# TOPOLOGY OF REAL ALGEBRAIC T-SURFACES

ILIA ITENBERG

ABSTRACT. The paper is devoted to algebraic surfaces which can be obtained using a simple combinatorial procedure called the T-construction. The class of T-surfaces is sufficiently rich: for example, we construct T-surfaces of an arbitrary degree in $\mathbf{R}P^3$ which are M-surfaces. We also present a construction of T-surfaces in $\mathbf{R}P^3$ with $dim\ H_1(\mathbf{R}X; \mathbf{Z}/2) > h^{1,1}(\mathbf{C}X)$, where $\mathbf{R}X$ and $\mathbf{C}X$ are the real and the complex point sets of the surface.

## 1. INTRODUCTION

The subject of the paper is T-surfaces, i. e. real algebraic surfaces which can be constructed in a simple combinatorial fashion : one can patchwork them from the pieces which essentially are planes.

The construction of combinatorial patchworking (or T-construction) works in any dimension. We restrict ourself here by the case of surfaces. The general T-construction can be formulated in a completely similar way (the combinatorial patchwork construction in the case of curves is described in [I-V], [I1], [I2]). The T-construction is a particular case of the Viro theorem (see [V2], [V3], [V5], [V6], [Ri]).

The results on topology of T-surfaces presented in the paper are concentrated around the following conjecture proposed by O. Viro ([V4]) : let $X$ be a nonsingular simply connected compact complex surface with an antiholomorphic involution $c : X \to X$; then $dim\ H_1(\mathbf{R}X; \mathbf{Z}/2) \le h^{1,1}(X)$, where $\mathbf{R}X$ is the fixed point set of the involution $c$ (for a detailed information on real algebraic surfaces see [Kh], [Si], [Wi]).

This conjecture is related to the Ragsdale conjecture (see [Ra]) concerning the topology of real algebraic curves. To formulate the Ragsdale conjecture, let us

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

denote the number of even ovals of a nonsingular real algebraic plane projective curve of degree $2k$ by $p$ (an oval of a nonsingular curve of an even degree is called *even* (resp. *odd*), if it lies inside of even (resp. odd) number of other ovals of this curve), and denote the number of odd ovals by $n$.

**Ragsdale conjecture.** *For a nonsingular real algebraic plane projective curve of degree $2k$*

$$p \leq \frac{3k^2 - 3k + 2}{2}, \quad n \leq \frac{3k^2 - 3k}{2}.$$

Any counter-example to the inequality $p \leq \frac{3k^2-3k+2}{2}$ produces a counter-example to Viro's conjecture: one can take a double plane ramified along the complex point set of a counter-example to the Ragsdale conjecture with an appropriate choice of a lifting of the involution of complex conjugation. Thus, the counter-examples to Ragsdale conjecture obtained in [I1] (see, also, [I2], [I-V]) show that Viro's conjecture is not true. The counter-examples to Ragsdale conjecture are constructed as T-curves. So, it is natural to try to use the combinatorial patchwork construction in order to construct counter-examples to Viro's conjecture which are real algebraic surfaces in $\mathbf{R}P^3$.

We show in sections 3 and 4 that under some conditions of "maximality" of the triangulation participating in the combinatorial patchwork construction, Viro's conjecture is true for the resulting T-surfaces. However, using a "nonmaximal" triangulation (see exact definitions in section 2), we can obtain a T-surface $X$ in $\mathbf{R}P^3$ with $dim\, H_1(\mathbf{R}X; \mathbf{Z}/2) > h^{1,1}(\mathbf{C}X)$ (see section 6).

We also construct T-surfaces of any degree in $\mathbf{R}P^3$ which are M-surfaces (it means that the total $\mathbf{Z}/2$-homology group of the real point set has the same rank as that of the complexification; see section 5).

I would like to thank V. Kharlamov and O. Viro for the useful discussions.

## 2. T-CONSTRUCTION

Let $m$ be a positive integer number (it would be the degree of the surface under construction) and $T$ be the tetrahedron in $\mathbf{R}^3$ with vertices $(0,0,0)$, $(0,0,m)$, $(0,m,0)$, $(m,0,0)$. Let us take a triangulation $\tau$ of $T$ with vertices having integer

coordinates. Suppose that a distribution of signs at the vertices of $\tau$ is given. The sign (plus or minus) at the vertex with coordinates $(i, j, l)$ is denoted by $\delta_{i,j,l}$.

Take the copies

$$T_x = s_x(T), \quad T_y = s_y(T), \quad T_z = s_z(T)$$

$$T_{xy} = s_x \circ s_y(T), \quad T_{xz} = s_x \circ s_z(T), \quad T_{yz} = s_y \circ s_z(T) \ , T_{xyz} = s_x \circ s_y \circ s_z(T)$$

of $T$, where $s_x$, $s_y$, $s_z$ are reflections with respect to the coordinate planes. Denote by $T_*$ the octahedron

$$T \cup T_x \cup T_y \cup T_z \cup T_{xy} \cup T_{xz} \cup T_{yz} \cup T_{xyz}.$$

Extend the triangulation $\tau$ to a symmetric triangulation of $T_*$, and the distribution of signs $\delta_{i,j,l}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate plane we preserve its sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If a tetrahedron of the triangulation of $T_*$ has vertices of different signs, select a piece of the plane (triangle or quadrangle) being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by $S$ the union of the selected pieces. It is a piecewise-linear surface contained in $T_*$. Glue by $s_x \circ s_y \circ s_z$ the facets of $T_*$. The resulting space $\tilde{T}$ is homeomorphic to the real projective space $\mathbf{R}P^3$. Denote by $\tilde{S}$ the image of $S$ in $\tilde{T}$.

Let us introduce an additional assumption: the triangulation $\tau$ of $T$ is *convex*. This means that there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbf{R}$ whose domains of linearity coincide with the tetrahedra of $\tau$. Sometimes, such triangulations are also called coherent (see [GKZ]) or regular (see [Zi]).

**Theorem 2.1 (O. Viro).** *Under the assumptions made above on the triangulation* $\tau$ *of* $T$, *there exist a nonsingular real algebraic surface* $X$ *of degree* $m$ *in* $\mathbf{R}P^3$ *and a homeomorphism* $\mathbf{R}P^3 \to \tilde{T}$ *mapping the set of real points* $\mathbf{R}X$ *of* $X$ *onto* $\tilde{S}$.

Moreover, a polynomial defining the surface $X$ can be written down explicitly: if $t$ is positive and sufficiently small, the polynomial

$$\sum_{(i,j,l) \in V} \delta_{i,j,l} x_0^i x_1^j x_2^l x_3^{m-i-j-l} t^{\nu(i,j,l)}$$

(where $V$ is the set of vertices of $\tau$) defines a surface with the properties described in Theorem 2.1.

We consider two special types of triangulations of $T$. A triangulation $\tau$ of $T$ is called *primitive* if all the tetrahedra of $\tau$ are of volume $1/6$. A T-surface constructed using a primitive triangulation is called *primitive*.

A triangulation $\tau'$ of $T$ is called *maximal* if all the integer points of $T$ are vertices of $\tau'$. Clearly, any primitive triangulation is maximal. The notions of primitive and maximal triangulations coincide in dimension 2. The situation is different in dimension 3 : there exist maximal triangulations of $T$ which are not primitive.

## 3. EULER CHARACTERISTIC OF T-SURFACE

Let us consider a k-dimensional simplex $Q$ having vertices with integer coordinates and belonging to the orthant $\{x_i \geq 0\}$ of $\mathbf{R}^n$. We call the simplex $Q$ *elementary* if the reductions modulo 2 of the vertices of $Q$ are independent (generate an affine space of dimension $k$ over $\mathbf{Z}/2$).

Suppose that a distribution of signs at the vertices of the simplex $Q$ is given. Let us take the distributions of signs at the vertices of the symmetric copies of $Q$ using the following generalization of the rule formulated in section 2 :

> the symmetric copy of a vertex $a$ in an orthant $b$ gets the sign $(-1)^{\vec{a}\cdot\vec{b}}sign(a)$, where $\vec{a}$ is the reduction modulo 2 of the vertex $a$ ; the $i$-th coordinate of the vector $\vec{b}$ in $(\mathbf{Z}/2)^n$ is equal to 0 (resp. to 1) if $x_i > 0$ (resp. $x_i < 0$) for a point $(x_1, \dots, x_n)$ in the interior of the orthant $b$; and $\vec{a} \cdot \vec{b}$ denotes the standard scalar product of two vectors in $(\mathbf{Z}/2)^n$.

We call a symmetric copy of $Q$ *nonempty* if it has vertices of different signs.

**Proposition 3.1.** *If the simplex $Q$ is elementary and does not belong to a coordinate hyperplane, then $Q$ has exactly $2^n - 2^{n-k}$ nonempty symmetric copies.*

*Proof.* Let us, first, remark that the map $\vec{a} \mapsto \vec{a}\cdot\vec{b}$ is linear over $\mathbf{Z}/2$. The following operations do not change the property of any symmetric copy of $Q$ to be nonempty:

(1) parallel translation of $Q$, ( ? )

(2) changing of signs at all the vertices of $Q$.

Thus, we can suppose that the reduction $\bar{v}_0$ modulo 2 of a vertex $v_0$ of $Q$ is 0 in $(\mathbf{Z}/2)^n$, and that the vertex $v_0$ has the sign "+". Denote the other vertices of $Q$ and their reductions modulo 2 by $v_1, \ldots, v_k$ and $\bar{v}_1, \ldots, \bar{v}_k$, respectively. The condition that the copy of $Q$ in an orthant $b$ is empty (i. e. is not nonempty) can be expressed by a system of linear equations

$$\bar{v}_1 \cdot \bar{b} = \varepsilon_1, \ \ldots \ , \bar{v}_k \cdot \bar{b} = \varepsilon_k,$$

where $\varepsilon_i = 0$ if the sign of the vertex $v_i$ is positive, and $\varepsilon_i = 1$ if the sign of $v_i$ is negative. The unknowns of the system are the coordinates of $\bar{b}$. A solution to the system does exist because the rank of the system is equal to $k$ (the simplex $Q$ is elementary). Moreover, the dimension of the space of solutions is equal to $n - k$. It means that the number of solutions is equal to $2^{n-k}$, in other words, the simplex $Q$ has exactly $2^n - 2^{n-k}$ nonempty copies. $\square$

Proposition 3.1 is similar to Lemma 1 in [I-R].

Now we are able to calculate the Euler characteristic of a primitive T-surface.

**Theorem 3.2.** *If $X$ is a primitive T-surface in $\mathbf{R}P^3$, then the Euler characteristic $\chi(\mathbf{R}X)$ of the real point set of $X$ is equal to the signature $\sigma(\mathbf{C}X)$ of the complex point set of $X$. In other words, if $X$ is a primitive T-surface of degree $m$ in $\mathbf{R}P^3$, then*

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3}.$$

*Proof.* Let us take an arbitrary primitive triangulation $\tau$ of the tetrahedron $T$ and an arbitrary distribution of signs at the integer points of $T$. The piecewise-linear surface $\tilde{S}$ has a natural cell subdivision: each cell is the intersection of $\tilde{S}$ with a simplex of the triangulation of $\tilde{T}$.

All the simplices of $\tau$ are elementary. The number of simplices of $\tau$ of any dimension is fixed (the number of simplices of any dimension contained in each face of $T$ is also fixed). Thus, we can calculate the Euler characteristic of $\tilde{S}$ according to Proposition 3.1.

The triangulation $\tau$ contains

$m^3$ tetrahedra,

$2m^3 + 2m^2$ triangles, and $4m^2$ of them are contained in the facets of $T$,

$7m^3/6 + 3m^2 + 11m/6$ edges, $6m^2$ of them are contained in the facets of $T$,

and $6m$ of them are contained in the edges of $T$,

$(m+1)(m+2)(m+3)/6$ vertices.

We obtain that the described cell subdivision of $\tilde{S}$ contains $7m^3$ two-dimensional cells, $12m^3$ edges and $14m^3/3 + 4m/3$ vertices. Thus,

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3} = \sigma(\mathbf{C}X). \quad \square$$

**Theorem 3.3.** *If $X$ is a $T$-surface constructed using a maximal triangulation of the tetrahedron $T$, then $\chi(\mathbf{R}X) \geq \sigma(\mathbf{C}X)$.*

*Proof.* Let us, first, remark that all simplices of dimension $\leq 2$ of a maximal triangulation $\tau'$ of $T$ are elementary. Denote by $q$ the number of tetrahedra of $\tau'$. If any tetrahedron of $\tau'$ is elementary than, repeating the calculation of the proof of Theorem 3.2, we obtain $\chi(\tilde{S}) = 2m^3/3 - q + 4m/3$.

Each nonelementary tetrahedron of $\tau'$ has at least 6 nonempty copies, because the rank of the corresponding system of linear equations (see the proof of Proposition 3.1) is equal to 2. Thus,

$$\chi(\mathbf{R}X) = \chi(\tilde{S}) \geq \frac{2m^3}{3} - q + \frac{4m}{3} - q',$$

where $q'$ is the number of nonelementary tetrahedra of $\tau'$. It remains to remark that $q + q' \leq m^3$, and we obtain

$$\chi(\mathbf{R}X) \geq -\frac{m^3}{3} + \frac{4m}{3} = \sigma(\mathbf{C}X). \quad \square$$

## CASE OF PRIMITIVE OR MAXIMAL TRIANGULATION

As we saw in section 3, the Euler characteristic of a primitive $T$-surface in $\mathbf{R}P^3$ is determined by the degree and is equal to the signature $\sigma(\mathbf{C}X)$ of the complex point set of the surface.

For a real algebraic surface $X$ (or, more generally, for a real algebraic variety of any dimension), we have Smith inequality (see, for example, [Wi]) :

$$b_*(\mathbf{R}X) \leq b_*(\mathbf{C}X)$$

*[handwritten: Is this true for arbitrary triangulation ...]*

between the ranks of total $\mathbf{Z}/2$-homology groups of the real and of the complex point sets of $X$. If $b_*(\mathbf{R}X) = b_*(\mathbf{C}X)$, the surface $X$ is called *an M-surface*. We denote by $b_i(Y)$ the rank of $i$-th homology group of $Y$ with $\mathbf{Z}/2$-coefficients.

Let us mention two congruences (see [Wi]).

**Rokhlin congruence.** *If $X$ is an M-surface, then*

$$\chi(\mathbf{R}X) \equiv \sigma(\mathbf{C}X) \bmod 16.$$

**Kharlamov-Gudkov-Krahnov congruence.** *If $X$ is an (M-1)-surface (in other words, if $b_*(\mathbf{R}X) = b_*(\mathbf{C}X) - 2$), then*

$$\chi(\mathbf{R}X) \equiv \sigma(\mathbf{C}X) \pm 2 \bmod 16.$$

Rokhlin congruence and Theorem 3.2 show that we can expect to construct primitive T-surfaces which are M-surfaces. We will see in section 5 that such surfaces do really exist in any degree. On the other hand, **there are no (M-1)-surfaces among primitive T-surfaces in $\mathbf{R}P^3$ according to Kharlamov-Gudkov-Krahnov congruence and Theorem 3.2.**

**Theorem 4.1.** *If $X$ is a primitive T-surface in $\mathbf{R}P^3$ then*

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X), \quad b_0(\mathbf{R}X) \leq h^{2,0}(\mathbf{C}X) + 1.$$

**Remarks.** Theorem 4.1 states that Viro's conjecture holds in the case of primitive T-surfaces.

The inequality $b_0(\mathbf{R}X) \leq h^{2,0}(\mathbf{C}X) + 1$ for primitive T-surfaces was proved by E. Shustin in [Sh].

*Proof of Theorem 4.1.* Using the Smith inequality

*[handwritten: This is true because of regularity !!!]*

$$b_*(\mathbf{R}X) \leq b_*(\mathbf{C}X) = m^3 - 4m^2 + 6m$$

*[handwritten: The surface $\mathbb{R}X$ is a real]*

(where $m$ is the degree of $X$) and the equality

$$\chi(\mathbf{R}X) = \sigma(\mathbf{C}X) = -\frac{m^3}{3} + \frac{4m}{3}$$

proved in Theorem 3.2, we immediately obtain

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X) = \frac{2m^3}{3} - 2m^2 + \frac{7m}{3}$$

and

$$b_0(\mathbf{R}X) \leq h^{2,0}(\mathbf{C}X) + 1 = \frac{m^3}{6} - m^2 + \frac{11m}{6}.$$

Viro's conjecture also holds in the case of T-surfaces constructed using maximal triangulations.

**Theorem 4.2.** *If $X$ is a T-surface constructed using a maximal triangulation of the tetrahedron $T$, then*

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X).$$

*Proof.* The Smith inequality and the inequality $\chi(\mathbf{R}X) \geq \sigma(\mathbf{C}X)$ proved in Theorem 3.3, give again the desired inequality

$$b_1(\mathbf{R}X) \leq h^{1,1}(\mathbf{C}X). \quad \square$$

## 5. M-SURFACES

We describe, first, a special primitive triangulation $\rho$ of $T$ suggested by O. Viro. We show that the T-construction using the triangulation $\rho$ and an appropriate distribution of signs at the integer points of $T$ gives an M-surface of degree $m$ in $\mathbf{R}P^3$. In fact, the surfaces given by the procedure described below are homeomorphic to ones constructed (not as T-surfaces) by O. Viro in [V1].

Let us divide the tetrahedron $T$ by the planes $z = l$, and denote by $P_l$ the polytope

$$\{(x, y, z) \in T \ : \ l \leq z \leq l + 1, \quad l = 0, \dots, m - 1\}.$$

Choose an arbitrary primitive convex triangulation of each triangle

$$T_l = T \cap \{z = l\}, \quad l = 0, \dots, m - 1$$

(a triangulation of the triangle $T_l$ is called *primitive* if all its triangles are of area $1/2$, or, equivalently, if all the integer points of $T_l$ are vertices of the triangulation).

Each polytope $P_l$ is triangulated as follows. If $l$ is even, take the join $J_l$ of the side of $T_l$ lying in the $xz$-coordinate plane and of the side of $T_{l+1}$ lying in the plane $x + y + z = m$. If $l$ is odd, take as $J_l$ the join of the side of $T_l$ lying in the plane $x + y + z = m$ and of the side of $T_{l+1}$ lying in the $xz$-coordinate plane. The join $J_l$ is naturally triangulated into the joins of segments

$$[(i,0,l),(i+1,0,l)],\quad [(m-(l+1)-j,j,l+1),(m-(l+1)-(j+1),j+1,l+1)],$$

$$i = 0,\dots,m-l-1,\quad j = 0,\dots,m-l-2$$

if $l$ is even, and $J_l$ is triangulated into the joins of segments

$$[(m-l-j,j,l),(m-l-(j+1),j+1,l)],\quad [(i,0,l+1),(i+1,0,l+1)],$$

$$i = 0,\dots,m-l-2,\quad j = 0,\dots,m-l-1$$

if $l$ is odd.

The polytope $P_l$ is the union of $J_l$ and of two tetrahedra $P_l^{(1)}$ and $P_l^{(2)}$. These tetrahedra can be triangulated into the cones over the triangles of the chosen triangulations of $T_l$ and of $T_{l+1}$.

Clearly, the described triangulation $\rho$ of $T$ is primitive. To explain that $\rho$ is convex, consider a triangulation of $T$ formed by the tetrahedra

$$J_l,\ P_l^{(1)},\ P_l^{(2)}\quad (l = 0,\dots,m-1).$$

The later triangulation is convex. Let $\nu' : T \to \mathbf{R}$ be a convex function certifying the convexity of this triangulation, and let $\nu_l : T_l \to \mathbf{R}$ $(l = 0,\dots,m-1)$ be a convex function certifying that the chosen triangulation of $T_l$ is convex. Consider a piecewise-linear function $\nu : T \to \mathbf{R}$ which is linear on each tetrahedron of $\rho$ and takes the value $\nu'(r_l) + \varepsilon\nu_l(r_l)$ at an integer point $r_l$ of $T_l$. It is easy to see that the function $\nu$ for a positive sufficiently small $\varepsilon$ certifies the convexity of $\rho$.

Choose the following distribution of signs at the integer points of $T$ :

a point $(i,j,l)$ gets the sign "+" if $i \equiv j \equiv l \equiv 0 \bmod 2$ or

$l \equiv 1 \bmod 2$ and $ij \equiv 0 \bmod 2$;

and it gets the sign "-" otherwise.

**Proposition 5.1.** *A T-surface $X$ constructed using the triangulation $\rho$ and the distribution of signs described is an M-surface. The real point set $\mathbf{R}X$ of $X$ is homeomorphic to the disjoint union of $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1$ spheres and a sphere with $\frac{m^3}{3} - m^2 + \frac{7m}{6}$ handles if $m$ is even or a projective plane with $\frac{m^3}{3} - m^2 + \frac{7m-3}{6}$ handles if $m$ is odd.*

*Proof.* It is easy to verify that any integer point $r$ lying strongly inside $T$ has a symmetric copy $s(r)$ with the following property : all the neighbouring vertices of $s(r)$ (i. e. vertices connected with $s(r)$ by an edge of the triangulation) have the same sign, and this sign is opposite to the sign of $s(r)$. It means that the surface $\tilde{S}$ has a connected component homeomorphic to a sphere contained in the star of $s(r)$.

We found $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 = h^{2,0}(\mathbf{C}X)$ components of $\tilde{S}$. There is at least one component of $\tilde{S}$ more, because the surface $\tilde{S}$ intersects the coordinate planes. On the other hand, according to Theorem 4.1, the number of connected components of $\mathbf{R}X$ does not exceed $h^{2,0}(\mathbf{C}X) + 1$. Thus, the real point set $\mathbf{R}X$ has exactly $h^{2,0}(\mathbf{C}X) + 1$ connected components.

Using the equalities

$$\chi(\mathbf{R}X) = \sigma(\mathbf{C}X), \quad b_0(\mathbf{R}X) = h^{2,0}(\mathbf{C}X) + 1,$$

we get $b_*(\mathbf{R}X) = b_*(\mathbf{C}X)$, i. e. $X$ is an M-surface. Furthermore,

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X),$$

and, thus, the topological type of $\mathbf{R}X$ coincides with one described in the statement of Proposition.    $\square$

## 6.  Counter-examples to Viro's conjecture

We saw in section 4 that Viro's conjecture is true for T-surfaces constructed using a maximal triangulation. Surprisingly enough, a nonmaximal triangulation of $T$ can produce a T-surface $X$ in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) > h^{1,1}(\mathbf{C}X)$.

Let us describe, first, the construction of *an extension* of a triangulation of the triangle $T_0 = T \cap \{z = 0\}$.

Suppose that $m$ is even and that a primitive triangulation $\tau_0$ of $T_0$ with the vertices having integer coordinates is given. Divide the tetrahedron $T$ into two parts $T \cap \{z \geq 2\}$ and $T \cap \{z \leq 2\}$ by the plane $z = 2$. Take in the first part the triangulation coinciding with the triangulation $\rho$ described in the construction of M-surfaces.

Divide now the second part $T \cap \{z \leq 2\}$ by the plane $x + y + kz = m$ (where $m = 2k$) into the tetrahedron $\bar{T}$ with vertices $(0,0,0)$, $(m,0,0)$, $(0,m,0)$, $(0,0,2)$ and the cone $C$ with the vertex $(0,0,2)$ over

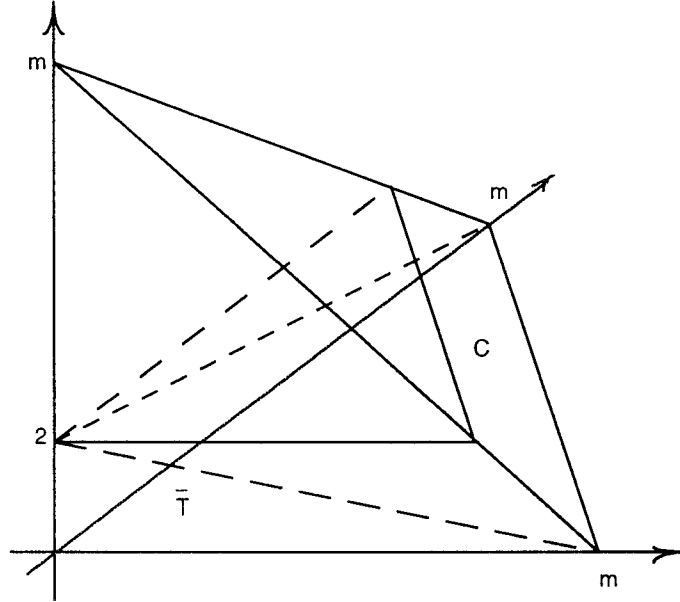$$\{(x,y,z) \in T \ : \ x + y + z = m, \ 0 \leq z \leq 2\}$$

(see Figure 1).



Figure 1

To triangulate the tetrahedron $\bar{T}$, we take the cones over all the triangles of $\tau_0$, and subdivide (in the unique possible way) the cones containing integer points of the plane $z = 1$ in order to obtain a maximal triangulation of $\bar{T}$.

To describe the triangulation of the cone $C$, let us consider the cone $\hat{C}$ with the vertex $(k + 1, 0, 1)$ over the triangle $T \cap \{x + y + kz = m\}$. The rest of the cone

$C$ is divided into two parts by the plane $z = 1$ (see Figure 2). Denote the lower part (contained in $C \cap \{0 \le z \le 1\}$) by $C_0$, and denote the upper part (contained in $C \cap \{1 \le z \le 2\}$) by $C_1$.
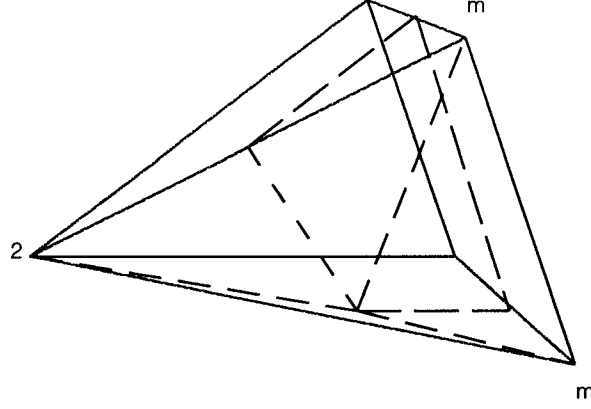


Figure 2

The triangulation of the triangle $T \cap \{x + y + kz = m\}$ is already fixed (it comes from the triangulation of $\bar{T}$). Thus, we can triangulate the cone $\hat{C}$ by the cones with the vertex $(k + 1, 0, 1)$ over the triangles of the triangulation of $T \cap \{x + y + kz = m\}$.

Subdivide $C_0$ taking the cone $C'$ with the vertex $(0, m, 0)$ over the facet of $C_0$ belonging to the plane $z = 1$, and the join $J'$ of segments $[(m, 0, 0), (0, m, 0)]$ and $[(k + 1, 0, 1), (m - 1, 0, 1)]$. Let us choose an arbitrary primitive convex triangulation of the quadrangle $C_0 \cap \{z = 1\}$. It gives a natural primitive triangulation of $C'$ (taking the cones over the triangles of the chosen triangulation of $C_0 \cap \{z = 1\}$). The join $J'$ is triangulated by the joins of segments $[(m - j, j, 0), (m - j - 1, j + 1, 0)]$ and $[(i, 0, 1), (i + 1, 0, 1)]$ (where $i = k + 1, \ldots, m - 2$ ; $j = 0, \ldots, m - 1$).

It remains to triangulate the part $C_1$. Subdivide $C_1$ into the join of segments $[(m - 1, 0, 1), (0, m - 1, 1)]$ and $[(0, 0, 2), (m - 2, 0, 2)]$ (triangulated by the joins of segments $[(m - j - 1, j, 1), (m - j - 2, j + 1, 1)]$ and $[(i, 0, 2), (i + 1, 0, 2)]$, where $i = 0, \ldots, m - 3$ ; $j = 0, \ldots, m - 2$) and the naturally triangulated cones : with the vertex $(0, 0, 2)$ (resp. $(0, m - 1, 1)$) over the quadrangle $C_1 \cap \{z = 1\}$ (resp. over

the triangle $T_2 = T \cap \{z = 2\}$).

The described maximal triangulation of $T$ is called *the extension* of the triangulation $\tau_0$ and is denoted by $ext(\tau_0)$.

Arguments, similar to ones used in the previous section to show that the triangulation $\rho$ is convex, prove that if $\tau_0$ is convex then $ext(\tau_0)$ is also convex. Almost all tetrahedra of $ext(\tau_0)$ are of volume $1/6$. The only tetrahedra of a greater volume (more precisely, of volume $1/3$) are the cones with the vertex $(0, 0, 2)$ over the odd triangles of $\tau_0$ (we call a triangle of $\tau_0$ *odd* if it does not have a vertex with the both even coordinates).

Suppose now that a distribution $\delta_0$ of signs at the integer points of $T_0$ is given. Let us describe a distribution $ext(\delta_0)$ of signs at the integer points of $T$ which we call *an extension* of $\delta_0$. In the part $T \cap \{z \geq 2\}$ we take the distribution of signs described in the construction of M-surfaces. It remains, thus, to fix a distribution of signs at the integer points of $T \cap \{z = 1\}$. We do it as follows:

> take an arbitrary distribution in $T \cap \{z = 1\} \cap \{x + y < k\}$,
>
> all the integer points of the segment $[(k, 0, 1), (0, k, 1)]$ but the point $(0, k, 1)$ get the sign "-",
>
> for the other points of $T_1$ we apply the rule : a point $(i, j, 1)$ gets the sign "-" if $i$ and $j$ are both odd, and the sign "+" otherwise.

Let us take a triangulation $\tau_0^1$ and a distribution $\delta_0^1$ of signs at the integer points of $T_0$ producing a counter-example to Ragsdale conjecture with $p = \frac{3k^2 - 3k + 2}{2} + 1$ (see [I1], [I2], [I-V]). The triangulation $\tau_0^1$ can be obtained placing the hexagon $H$ shown in Figure 3 inside of $T_0$ (on suppose that $m \geq 10$) in such a way that the center of $H$ has both the nonzero coordinates odd, and extending, then, the triangulation of $H$ to a primitive convex triangulation of $T_0$. To obtain a distribution of signs at the integer points of $T_0$, we complete the distribution presented in Figure 3 by the rule :

> a point $(i, j, 0)$ gets the sign "-" if $i$ and $j$ are even, and $i + j < m$,
>
> a point $(i, j, 0)$ gets the sign "+" otherwise.

Remark that this distribution of signs at the integer points of $T_0$ is slightly different
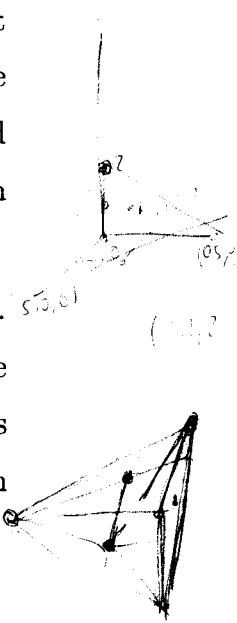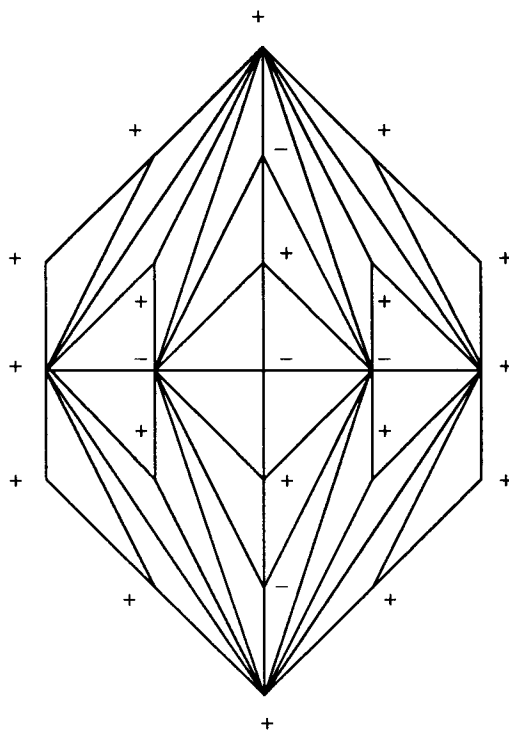
form the distribution described in [I1], [I2], [I-V].

**Figure 3**

**Proposition 6.1.** *The maximal triangulation* $ext(\tau_0^1)$ *and a distribution of signs* $ext(\delta_0^1)$ *produce a T-surface $X$ of degree $m$ in* $\mathbf{R}P^3$ *with*

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3}, \quad b_0(\mathbf{R}X) = h^{2,0}(\mathbf{C}X) - 2.$$

*The real point set* $\mathbf{R}X$ *of $X$ is homeomorphic to the disjoint union*

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5\right) S^2 \coprod S_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}$$

*of* $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5$ *spheres,    a sphere with 2 handles    and a sphere with* $\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5$ *handles.*

*Proof.* Let us, first, calculate $\chi(\mathbf{R}X)$. It was already remarked that almost all tetrahedra of $ext(\tau_0^1)$ are of volume $1/6$. The only tetrahedra of greater volume (of

volume $1/3$) are the cones over the odd triangles of $\tau_0^1$. Each of these tetrahedra of volume $1/3$ has 6 nonempty symmetric copies (a tetrahedron of volume $1/3$ of a maximal triangulation has 6 nonempty copies if the product of signs at its vertices is positive, and it has 8 nonempty copies if the product of signs is negative). Thus, the arguments of the proof of Theorems 3.2 and 3.3 show that $\chi(\mathbf{R}X) = \sigma(\mathbf{C}X)$.

Calculate now the number of connected components of $\tilde{S}$. Exactly as in the proof of Theorem 5.1, any integer point lying strongly inside $(T \cap \{z \geq 2\}) \cup C$ has a symmetric copy with the star containing a component of $\tilde{S}$ homeomorphic to a sphere. It is easy to see that the stars of integer points lying strongly inside $T$ and belonging to the segment $[(k,0,1),(0,k,1)]$ also contain the components of $\tilde{S}$ homeomorphic to a sphere. Consider the integer points lying strongly inside the tetrahedron $\bar{T}$. Let us call *even interior points of $T_0$* the integer points $(i,j,0)$ such that $i > 0$, $j > 0$, $i + j < m$, $i$ and $j$ are both even. There is a correspondence between the even interior points of $T_0$ and the points of $Int(\bar{T}) \cap \mathbf{Z}^3$ : any integer point lying strongly inside $\bar{T}$ is a middle point of a segment joining the point $(0,0,2)$ and an even interior point of $T_0$. We denote the middle point of a segment $[(0,0,2),r]$ (where $r$ is an even interior point of $T_0$) by $f(r)$.

Suppose that an even interior point $r$ does not belong to the hexagon $H$. Then $r$ has the sign "-". If $f(r)$ has also the sign "-", then the union of stars of $r$ and of $f(r)$ (in the triangulation of $T_*$) contains a component of $\tilde{S}$ homeomorphic to a sphere. If $f(r)$ has the sign "+", then the union of stars of $r$ and of $s_z(f(r))$ contains a component of $\tilde{S}$ homeomorphic to a sphere.

We have found $h^{2,0}(\mathbf{C}X) - 4$ spheres of $\tilde{S}$ (a sphere was associated to any integer point lying strongly inside of $T$ except 4 points of the form $f(r)$, where $r$ is an even interior point of $T_0$ belonging to the hexagon $H$). There are two connected components of $\tilde{S}$ more. One component is homeomorphic to a sphere with two handles and lies inside of $\bar{H} \cup s_z(\bar{H})$, where $\bar{H}$ is a cone with the vertex $(0,0,2)$ over $H$. The remaining part of $\tilde{S}$ is connected. The number $b_1(\tilde{S})$ can be calculated via the Euler characteristic.    $\square$

**Theorem 6.2.** *If $m$ is an even integer number not less than 10, then there exists an (M-2)-surface $X$ of degree $m$ in $\mathbf{R}P^3$ such that $b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2$.*

*Proof.* Let us take the triangulation $ext(\tau_0^1)$ of $T$ and the distribution of signs $ext(\delta_0^1)$ at the integer points of $T$. According to Proposition 6.1 the resulting surface $\tilde{S}$ is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5\right) S^2 \coprod S_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}.$$

Remove now 4 vertices of the form $f(r)$, where $r$ is an even interior point of $T_0$ belonging to $H$ (see the proof of Proposition 6.1), with all the adjacent edges. Denote the new triangulation (which is nonmaximal) by $ext'(\tau_0^1)$ and consider the surface $\tilde{S}'$ constructed using $ext'(\tau_0^1)$ and the restriction $ext'(\delta_0^1)$ of the distribution $ext(\delta_0^1)$ to the set of vertices of $ext'(\tau_0^1)$. Clearly, the surface $\tilde{S}'$ is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5\right) S^2 \coprod S_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 1}$$

because we added 4 handles to the component homeomorphic to $S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}$. Thus, the number of $b_1(\tilde{S}')$ is equal to

$$\frac{2m^3}{3} - 2m^2 + \frac{7m}{3} + 2. \quad \square$$

Using counter-examples of degree $2k$ to Ragsdale conjecture with more than $\frac{3k^2 - 3k + 2}{2} + 1$ even ovals (see [I1], [I2], [I-V]), one can construct surfaces $X$ of degree $2k$ in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) > h^{1,1}(\mathbf{C}X) + 2$.

**Theorem 6.3.** *If* $m = 2k$ *is an even integer not less than 10, then there exists a surface* $X$ *of degree* $m$ *in* $\mathbf{R}P^3$ *such that*

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2\left[\frac{(k-3)^2 + 4}{8}\right]$$

*(where [u] denotes the greatest integer which does not exceed u).*

*Proof.* We start from a triangulation $\tau_0^a$ and a distribution $\delta_0^a$ of signs at the integer points of $T_0$ giving a counter-example to Ragsdale conjecture with

$$p = \frac{3k^2 - 3k + 2}{2} + a,$$

where $a = \left[\frac{(k-3)^2 + 4}{8}\right]$ (see [I1], [I2], [I-V]). The triangulation $\tau_0^a$ can be obtained in the following way. Consider the partition of the triangle $T_0$ shown in Figure 4. Let

us take in each shadowed hexagon the triangulation (and the signs) of the hexagon $H$. The triangulation of the union of the shadowed hexagons can be extended to a primitive convex triangulation $\tau_0^a$ of $T_0$. To obtain the distribution $\delta_0^a$ of signs at the integer points of $T_0$, we choose the signs outside of the union of the shadowed hexagons again using the rule :

a point $(i,j,0)$ gets the sign "-" if $i$ and $j$ are even, and $i + j < m$,
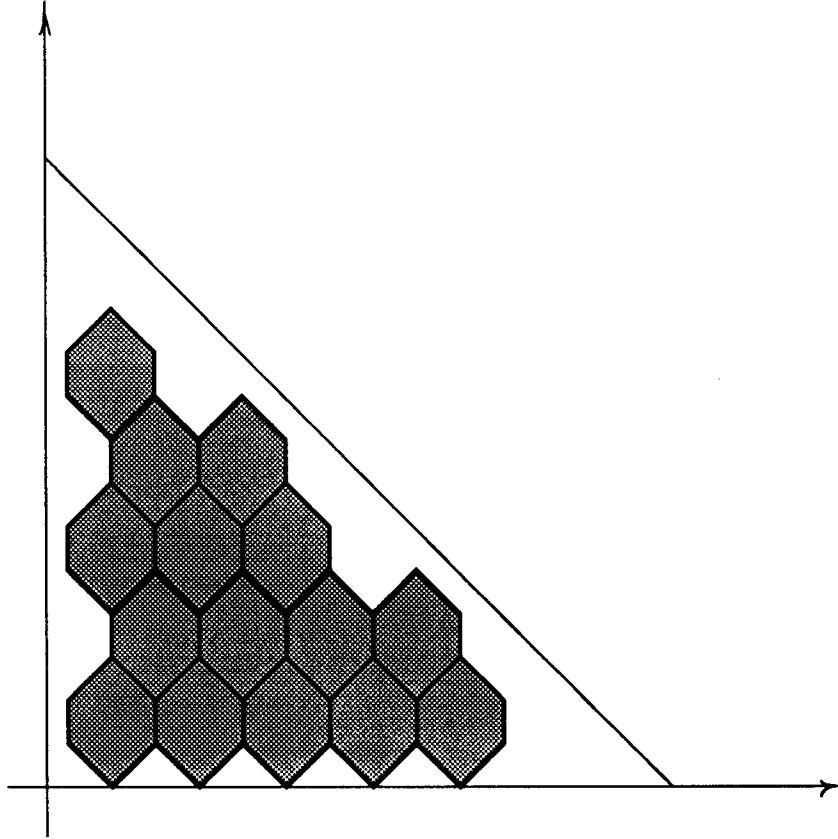
a point $(i,j,0)$ gets the sign "+" otherwise.



Figure 4

Consider the triangulation $ext(\tau_0^a)$ of $T$ and the distribution $ext(\delta_0^a)$ of signs at the integer points of $T$. The resulting surface $\tilde{S}$ is homeomorphic to

$$\left( \frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a \right) S^2 \coprod aS_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5a}.$$

Remove now the vertices of the triangulation $ext(\tau_0^a)$ (with adjacent edges) of the

form $f(r)$, where $r$ is an even interior point of $T_0$ belonging to one of the shadowed hexagons, and take the restriction $ext'(\delta_0^a)$ of the distribution $ext(\delta_0^a)$ to the vertices of the new triangulation $ext'(\tau_0^a)$. We obtain a surface $\tilde{S}'$ homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \coprod aS_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - a}$$

with

$$b_1(\tilde{S}') = \frac{2m^3}{3} - 2m^2 + \frac{7m}{3} + 2a. \quad \square$$

**Remarks.**

1. Removing, if necessary, some of the shadowed hexagons in the construction of Theorem 6.3, we get counter-examples to Viro's conjecture with the real point set homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \coprod aS_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - a},$$

where $a = 1, \ldots, \left[\frac{(k-3)^2 + 4}{8}\right]$.

2. The counter-example of the smallest degree in $\mathbb{R}P^3$ given by Theorems 6.2 and 6.3 is a surface of degree 10. The real point set of this surface is homeomorphic to

$$80S^2 \coprod S_2 \coprod S_{244}.$$

It is unknown if there exist counter-examples of degree less than 10. The smallest degree we can expect for a counter-example to Viro's conjecture is degree 5.

3. Repeating the procedure described above for the new counter-examples to the Ragsdale conjecture constructed by B. Haas [Ha], one can construct surfaces $X$ of degree $2k$ in $\mathbb{R}P^3$ with

$$b_1(\mathbb{R}X) = h^{1,1}(\mathbb{C}X) + 2a',$$

where $a' = \left[\frac{k^2 - 7k + 16}{6}\right]$.

4. We can obtain counter-examples to Viro's conjecture which are asymptotically better than the examples described above: there exist T-surfaces $X$ of degree $2k$ in $\mathbb{R}P^3$ with $b_1(\mathbb{R}X) = h^{1,1}(\mathbb{C}X) + 2A$, where $A = k^3/24+$ terms of smaller degrees. To construct such surfaces, we divide the tetrahedron $T$ by the planes $z = 2l$ (where $l = 1, \ldots, k - 1$), and define a triangulation and a distribution of signs in each part of the subdivision using the procedure described in the proof of Theorem 6.3 for $T \cap \{0 \leq z \leq 2\}$.

## References

[GKZ]  I. Gelfand, M. Kapranov and A. Zelevinski, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.

[Ha]  B. Haas, *Les multilucarnes: Nouveaux contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris (to appear).

[I1]  I. Itenberg, *Contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris, Ser. 1. **317** (1993), 277-282.

[I2]  I. Itenberg, *Counter-examples to Ragsdale Conjecture and T-curves*, Contemporary Mathematics **182** (1995), 55-72.

[I-R]  I. Itenberg, M.-F. Roy, *Multivariate Descartes' rule*, Beiträge zur Algebra und Geometrie (to appear).

[I-V]  I. Itenberg, O. Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Preprint, Uppsala University, 1995.

[Kh]  V. Kharlamov, *Real algebraic surfaces*, Proc. Internat. Congress Math., vol. 1, Helsinki, 1978, pp. 421-428. (Russian)

[Ra]  V. Ragsdale, *On the arrangement of the real branches of plane algebaric curves*, Amer. J. Math. **28** (1906), 377-404.

[Ri]  J.-J. Risler, *Construction d'hypersurfaces réelles [d'après Viro]*, Séminaire N.Bourbaki, no. 763, vol. 1992-93, Novembre 1992.

[Sh]  E. Shustin, *Critical points of real polynomials, subdivisions of Newton polyhedra and topology of real algebraic hypersurfaces*, Amer. Math. Soc. Transl. (2) **173** (1996), 203-223.

[Si]  R. Silhol, *Real Algebraic Surfaces*, Lect. Notes in Math., vol. 1392, Springer-Verlag, Berlin Heidelberg, 1989.

[V1]  O. Viro, *Construction of M-surfaces*, Functional Anal. Appl. **13** (1979).

[V2]  O. Viro, *Gluing of algebraic hypersurfaces, smoothing of singularities and construction of curves*, Proc. Leningrad Int. Topological Conf. (Leningrad, Aug. 1983), Nauka, Leningrad, 1983, pp. 149-197. (Russian)

[V3]  O. Viro, *Gluing of plane real algebraic curves and construction of curves of degrees 6 and 7*, Lect. Notes in Math., vol. 1060, Springer-Verlag, Berlin Heidelberg, 1984, pp. 187-200.

[V4]  O. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Rus. Math. Surv. **41** (1986), no. 3, 55-82.

[V5]  O. Viro, *Real algebraic plane curves: constructions with controlled topology*, Leningrad Math. J. **1** (1990), 1059-1134.

[V6]  O. Viro, *Patchworking real algebraic varieties*, Preprint, Uppsala University, 1994.

[Wi]  G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), no. 1, 53-73.

[Zi]  G. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York, 1994.

INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES, CAMPUS DE BEAULIEU 35042 RENNES CEDEX FRANCE. *E-mail:* ILIA.ITENBERG@UNIV-RENNES1.FR