl} = in_{\succ'}(g_{ijkl})$ since $c_{ik} + c_{jl} > c_{il} + c_{jk}$. Thus $in_{\succ_{\mathbf{c}}}(I_{A_d}) = in_{\succ'}(I_{A_d})$, which implies that the reduced Gröbner basis for I_{A_d} with respect to $\succ_{\mathbf{c}}$ is same as the basis (1). □

Theorem 4. *Let $\mathbf{c} = (c_{ij})_{1 \le i < j \le d}$ be the cost vector such that $c_{ij} = j - i$, and $\succ$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \iff j - i < l - k \text{ or } (j - i = l - k \text{ and } i < k).$$

Then the reduced Gröbner basis for I_{A_d} with respect to $\succ_{\mathbf{c}}$ is

$$\{g_{ijk} := x_{ij}x_{jk} - x_{ik} : i < j < k\} \cup \{g_{ijkl} := x_{il}x_{jk} - x_{ik}x_{jl} : i < j < k < l\}. \tag{2}$$

In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{3} + \binom{d}{4}$.

The set $\{g_{ijk} : i < j < k\}$ corresponds to all of the circuits of length three in D_d, and $\{g_{ijkl} : i < j < k < l\}$ corresponds to the set of circuits of length four same as in Figure 1 but the direction of each circuit is opposite.

Proof. Any binomial corresponds to a circuit of length 3 is contained in $\{g_{ijk}\}$.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i, j, k, l, i)$, $C_2 := (i, j, l, k, i)$, $C_3 := (i, k, j, l, i)$ and their opposites. The binomial which corresponds to C_1 or its opposite is $x_{ij}x_{jk}x_{kl} - x_{il}$, whose initial term $x_{ij}x_{jk}x_{kl}$

is divisible by $in_{\succ_c}(g_{ijk})$. The binomial which corresponds to C_2 or its opposite is $\pm(x_{ij}x_{jl} - x_{ik}x_{kl})$. If its initial term is $x_{ij}x_{jl}$, it is divisible by $in_{\succ_c}(g_{ijl})$. On the other hand, if initial term is $x_{ik}x_{kl}$, it is divisible by $in_{\succ_c}(g_{ikl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

Let C be a circuit of length more than five. Let (v_1, v_2) $(v_1 < v_2)$ be the arc which the difference of labels is minimum in C, and $C := (v_1, v_2, \dots, v_p, v_1)$. Let f_C be the binomial corresponding to C, then $in_{\succ_c}(f_C)$ is the product of all variables whose associated arcs have the same direction with (v_1, v_2) on C.

If $v_2 < v_3$, then both $x_{v_1 v_2}$ and $x_{v_2 v_3}$ appear in $in_{\succ_c}(f_C)$, and $in_{\succ_c}(f_C)$ is divisible by $in_{\succ_c}(g_{v_1 v_2 v_3})$. Similarly, if $v_p < v_1$, then $in_{\succ_c}(f_C)$ is divisible by $in_{\succ_c}(g_{v_p v_1 v_2})$.

Let $v_3 < v_2$ and $v_1 < v_p$. Then $v_3 < v_1 < v_2 < v_p$ by the definition of v_1 and v_2. If there exists some q such that $v_q < v_{q+1} < v_{q+2}$, then $in_{\succ_c}(f_C)$ is divisible by $in_{\succ_c}(g_{v_q v_{q+1} v_{q+2}})$. Consider the case that there does not exist such q. Let $v_s < v_1 < v_{s+1} < v_2$ (Figure 3 left). Then $v_{s+2} < v_{s+1}$ by assumption, and $v_{s+2} < v_1$ by the definition of v_1 and v_2. Thus there must be some r $(3 \leq r \leq p-1)$ such that $v_r < v_1 < v_2 < v_{r+1}$ (Figure 3 right) since $v_3 < v_1 < v_2 < i_p$.. Then $in_{\succ}(f_C)$ is divisible by $in_{\succ_c}(g_{v_r v_1 v_2 v_{r+1}})$. □

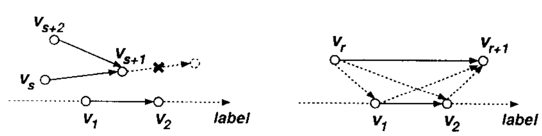

Fig. 3. If $v_s < v_1 < v_{s+1} < v_2$, $v_{s+2} < v_1$ (left). Then $v_r < v_1 < v_2 < v_{r+1} (^\exists r)$ (right).

Corollary 2. *Let $\succ$ be any term order and $\mathbf{c} = (c_{12}, \dots, c_{1d}, c_{23}, \dots, c_{d-1,d}) \in \mathbb{R}^n$ satisfy $c_{ij} + c_{jk} > c_{ik}$ for any $i < j < k$ and $c_{il} + c_{jk} > c_{ik} + c_{jl}$ for any $i < j < k < l$. Then the reduced Gröbner basis for I_{A_d} with respect to $\succ_{\mathbf{c}}$ is same as the basis (2) in Theorem 4.*

Proof. Let $\succ'$ be the term order defined in Theorem 4. Then $in_{\succ_{\mathbf{c}}}(g_{ijk}) = x_{ij}x_{jk} = in_{\succ'}(g_{ijk})$ since $c_{ij} + c_{jk} > c_{ik}$, and $in_{\succ_{\mathbf{c}}}(g_{ijkl}) = x_{il}x_{jk} = in_{\succ'}(g_{ijkl})$ since $c_{il} + c_{jk} > c_{ik} + c_{jl}$. Thus $in_{\succ_{\mathbf{c}}}(I_{A_d}) = in_{\succ'}(I_{A_d})$, which implies that the reduced Gröbner basis for I_{A_d} with respect to $\succ_{\mathbf{c}}$ is same as the basis (2). □

Theorem 5. *Let $\mathbf{c} = (c_{ij})_{1 \le i < j \le d}$ be the cost vector such that $c_{ij} = j - i$, and $\succ$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \iff i < k \text{ or } (i = k \text{ and } j > l).$$

Then the reduced Gröbner basis for I_{A_d} with respect to $\succ_{\mathbf{c}}$ is

$$\{g_{ij} := x_{ij} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j} : i < j - 1\}. \tag{3}$$

In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{2} - (d-1)$.

The set $\{g_{ij} : i < j - 1\}$ corresponds to all of the fundamental circuits of D_d for the spanning tree $T := \{(i, i+1) : 1 \le i < d\}$.

Proof. Let C be a circuit which is not a fundamental circuit of T. Let v_1 be the vertex whose label is minimum in C, and $C := (v_1, v_2, \dots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Then the variable $x_{v_1 v_p}$ appears in the initial term of the associated binomial f_C. Thus $in_{\succ_{\mathbf{c}}}(f_C)$ is divisible by $in_{\succ_{\mathbf{c}}}(g_{v_1 v_p})$. □

Corollary 3. *Let $\succ$ be any term order and* $\mathbf{c} = (c_{12}, \dots, c_{1d}, c_{23}, \dots, c_{d-1,d}) \in \mathbb{R}^n$ *satisfy* $c_{ij} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$ *for any* $i < j-1$. *Then the reduced Gröbner basis for* I_{A_d} *with respect to* $\succ_\mathbf{c}$ *is same as the basis* (3) *in Theorem 5.*

Proof. Let $\succ'$ be the term order defined in Theorem 5. Then $in_{\succ_\mathbf{c}}(g_{ij}) = x_{ij} = in_{\succ'}(g_{ij})$ since $c_{ij} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$. Thus $in_{\succ_\mathbf{c}}(I_{A_d}) = in_{\succ'}(I_{A_d})$, which implies that the reduced Gröbner basis for I_{A_d} with respect to $\succ_\mathbf{c}$ is same as the basis (3). □

4. Bounds for the Size of Gröbner Bases

The *degree* of reduced Gröbner basis is the maximum of the degree of polynomials in the Gröbner basis. Generally the degree of any reduced Gröbner basis for toric ideals is of exponential order with respect to the number of rows in the matrix [15], but the cardinality is not well understood. For the case of the toric ideals of acyclic tournament graphs, since those vertex-arc incidence matrices are unimodular, the degrees and the cardinalities of the reduced Gröbner bases may be bounded.

4.1. Bounds for the Degrees – the Case of Graphical Grading

Proposition 7. *The lower bound for the degrees of reduced Gröbner bases for* I_{A_d} *with respect to the graphical grading is* $d-2$.

Proof. Because of the definition of Gröbner basis, any reduced Gröbner basis contains some element g such that its initial term $in(g)$ divides the initial term of the binomial $f := x_{1,d-1}x_{d-1,d} - x_{1d}$ which corresponds to the cycle $(1, d-1, d, 1)$.

If the initial term $in(f)$ of f is $in(f) = x_{1d}$, then $in(g) = x_{1d}$ and the degree of $in(g)$ is equal to $d-1$. If $in(f) = x_{1,d-1}x_{d-1,d}$, then $in(g)$ must contain the variable $x_{1,d-1}$. In fact, if $in(g)$ does not contain $x_{1,d-1}$, then $in(g) = x_{d-1,d}$. But since any cycle which passes the arc $(d-1, d)$ always passes at least one of the arc $(i, d-1)$ $(1 \leq i \leq d-2)$ from the vertex i to the vertex $d-1$, $in(g)$ contains the variable $x_{i,d-1}$, this is contradiction. Thus $\deg(g) \geq d-2$. □

Proposition 8. *The upper bound for the degrees of reduced Gröbner bases for I_{A_d} with respect to the graphical grading is $(d-1)^2$.*

Proof. The length of each circuit in D_d is at most d. But the direction of at least one arc is opposite since D_d is acyclic. Thus each term of elements in reduced Gröbner bases contains at most $d-1$ variables. Since the degree of each variable is less than $d-1$, the degree of each element in the reduced Gröbner bases is at most $(d-1)^2$. □

4.2. Bounds for the Degrees – the Case of Standard Grading

Proposition 9. *The minimum degree of the reduced Gröbner bases for I_{A_d} with respect to the standard grading is 2. The basis in Theorem 3 is an example achieving this degree.*

Proof. The length of a circuit in D_d is at least 3, but the direction of at least one arc is opposite. Thus the degree of reduced Gröbner bases is at least 2. □

Proposition 10. *The maximum degree of the reduced Gröbner bases for I_{A_d} with respect to the standard grading is $d-1$. The basis in Theorem 5 is an example achieving this degree.*

Proof. The length of a circuit in D_d is at most d. But the direction of at least one arc is opposite since D_d is acyclic. Thus, in any circuit the number of arcs whose directions are the same is at most $d-1$, which implies the upper bound of the degree is $d-1$. □

4.3. Bounds for the Cardinalities of Gröbner Bases

Proposition 11. *The minimum cardinality of the reduced Gröbner bases for I_{A_d} is $\binom{d}{2}-(d-1)$. The basis we have shown in Theorem 5 is the example achieving this cardinality.*

Proof. Because of Proposition 2, the cardinality of the reduced Gröbner basis is more than that of the basis for I_{A_d}. Since I_{A_d} corresponds to the cycle space of D_d, the cardinality of the basis for I_{A_d} is equal to the dimension of the cycle space, which is $\binom{d}{2}-(d-1)$. □

To analyze the upper bound for the cardinalities of the reduced Gröbner bases, we calculate all reduced Gröbner bases for small d using TiGERS [10]. Table 1 is the result for $d = 4, 5, 6, 7$.

For the case of $d = 7$, the number of reduced Gröbner bases and the maximum of the cardinality are both too large, so we could not know the correct values. For $d \leq 5$, the reduced Gröbner basis in Theorem 3 is the example achieving

d	# GB	max cardinality	min cardinality
4	10	5	3
5	211	15	6
6	48312	37	10
7	$\geq$ 37665	$\geq$ 75	15

Table 1. Number of reduced Gröbner bases, maximum and minimum of cardinality.

maximum cardinality, but for $d \geq 6$ the maximum cardinality is little larger than the cardinality of Gröbner basis in Theorem 3. For $d = 6$, we do not know what cost vectors produce the Gröbner bases of cardinality 37. The reduced Gröbner bases which achieve the maximum cardinality seem to be complicated and difficult to characterize.

Problem 1. Are the cardinalities of reduced Gröbner bases for I_{A_d} of polynomial order with respect to d?

5. Applications to the Minimum Cost Flow Problems

5.1. Conti-Traverso Algorithm

Conti and Traverso [5] introduced a Gröbner basis for solving $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$. We describe the condensed version of Conti-Traverso Algorithm(See [16]). This version is useful for highlighting the main computational step involved.

Algorithm 1 (Conti-Traverso Algorithm).

Input: $A \in \mathbb{Z}^{d\times n}$, $\mathbf{b} \in \mathbb{Z}^d$, $\mathbf{c} \in \mathbb{R}_0^n$ and a cost vector $\mathbf{c}$

Output: A solution $\mathbf{u}$ for $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$

1. Compute the reduced Gröbner basis $\mathcal{G}_{\succ_{\mathbf{c}}}$ of I_A with respect to $\succ_{\mathbf{c}}$.

2. Find any feasible solution $\mathbf{v}$ of $IP_{A,\mathbf{c}}(\mathbf{b})$.

3. Compute the normal form $\mathbf{x}^{\mathbf{u}}$ of $\mathbf{x}^{\mathbf{v}}$ by $\mathcal{G}_{\succ_{\mathbf{c}}}$.

4. Output $\mathbf{u}$. $\mathbf{u}$ is the solution of $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$.

5.2. Applications to Minimum Cost Flow Problems

Using Algorithm 1, the reduced Gröbner bases for I_{A_d} can be applied to the minimum cost flow problems on D_d or the subgraphs of D_d, or to the transportation problem on the bipartite graphs B_{d_1,d_2}. In this case, this algorithm is similar to the *cycle canceling algorithm*, that is, for a feasible flow the algorithm iteratively finds a negative cost directed cycle in the residual network and augments flows on this cycle. If the residual network contains no negative cost cycle, then the flow is the minimum cost flow [1]. The *minimum mean cycle-canceling algorithm* [9] is known as a strongly polynomial time algorithm, which depends only on the number of vertices d and arcs n. Using this algorithm, from any feasible flow, we can obtain the minimum cost flow by canceling minimum mean cycle at most $O(dn^2 \log d)$ times. The cycle canceling algorithm can be considered as a special case of Conti-Traverso algorithm to use the universal Gröbner basis in each step.

Conti-Traverso algorithm for the minimum cost flow problem shows that we can obtain the minimum cost flow by augmenting flows only on the negative cost directed cycles which correspond to the reduced Gröbner bases. Although the result in Section 4 shows that it is difficult to analyze the size of reduced

Gröbner bases, Conti-Traverso algorithm is efficient since the number of cycles to augment (i.e. the cardinality of the reduced Gröbner basis) is much smaller than that in other cycle canceling algorithms (i.e. the number of all cycles in the graph).

Problem 2. Can the time complexity of Conti-Traverso algorithm for the minimum cost flow problem be bounded with respect to d?

6. Applications to the Hypergeometric Systems on the Group of Unipotent Matrices

In this section we consider A_d as a set of integer points

$$A_d = \{\mathbf{a}_{12}, \dots, \mathbf{a}_{1d}, \mathbf{a}_{23}, \dots, \mathbf{a}_{d-1,d}\}$$

in $\mathbb{R}^d$ where $\mathbf{a}_{ij}$ corresponds to the arc (i, j).

We consider the following system of differential equation with coordinates z_{ij}, $0 \le i < j \le d$.

$$-\sum_{i=1}^{j-1} z_{ij} \frac{\partial f}{\partial z_{ij}} + \sum_{k=j+1}^{d} z_{jk} \frac{\partial f}{\partial z_{jk}} = \alpha_j f, \quad j = 1, 2, \dots, d \tag{4}$$

$$\frac{\partial f}{\partial z_{ik}} = \frac{\partial^2 f}{\partial z_{ij} \partial z_{jk}}, \qquad 0 \le i < j < k \le d \tag{5}$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$ such that $\sum_{j=1}^{d} \alpha_j = 0$. Gelfand, Graev and Postnikov [8] showed that the number of linearly independent solutions of (4), (5) in a neighborhood of a generic point is equal to the normalized volume of $\text{conv}(A_d \cup \{\mathbf{0}\})$ ($\mathbf{0}$ is the origin of $\mathbb{R}^d$), and this normalized volume is equal to the Catalan number

$$C_{d-1} = \frac{1}{d}\binom{2(d-1)}{d-1}.$$

Using our result, we can give an alternative proof from the viewpoint of Gröbner bases of acyclic tournament graphs. For more details and proofs, see [12].

6.1. Regular Triangulations, Standard Trees and Anti-standard Trees

Assume that a subset $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ is the set of n points in $\mathbb{R}^d$.

Definition 3. *Let* $q = \dim \operatorname{conv}(A)$. $T = \{T_1, \ldots, T_p\}$ *is a* triangulation *of* $\operatorname{conv}(A)$ *if*

1. $T_i \subseteq A,\ |T_i| = q+1,\ \dim \operatorname{conv}(T_i) = q.$
2. $\cup_{i=1}^{p} \operatorname{conv}(T_i) = \operatorname{conv}(A).$
3. $\operatorname{conv}(T_i) \cap \operatorname{conv}(T_j) = \operatorname{conv}(T_i \cap T_j) \quad (i \neq j).$

Sturmfels [16, Chapter 8] showed that for each point set A and generic term order $\succ$, we can define a triangulation of A with respect to $\succ$.

Definition 4. *Let* $\succ$ *be a generic term order and* $\sqrt{in_{\succ}(I_A)}$ *a radical ideal of the initial ideal* $in_{\succ}(I_A)$. *Then we can define the triangulation* $\Delta_{\succ}(I_A)$ *as follows:*

$$\Delta_{\succ}(I_A) := \left\{ \operatorname{conv}(F) \colon F \subseteq A,\ \prod_{i\colon \mathbf{a}_i \in F} x_i \notin \sqrt{in_{\succ}(I_A)} \right\}.$$

We call $\Delta_{\succ}(I_A)$ *the* regular triangulation *of* A *with respect* $\succ$.

Definition 5. *Let* $q = \dim \operatorname{conv}(A)$. *Then we define the* normalized volume *of* $\operatorname{conv}(A)$ *as* $q!$ *times the Euclidean volume of* $\operatorname{conv}(A)$.

Hilbert polynomial $H_A(r)$ of $k[x_1, \dots, x_d]/I_A$ is the k-dimension of the r-th graded component of $k[x_1, \dots, x_d]/I_A$ for $r \gg 0$.

Theorem 6 ([16]). *Let $q = \dim \operatorname{conv}(A)$. Then $q!$ times the leading coefficient of the Hilbert polynomial $H_A(r)$ of $k[x_1, \dots, x_d]/I_A$ is equal to the normalized volume of* conv(A).

Definition 6. *Triangulation $T = \{T_1, \dots, T_p\}$ of* conv(A) *is* unimodular *if for any $T_i \in T$, the normalized volume of T_i is equal to 1. The matrix A is also called* unimodular *if all triangulations of* conv(A) *are unimodular.*

Proposition 12 ([16]). *Suppose that I_A be a homogeneous with respect to the standard grading. Then the initial ideal $in_\succ(I_A)$ is square-free (i.e. $\sqrt{in_\succ(I_A)} = in_\succ(I_A)$) if and only if the corresponding regular triangulation $\Delta_\succ(I_A)$ of* conv(A) *is unimodular.*

Definition 7. *Let T be a tree on the set $\{1, 2, \dots, d\}$.*

- T *is* admissible *if there are no $1 \le i < j < k \le d$ such that both (i, j) and (j, k) are edges of T.*
- T *is* standard *if T is admissible and there are no $1 \le i < j < k < l \le d$ such that both (i, k) and (j, l) are edges of T.*
- T *is* anti-standard *if T is admissible and there are no $1 \le i < j < k < l \le d$ such that both (i, l) and (j, k) are edges of T.*

Theorem 7 ([8]). *Let*

$$T_{ST} = \left\{ \mathrm{conv}\left(\bigcup_{(i,j)\in ST} \mathbf{a}_{ij} \cup \{\mathbf{0}\} \right) : ST \text{ is standard tree on } \{1,\dots,d\} \right\}$$

$$T_{AT} = \left\{ \mathrm{conv}\left(\bigcup_{(i,j)\in AT} \mathbf{a}_{ij} \cup \{\mathbf{0}\} \right) : AT \text{ is anti-standard tree on } \{1,\dots,d\} \right\}.$$

Then both T_{ST} and T_{AT} give unimodular triangulations of $A_d \cup \{\mathbf{0}\}$.

Theorem 8 ([14]). *The number of standard trees (resp. anti-standard trees) on the set $\{1, 2, \dots, d\}$ is equal to the Catalan number*

$$C_{d-1} = \frac{1}{d}\binom{2(d-1)}{d-1}.$$

6.2. Dimension of Hypergeometric Systems on the Group of Unipotent Matrices

We consider the $(d+1) \times (n+1)$ matrix

$$A'_d := \begin{pmatrix} A_d & \mathbf{0} \\ {}^t\mathbf{1} & 1 \end{pmatrix} \subset \mathbb{R}^{d+1}$$

where ${}^t\mathbf{1}$ is a row vector whose components are all 1, and $\mathbf{0}$ is a column zero vector. Then the toric ideal of A'_d is homogeneous for the standard grading. Thus we can relate the Gröbner bases of acyclic tournament graphs with the regular triangulations of $\mathrm{conv}(A'_d)$.

In the rest of this section, we discuss the toric ideal of A'_d. We associate the point $\mathbf{a}_{ij} \in A_d$ with the point $\mathbf{a}'_{ij} := \binom{\mathbf{a}_{ij}}{1} \in A'_d$ and the variable x_{ij} in the polynomial ring $k[\mathbf{x}, x_0] := k[x_{12}, \dots, x_{1d}, x_{23}, \dots, x_{d-1,d}, x_0]$, and the point $\mathbf{0} \in \mathbb{R}^n$ with $\mathbf{a}'_0 := \binom{0}{1} \in A'_d$ and the variable x_0 in $k[\mathbf{x}, x_0]$.

Remark 1.

1. The triangulation Δ of $\text{conv}(A'_d)$ can be associated with the triangulation of $\text{conv}(A_d \cup \{\mathbf{0}\})$ by projecting Δ to the hyperplane $x_{d+1} = 0$ in $\mathbb{R}^{d+1}$. Thus the normalized volume of $\text{conv}(A'_d)$ is equal to that of $\text{conv}(A_d \cup \{\mathbf{0}\})$, and these number is equal to the number of standard trees (or anti-standard trees) on $\{1, \dots, d\}$ by Theorem 7.
2. If $\mathbf{x^u} - \mathbf{x^v} \in I_{A_d}$ and $\deg(\mathbf{x^u}) - \deg(\mathbf{x^v}) = k$, then $\mathbf{x^u} - \mathbf{x^v} x_0^k \in I_{A'_d}$. Conversely, if $\mathbf{x^u} - \mathbf{x^v} x_0^k \in I_{A'_d}$, then $\mathbf{x^u} - \mathbf{x^v} \in I_{A_d}$.

Remark 2. When $d = 3$, then A'_3 is a unimodular matrix. On the other hand, A'_d is not unimodular for $d \geq 4$. For example, in the case of $d = 4$,

$$\{x_{12}x_{24}^2 - x_{14}x_{23}x_{34}, x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{34} - x_{12}x_{24},$$

$$x_{13}x_0 - x_{12}x_{23}, x_{14}x_0 - x_{12}x_{24}, x_{24}x_0 - x_{23}x_{34}\}$$

is a reduced Gröbner basis for $I_{A'_4}$ with respect to the lexicographic term order induced by the variable ordering

$$x_0 \succ x_{13} \succ x_{24} \succ x_{12} \succ x_{14} \succ x_{23} \succ x_{34},$$

thus even among the variables $x_{12}, \dots, x_{d-1,d}$, unimodularity is broken.

Theorem 3 shows that $I_{A'_d}$ is generated by $\{x_{ij}x_{jk} - x_{ik}x_0 \colon 1 \leq i < j < k \leq d\} \cup \{x_{ik}x_{jl} - x_{il}x_{jk} \colon 1 \leq i < j < k < l \leq d\}$, and Theorem 4 shows that $I_{A'_d}$ is generated by $\{x_{ij}x_{jk} - x_{ik}x_0 \colon 1 \leq i < j < k \leq d\} \cup \{x_{il}x_{jk} - x_{ik}x_{jl} \colon 1 \leq i < j < k < l \leq d\}$. Thus in these cases, we can extend each term order $\succ$ in Theorem 3 (resp. Theorem 4) to the term order $\succ'$ on $k[\mathbf{x}, x_0]$ such that $in_{\succ}(I_{A_d}) = in_{\succ'}(I_{A'_d})$.

Corollary 4.

(*i*) *For the lexicographic term order induced by the variable ordering*

$$x_{ij} \succ x_{kl} \Leftrightarrow i < k \text{ or } (i = k \text{ and } j < l),\ x_{ij} \succ x_0 \text{ for any } i < j,$$

the initial ideal of $I_{A'_d}$ *is* $\langle \{x_{ij}x_{jk} \colon 1 \le i < j < k \le d\} \cup \{x_{ik}x_{jl} \colon 1 \le i < j < k < l \le d\}\rangle$.

(*ii*) *For the lexicographic term order induced by the variable ordering*

$$x_{ij} \succ x_{kl} \Leftrightarrow j - i < l - k \text{ or } (j - i = l - k \text{ and } j < l),\ x_{ij} \succ x_0 \text{ for any } i < j,$$

the initial ideal of $I_{A'_d}$ *is* $\langle \{x_{ij}x_{jk} \colon 1 \le i < j < k \le d\} \cup \{x_{il}x_{jk} \colon 1 \le i < j < k < l \le d\}\rangle$.

Thus we get two regular unimodular triangulations $\Delta_1,\ \Delta_2$ of A'_d by applying Definition 4. The normalized volume of $\mathrm{conv}(A'_d)$ can be obtained by calculating the Hilbert polynomial of $k[\mathbf{x}, x_0]/I_{A'_d}$.

As a matter of fact, $F \subset A'_d$ is the face of Δ_1 (resp. Δ_2) if and only if the set $\{(i,j) \colon \mathbf{a}_{ij} \in F\}$ is standard tree (resp. anti-standard tree). Thus by Theorem 7, we obtain the result of Gelfand, Graev and Postnikov [8] about the number of linearly independent solutions of the system (4), (5) in a neighborhood of a generic point.

Theorem 9. *The number of linearly independent solutions of the system* (4), (5) *in a neighborhood of a generic point is equal to the Catalan number*

$$C_{d-1} = \frac{1}{d}\binom{2(d-1)}{d-1}.$$

Acknowledgement

The authors thank Hidefumi Ohsugi, Fumihiko Takeuchi, Masahiro Hachimori, Ryuichi Hirabayashi, Yoshiko T. Ikebe, Kenji Kashiwabara and Masafumi Itoh for their useful comments.

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