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Towards the Maximal Number of Components of a Nonsingular Surface of Degree 5 in $\mathbb{R}P^3$

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§1. Introduction

The problem of determining the maximal number of connected components of a surface of given degree m in $\mathbb{R}P^3$ was posed by Hilbert in 1900 (see the 16th problem of his famous list). Despite developments in the last decades in the topology of real algebraic varieties, the answer is still unknown, except in the trivial cases $m \leqslant 3$ and the case $m = 4$. In this last case the maximal number of components is equal to 10 (surfaces with 10 components were constructed by Rohn **[Ro]** in 1886; a proof of the maximality was given by Kharlamov **[Kh1]** in 1972).

It is well known that to determine the maximal number of components it suffices to consider nonsingular surfaces: by a small variation, any singular surface can be replaced by a nonsingular one with at least the same number of components.

A standard application of the Smith and Comessatti inequalities (see §3) gives the following estimate: the number of components of a nonsingular surface of degree m in $\mathbb{R}P^3$ is less than or equal to $(5m^3 - 18m^2 + 25m)/12$. In particular, it cannot be more than 25 for $m = 5$.

Kharlamov **[Kh2]** constructed a surface of degree 5 in $\mathbb{R}P^3$ with 21 components. (The surface constructed is an M-surface: the total $\mathbb{Z}_2$-homology group has the same rank as that of its complexification; see §3)

In the present paper we construct a nonsingular surface of degree 5 in $\mathbb{R}P^3$ with 22 components. We follow the scheme of **[Kh2]** and use, in addition, some elements of Itenberg's recent construction **[It]** of counter-examples to the Ragsdale conjecture (see **[Ra]**).

§2. Construction

2.1. An equivariant analog of Horikawa's theorem. By a real algebraic (or analytic) manifold we mean a complex manifold supplied with complex conjugation. For a real variety X we denote the set of its real points by $\mathbb{R}X$ and the set of complex points by $\mathbb{C}X$.

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Let Σ_2 be the standard nonsingular model of the real cone defined in P^3 by the equation $x_0^2 + x_1^2 = x_2^2 + x_3^2$. Following Horikawa **[Ho]**, consider an irreducible curve B on Σ_2 satisfying the following conditions:

(i) its intersection number with a linear generator of Σ_2 is equal to 6;
(ii) its intersection number with the inverse image of the vertex of the cone is equal to 1;
(iii) it has only two singular points, these points are ordinary triple points and they both lie on the same linear generator L.

We shall call it the *Horikawa curve*.

Denote by $\widetilde{W}$ the surface obtained from Σ_2 by blowing-up the two singular points of the Horikawa curve B, and by $\widetilde{L}$ and $\widetilde{B}$ the proper transforms of L and B under this blowing-up. Then take a double covering $\widetilde{S} \to \widetilde{W}$ with branch locus $B \cup L$. Such a covering exists because of (i)–(iii) and is unique.

The inverse image of L is a nonsingular rational curve with self-intersection number -1. Contracting it to a point, we get a nonsingular surface; denote it by S.

If the Horikawa curve is real, the surface $\widetilde{S}$ acquires, in the usual way, two canonical real structures. They are liftings of the complex conjugation of $\widetilde{W}$. They differ by the covering transformation and both can be projected to S; we also call *canonical* the two resulting real structures on S.

PROPOSITION 1. *Let the Horikawa curve B be real and let S be supplied with one of its canonical real structures. Then there exists an equivariant deformation of S to a nonsingular surface of degree 5 in $\mathbb{R}P^3$.*

PROOF. Take a versal deformation $p\colon L \to M$ of S. By Horikawa's theorem (see **[Ho**, Theorem 3]), M consists of two smooth irreducible components M_0 and M_1 intersecting normally; $\dim_{\mathbb{C}} M_0 = \dim_{\mathbb{C}} M_1 = 40$, $\dim_{\mathbb{C}} M_0 \cap M_1 = 39$. Points of $M_0 \setminus M_0 \cap M_1$ correspond to quintic surfaces and points of $M_1 \setminus M_0 \cap M_1$ to coverings of $\mathbb{P}^1 \times \mathbb{P}^1$. The standard versality arguments show that the deformation may be made equivariant. It remains to notice that the corresponding antiholomorphic involution does not interchange irreducible components of M (they are of different nature) and that $\mathbb{R}M_0$ and $\mathbb{R}M_1$, as fixed point sets of an antiholomorphic involution on smooth complex manifolds, are smooth connected manifolds; they intersect normally and

$$\dim_{\mathbb{R}} M_0 = \dim_{\mathbb{R}} M_1 = 40, \qquad \dim_{\mathbb{R}} M_0 \cap M_1 = 39. \qquad \square$$

2.2. A special case of Viro's theorem. Let P be a convex polygon in $\mathbb{R}^2$ with integer vertices that verifies the following condition: it is contained in the triangle

$$\Delta = \{x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant m\}, \qquad m \in \mathbb{N},$$

and it contains the vertices $x = 0$, $y = m$ and $x = m$, $y = 0$. In the sequel, such a polygon will serve as a Newton polygon of curves of degree m with a singularity at the origin prescribed by the Newton polygonal line $\Gamma(P)$, which is, by definition, the union of sides of P facing the origin.

Suppose that P is triangulated, that the vertices of the triangles are integer and that some distribution of signs, $a_{i,j} = \pm$, at the vertices of the triangulation is given. Then there arises a naturally associated piecewise-linear curve L in $\mathbb{R}P^2$.

The construction of L is the following.

Take the copies

$$P_x = s_x(P), \quad P_y = s_y(P), \quad P_{xy} = s(P),$$
$$\Delta_x = s_x(\Delta), \quad \Delta_y = s_y(\Delta), \quad \Delta_{xy} = s(\Delta)$$

of P and Δ, where $s = s_x \circ s_y$ and s_x, s_y are reflections with respect to the coordinate axes. Extend the triangulation of P to a symmetric triangulation of $P_* = P \cup P_x \cup P_y \cup P_{xy}$ and extend the distribution of signs to a distribution at the vertices of the extended triangulation so that it verifies the modular property: $g^*(a_{i,j}x^i y^j) = a_{g(i,j)}x^i y^j$ for $g = s_x, s_y$, and s (in other words, the sign at a vertex is the sign of the corresponding monomial in the quadrant containing the vertex).

If a triangle of the triangulation has vertices of different signs, select a midline separating them. If a midline comes to $\Gamma(P)$ at a point b, select also the segment joining b to the origin. Denote by L_* the union of the selected midlines and segments. It is contained in $T_* = \Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$. Glue the sides of T_* by s. The resulting space T is homeomorphic to $\mathbb{R}P^2$. Take the curve L to be the image of L_* in T.

A pair (T, L) is called a *chart* of a real plane algebraic curve A if there exists a homeomorphism $(T, L) \to (\mathbb{R}P^2, \mathbb{R}A)$ that maps each segment of L joining the origin and an edge of $\Gamma(P)$ to the branch of A corresponding to this edge.

A curve A is called *regular* if it does not have singular points outside of the origin.

Let us introduce two additional assumptions: the triangulation of P considered is *primitive* and *convex*. The first condition means that all triangles are of area $1/2$ (or, equivalently, that all integer points of P are vertices of the triangulation). The second one means that there exists a convex piecewise-linear function $P \to \mathbb{R}$ which is linear on each triangle of the triangulation and not linear on the union of two triangles.

The following statement is a special case of Viro's theorem **[Vi1,** Theorem 1.4]:

PROPOSITION 2. *Under the assumptions made above concerning the polygon P and its triangulation, there exists a real regular curve A in $\mathbb{R}P^2$ with chart (T, L).*

2.3. A lemma. We say that a real curve in P^2 is *of class H*, if

(i) its Newton diagram is the pentagon Π presented in Figure 1,
(ii) there are three branches corresponding to the edge BC of the diagram, they are smooth and each of them is tangent with simple inflection to another.

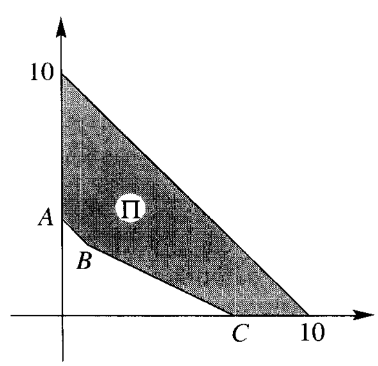

FIGURE 1

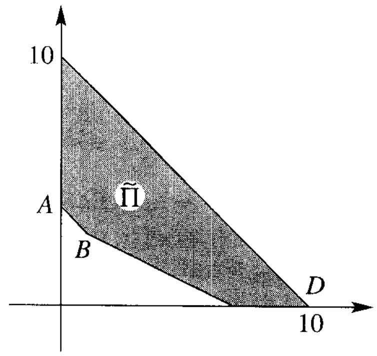

FIGURE 2

We say that a real curve in P^2 is *of class* $\widetilde{H}$, if

(i) its Newton diagram is the quadrangle $\widetilde{\Pi}$ presented in Figure 2,

(ii) the truncation f_{BD} of the polynomial f defining the curve is equal to
$$\gamma x(y - ax^3)(y - bx^3)(y - cx^3),$$
where $\gamma \neq 0$, a, b, c are real numbers and a, b, c are pairwise different.

Curves of class $\widetilde{H}$ have 4 branches at the origin, three of them are smooth and pairwise tangent with simple inflection; the same is true for that class H. (The principal difference between these two classes is that a curve of class H is degenerate at the origin with respect to the Newton diagram and a curve of class $\widetilde{H}$ is nondegenerate.)

LEMMA 3. *Up to a homeomorphism of the plane, each real regular curve $\widetilde{C}$ of class $\widetilde{H}$ is equivalent to a real regular curve C of class H. Moreover, a homeomorphism may be chosen to transform three tangent branches of $\widetilde{C}$ to three tangent branches of C.*

PROOF. Let
$$Q(x, y) = \sum_{(i,j)\in\widetilde{\Pi}} a_{i,j} x^i y^j$$
be a polynomial which defines a real regular curve of class $\widetilde{H}$, and let
$$\Gamma(x, y) = \gamma x(y - ax^3)(y - bx^3)(y - cx^3)$$
be the truncation of Q on BD.

Take a linear function ν in two variables with integer coefficients vanishing at each point of BD and positive at the other points of $\widetilde{\Pi}$ and put
$$Q'_t(x, y) = \sum_{(i,j)\in\widetilde{\Pi}} a_{i,j} x^i y^j t^{\nu(i,j)} + \gamma x(y - x^2 - ax^3)(y - x^2 - bx^3)(y - x^2 - cx^3) - \Gamma(x, y).$$
For any real t, the curve $Q'_t = 0$ is of class H. Following the same lines as in **[Vi2]** in the proof of the theorem on the smoothing of quasi-homogeneous singularities, one verifies that for any sufficiently small positive value of t there exist two radii, $r_2 > r_1 > 0$, such that the curve $Q'_t = 0$ is approximated

(a) inside of the disc D_1 of radius r_1 centered at the origin, by the curve
$$\gamma x(y - x^2 - ax^3)(y - x^2 - bx^3)(y - x^2 - cx^3) = 0;$$

(b) outside of the disc D_2 of radius r_2 centered at the origin, by the curve $Q = 0$;
(c) in the annulus $D_2 \setminus D_1$, by the curve $\Gamma = 0$.

Thus for a sufficiently small positive t the curve $Q'_t = 0$ is regular and topologically equivalent to the initial curve $Q = 0$ and a homeomorphism of the plane, mapping one into another, may be chosen to transform tangent branches into tangent branches. □

2.4. The curve.

PROPOSITION 4. *There exists a real regular curve of class H of the isotopy type represented in Figure 3 (the letters a, b, c mark the three branches with common tangent).*

PROOF. By Lemma 3, it suffices to realize the given isotopy type by a real regular curve of class $\tilde{H}$.

Any convex primitive triangulation of a convex part of a convex polygon is extendible to a convex primitive triangulation of the polygon. Inside the part $BKMN$ of the quadrangle $\tilde{\Pi}$, take the convex primitive triangulation shown in Figure 4 and extend it to $\tilde{\Pi}$.

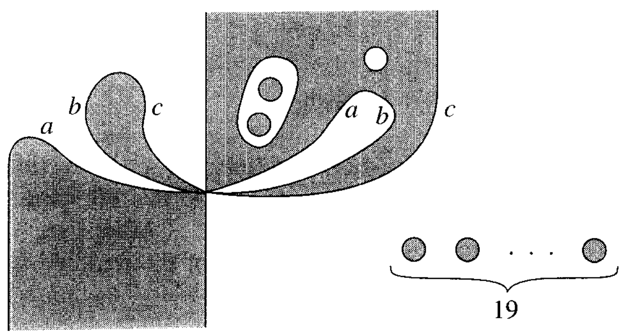

FIGURE 3

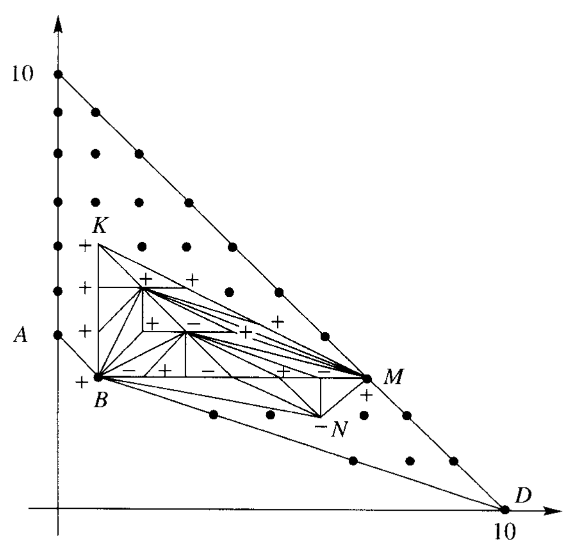

FIGURE 4

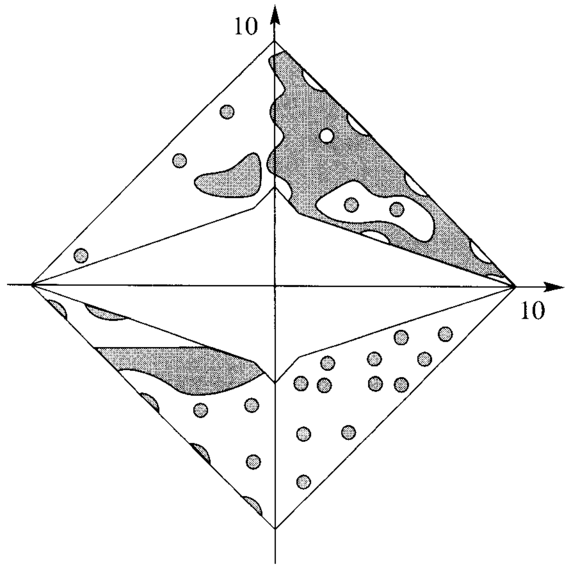

FIGURE 5

To apply Proposition 2, we need to choose signs on the vertices in $\widetilde{\Pi}$. Inside $BKMN$ put signs according to Figure 4, outside, use the following rule: the vertex (i, j) acquires sign "$-$" if i, j are even, and the sign "+" otherwise.

The corresponding piecewise-linear curve L is of the required isotopy type (see Figure 5) and Proposition 2 gives the desired result. □

2.5. The surface.

THEOREM 5. *There exists a nonsingular surface of degree* 5 *in* $\mathbb{R}P^3$ *having* 21 *connected components homeomorphic to the sphere and one component homeomorphic to the sphere with* 7 *Möbius bands.*

PROOF. To each real regular curve of class H corresponds a Horikawa curve: make two consecutive blowing-ups at the origin, the second one corresponding to the direction of the tangent line l to the parabolic branches, and then contract the proper transform of l to a point; thus we get Σ_2 and a Horikawa curve on it.

Let us start with the curve A constructed in 2.4. Then, applying Proposition 1, we obtain a nonsingular surface of degree 5 in $\mathbb{R}P^3$ homeomorphic to $\mathbb{R}S$ (see 2.1). The surface $\mathbb{R}S$ is the real part of a nonsingular real model of the two-sheeted covering Y of P^2, ramified along A. Choosing the appropriate real structure on Y from the two canonical ones (namely, take the one for which $\mathbb{R}S$ is situated over the dark regions in Figure 3), we get, according to Proposition 4, exactly 21 connected components homeomorphic to the sphere and one additional component. It now suffices to note that this component is not orientable and to calculate its Euler characteristic by retracing blowing-ups:

$$\chi = 2 - 1 + 2(1-2) - 3 + 0 + (1-2) = -5$$

(on the nonsingular model $\mathbb{R}S$ of $\mathbb{R}Y$, the singular point is replaced by a wedge of two circles). □

§3. Limits of the method

3.1. Known restrictions on the topological type of a real surface. We mention three well-known results (see, for example, the survey articles **[Wi, Kh3]**): for a nonsingular real projective surface X,

(a) $\chi(\mathbb{R}X) \leqslant h^{1,1}(\mathbb{C}X) - 2(\rho - 1)$ (Comessatti inequality),
where χ is the Euler characteristic and ρ is the number of linearly independent real algebraic classes in $H_2(\mathbb{C}X; \mathbb{R})$;

(b) $\beta_*(\mathbb{R}X) \leqslant \beta_*(\mathbb{C}X) - 2\phi$ (Smith inequality),
where β_* is the rank of the total $\mathbb{Z}_2$-homology group and $\rho + \phi$ is the number of linearly independent algebraic classes (not only real ones) in $H_2(\mathbb{C}X; \mathbb{R})$;

(c) if $\beta_*(\mathbb{R}X) = \beta_*(\mathbb{C}X)$, then

$$\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X) \bmod 16 \qquad \text{(Rokhlin congruence)}.$$

3.2. Application to surfaces of degree 5. If X is a nonsingular surface of degree 5, then

$$h^{1,1}(\mathbb{C}X) = 45, \quad \beta_*(\mathbb{C}X) = 55, \quad \text{and} \quad \sigma(\mathbb{C}X) = -35.$$

Thus, according to the Smith and Comessatti inequalities, the number of components of a surface of degree 5 in $\mathbb{R}P^3$ is not greater than 25.

Proposition 6. *The real part $\mathbb{R}S$ of a Horikawa surface S cannot have more than 24 connected components. If the singular points of a Horikawa curve B are real, then $\mathbb{R}S$ has no more than 23 components.*

Proof. First, consider the case when the singular points are real.

Then the surface $\widetilde{W}$ has at least 4 independent real algebraic cycles: the inverse image of the vertex of the cone, the inverse images of the singular points of B and the hyperplane section. So this is also the case for $\widetilde{S}$. Thus

$$\chi(\mathbb{R}S) = 1 + \chi(\mathbb{R}\widetilde{S}) \leqslant 1 + (h^{1,1}(\mathbb{C}\widetilde{S}) - 2 \cdot 3) = h^{1,1}(\mathbb{C}S) - 4 = 41,$$
$$\beta_0(\mathbb{R}S) = (\chi(\mathbb{R}S) + \beta_*(\mathbb{R}S))/4 \leqslant 24.$$

Moreover, if $\beta_0(\mathbb{R}S) = 24$, then $\beta_*(\mathbb{R}S) = 55$ and $\chi(\mathbb{R}\widetilde{S}) = 41$. The last combination contradicts the Rokhlin congruence.

If the singular points are imaginary, then $\rho \geqslant 3$ and $\phi \geqslant 1$. Thus

$$\chi(\mathbb{R}S) \leqslant 1 + (h^{1,1}(\mathbb{C}\widetilde{S}) - 2 \cdot 2) = 43, \qquad \beta_*(\mathbb{R}S) \leqslant \beta_*(\mathbb{C}S) - 2 = 53$$

and we obtain the bound $\beta_0(\mathbb{R}S) \leqslant 24$ again. □

3.3. Concluding remarks. A. It was conjectured by V. Arnold (see **[Vi3]**) that a nonsingular surface of degree m in $\mathbb{R}P^3$ has at most

$$(m^3 - m + 3((-1)^{m+1} + 1))/6$$

components. Viro **[Vi3]** showed that for any even $m \geqslant 6$ the conjecture is not true. The surface constructed in the present paper provides a counter-example for $m = 5$ (for $m \leqslant 4$ the conjecture is true).

B. Real double planes $\mathbb{R}Y$ ramified along real plane curves constructed by Itenberg in **[It]** have more than $(2 + h^{1,1}(\mathbb{C}Y))/2$ components. For a surface X of degree 5 in $\mathbb{R}P^3$

$$(2 + h^{1,1}(\mathbb{C}X))/2 = 23.5,$$

and one may expect that a clever direct application of Viro's construction can give examples of surfaces of degree 5 with at least 24 components.

C. The case of M-surfaces, $\beta_*(\mathbb{R}X) = \beta_*(\mathbb{C}X)$, is always of particular interest. By 3.1, an M-surface of degree 5 in $\mathbb{R}P^3$ may have 5, 9, 13, 17, 21, or 25 connected components. Examples with 5, 94, 13, 17, and 21 components were constructed by Kharlamov [**Kh2**]. If M-surfaces with 25 components really exist, then, again according to 3.1, they may be only of the following topological types:

$$24S \amalg P(2), \quad 23S \amalg S(1) \amalg P(1), \quad 23S \amalg S(2) \amalg P, \quad 22S \amalg S(1) \amalg S(1) \amalg P,$$

where S is the sphere, P is the projective plane, $S(q)$ and $P(q)$ are the sphere and the projective plane with q handles. The two last types are not realizable (the fourth was excluded by Viro, the third by Kharlamov; see [**Kh4**]). The problem of the existence of M-surfaces of degree 5 of the two other topological types $24S \amalg P(2)$, $23S \amalg S(1) \amalg P(1)$ is open.

D. Taking the other canonical real structure (see 2.1) on the Horikawa surface constructed in 2.5, one gets a surface with real part homeomorphic to $S \amalg S(2) \amalg P(20)$. In particular, here $\beta_1 = 45 = h^{1,1}$. The same value is given by M-surfaces having 5 components (see C above). It would be interesting to construct surfaces of degree 5 with larger β_1.

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Translated by THE AUTHORS

Chevallier has compiled degree 5 surfaces!!!