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Counter-examples to Ragsdale Conjecture and T-curves

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ABSTRACT. The paper is devoted to the construction of counter-examples to the Ragsdale conjecture, an old and an important conjecture in the topology of real algebraic curves. The constructed counter-examples are T-curves, i.e. curves which can be obtained by a combinatorial procedure based on Viro's method of construction of real algebraic varieties with prescribed topology. T-curves are also used in the paper to obtain a classification of M-curves of degree $4l+2$ with one non-empty oval, and to show the sharpness of the second Petrovsky inequality.

1. Introduction

In 1906 V. Ragsdale [8] analyzing the results of Harnack's and Hilbert's constructions proposed an important conjecture on the topology of real algebraic curves. To formulate it let us consider a real algebraic plane projective curve of even degree $m = 2k$, i.e. a real homogeneous polynomial of degree $2k$ in three variables defined up to multiplication by a non-zero real number. We suppose a curve to be non-singular, which means that a polynomial does not have singular points in $\mathbf{R}^3 \setminus 0$.

Such a curve A has a well defined zero locus $\mathbf{R}A$ in the real projective plane $\mathbf{R}P^2$. The set $\mathbf{R}A$ is a union of non-intersecting circles embedded in $\mathbf{R}P^2$. The topological type of the pair $(\mathbf{R}P^2, \mathbf{R}A)$ is defined by the scheme of disposition of the components of $\mathbf{R}A$. This scheme is called *the real scheme of curve A*.

The real point set $\mathbf{R}A$ of the curve A divides the real projective plane $\mathbf{R}P^2$ in two parts with a common boundary $\mathbf{R}A$ (these parts are the subsets of $\mathbf{R}P^2$ where a polynomial has positive or, respectively, negative values). One of these parts is non-orientable, we will denote it by $\mathbf{R}P^2_-$. The other one will be denoted by $\mathbf{R}P^2_+$.

The topology of $\mathbf{R}P^2_-$ and $\mathbf{R}P^2_+$ is closely connected with the topological type of the pair $(\mathbf{R}P^2, \mathbf{R}A)$. Let p be the number of connected components of

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The question on the sharp upper bound for the numbers p, n of curves of even degrees is still open. One can obtain an upper bound using Harnack and Petrovsky inequalities. However, the known examples are far from this estimation.

The counter-examples constructed in this paper are T-curves, i.e. the curves which can be obtained by a combinatorial procedure based on Viro's method of construction of real algebraic varieties (see, for example, [9], [10], [12], [13], [15]). The definition of T-curves is given in the section 2, some of their properties are discussed in the section 7.

We give two other applications of T-curves : classification of M-curves (i.e. curves having the maximal possible number of connected components of the real point set for a given degree) of degree $4l+2$ with one non-empty oval (in the section 5, see also [4]), and construction of examples of curves showing the sharpness of the second Petrovsky inequality (in the section 6).

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2. T-curves

Let m be a positive integer number and T be the triangle

$$\{(x,\ y) \in \mathbb{R}^2,\ \ x \geq 0,\ \ y \geq 0,\ \ x+y \leq m\}.$$

Suppose that T is triangulated in such a way that the vertices of the triangles are integer, and that some distribution of signs, $a_{i,j} = \pm$ at the vertices of the triangulation, is given. Then there arises a naturally associated piecewise-linear curve L in $\mathbf{R}P^2$.

The construction of L is the following.

Take copies $T_x = s_x(T)$, $T_y = s_y(T)$, $T_{xy} = s(T)$ of T, where $s = s_x \circ s_y$ and s_x, s_y are reflections with respect to the coordinate axes. Extend the triangulation of T to a symmetric triangulation of $T \cup T_x \cup T_y \cup T_{xy}$ and extend the distribution of signs to a distribution at the vertices of the extended triangulation which verifies the modular property: $g^*(a_{i,j}x^iy^j) = a_{g(i,j)}x^iy^j$ for $g = s_x,\ s_y,\ s$ (other words, the sign at a vertex is the sign of the corresponding monomial in the quadrant containing the vertex).

If a triangle of the triangulation has vertices of different signs, select a midline separating them. Denote by L' the union of the selected midlines. It is contained in $T \cup T_x \cup T_y \cup T_{xy}$. Glue by s the sides of $T \cup T_x \cup T_y \cup T_{xy}$. The resulting space T_* is homeomorphic to $\mathbf{R}P^2$. Let us take the curve L to be the image of L' in T_*.

A pair (T_*, L) is called *a chart* of a real algebraic plane projective curve A, if there exists a homeomorphism of pairs $(T_*, L) \longrightarrow (\mathbf{R}P^2, \mathbf{R}A)$.

Let us introduce two additional assumptions : the considered triangulation of T is *primitive* and *convex*. The first condition means that all triangles are of area $1/2$ (or, equivalently, that all integer points of T are vertices of the triangulation). The second one means that there exists a convex piecewise-linear function $T \longrightarrow \mathbf{R}$ which is linear on each triangle of the triangulation and not linear on the union of two triangles.

The following statement is the special case of Viro's theorem [13, Th. 1.4].

$\mathbf{R}P^2_+$, and $n+1$ be the number of connected components of $\mathbf{R}P^2_-$ (exactly one component of $\mathbf{R}P^2_-$ is non-orientable).

The numbers p and n can be described in another way. Each connected component of the real point set $\mathbf{R}A$ of a curve of even degree is called *an oval.* It divides $\mathbf{R}P^2$ in two parts. We call the part homeomorphic to a disk *the interior* of the oval.

An oval of a curve is called *even* (resp. *odd*) if it lies inside of an even (resp. odd) number of other ovals of this curve.

It is easy to see that p is the number of even ovals of a curve, and n is the number of odd ovals.

The statement of the Ragsdale conjecture is the following : for any curve of degree $2k$

$$p \leq \frac{3k^2-3k+2}{2}, \quad n+1 \leq \frac{3k^2-3k+2}{2}$$

Ragsdale also proposed the other conjecture :

$$p-n \leq \frac{3k^2-3k+2}{2}, \quad n-p+1 \leq \frac{3k^2-3k+2}{2}$$

So, the first Ragsdale conjecture is a statement on the maximal possible number of connected components of $\mathbf{R}P^2_+$ and $\mathbf{R}P^2_-$. The second conjecture is a statement on the maximal value of Euler characteristic of $\mathbf{R}P^2_+$ and $\mathbf{R}P^2_-$.

In 1938 I. Petrovsky [7] proved the second Ragsdale conjecture (the inequalities of this conjecture are called now the Petrovsky inequalities) and also proposed a conjecture similar to the first one :

$$p \leq \frac{3k^2-3k+2}{2}, \quad n \leq \frac{3k^2-3k+2}{2}$$

In 1980 O. Viro [11] constructed curves of degree $2k$ with $n = \frac{3k^2-3k+2}{2}$ for any even $k \geq 4$. These curves are counter-examples to the original Ragsdale conjecture, but not to the conjecture of Petrovsky.

The following theorem gives counter-examples to the "corrected" Ragsdale conjecture (or to the conjecture of Petrovsky) (see also [3], [4]).

THEOREM 1.1. *For any integer number $k \geq 1$*

a) there exists a non-singular real algebraic plane projective curve of degree $2k$ with

$$p = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right]$$

(where $[a]$ denotes the maximal integer not greater than a),

b) there exists a non-singular real algebraic plane projective curve of degree $2k$ with

$$n = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right] - 1$$

The proof of this theorem will be given in the section 4.

THEOREM 2.1 (O. VIRO). *Under the assumptions made above on the triangulation of the triangle T, there exists a non-singular real algebraic plane projective curve A of degree m with the chart (T_*, L).*

A curve having the chart (T_*, L) is called *a T-curve*. This notion was introduced by S. Orevkov [6].

3. Construction of Harnack curves

Recall that an M-curve is a curve having the maximal possible number of connected components of the real point set for a given degree. Harnack [2] proved that this maximal number is equal to $\frac{(m-1)(m-2)}{2} + 1$ for the degree m.

In this section we will describe, using Theorem 2.1, the construction of some M-curves (a special case of Harnack curves). This construction will play an important role in the sections 4, 5, 6.

Let $m = 2k$ be a positive even number, and T again be the triangle

$$\{(x,\ y) \in \mathbb{R}^2,\ \ x \geq 0,\ \ y \geq 0,\ \ x + y \leq m\}.$$

An integer point (i, j) of T is called *even*, if i, j are both even, and *odd* if not.

The following distribution of signs at the integer points of the triangle T is called *Harnack distribution* :

even points get sign "-", and odd points get sign "+".

We use the system of notations for the real schemes of non-singular curves suggested by Viro [11]. The scheme consisting of a single oval is denoted by the symbol $< 1 >$, the empty scheme - by the symbol $< 0 >$. If a symbol $< A >$ stands for some set of ovals, then the set of ovals obtained by addition of an oval surrounding all old ovals is denoted by $< 1 < A >>$. If a scheme is the union of two non-intersecting sets of ovals denoted by $< A >$ and $< B >$ respectively with no oval of one set surrounding an oval of the other set, then this scheme is denoted by the symbol $< A \cup B >$. Besides, if A is the notation for some set of ovals then a part $A \cup \ \ldots\ \cup A$ of another notation where A repeats n times is denoted by $n \times A$; a part $n \times 1$ is denoted by n.

PROPOSITION 3.1. *An arbitrary primitive convex triangulation of T with the Harnack distribution of signs at the vertices produces a T-curve which is an M-curve of degree $m = 2k$ with the real scheme*

$$< \frac{3k^2 - 3k}{2} \cup 1 < \frac{(k-1)(k-2)}{2} >>$$

PROOF. Let us, first, notice that the number of interior (i.e. lying strongly inside of the triangle T) integer points is equal to $\frac{(m-1)(m-2)}{2}$, the number of even interior points is equal to $\frac{(k-1)(k-2)}{2}$, and the number of odd interior points is equal to $\frac{3k^2-3k}{2}$.

If an interior vertex (i,j) is even, then its star $St(i,j)$ contains an oval of the curve L (this oval surrounds "-"). If an interior vertex (i,j) is odd, then one of the symmetric copies of the star $St(i,j)$ (namely, the copy in the quadrant $\{(x,y) \mid sign(x) = (-1)^j,\ sign(y) = (-1)^i\}$) contains an oval of L (this oval surrounds "+").

We have found $\frac{(m-1)(m-2)}{2}$ ovals and, thus, the curve L can have only one oval more. This oval exists, because, for example, the curve L intersects the coordinate axes.

To finish the proof it remains to notice that the union of the segments

$$\{x - y = -m,\ -m \le x, y \le m\} \ \cup$$

$$\{x \le 0,\ y = 0,\ -m \le x, y \le m\} \ \cup \ \{x = 0,\ y \le 0,\ -m \le x, y \le m\}$$

is not contractable in T_* and contains only the signs "-". It means that $\frac{3k^2-3k}{2}$ ovals corresponding to odd interior points and containing the sign "+" inside of them are situated outside of the non-empty oval. □

4. Construction of counter-examples to Ragsdale conjecture

Now we are able to describe a construction of counter-examples to Ragsdale conjecture.

PROOF OF THEOREM 1.1. We will construct T-curves with the stated properties. Let us show, first, how to construct a curve of degree $m = 2k$ with $p = \frac{3k^2-3k+2}{2} + 1$.

Suppose that the hexagon S shown in Figure 1 is placed inside of the triangle $T = \{x \ge 0,\quad y \ge 0,\quad x + y \le m\}$ in such a way that the center of S has both coordinates odd. Any convex primitive triangulation of a convex part of a convex polygon is extendable to a convex primitive triangulation of the polygon. Let us extend to T the convex primitive triangulation of S. Extend also the distribution of signs using the Harnack distribution outside of S.

The corresponding piecewise-linear curve L has $\frac{3k^2-3k+2}{2} + 1$ even ovals, and its real scheme is the following :

$$< \frac{3k^2 - 3k - 2}{2} \cup 1 < \frac{(k-1)(k-2) - 8}{2} \cup 1 < 2 >>>$$

Suppose now that each marked hexagon presented in Figure 2 has the triangulation and the signs of S. The triangulation of the union $\widetilde{S}$ of the marked hexagons can be extended to a primitive convex triangulation of T. Let us fix such an extension. Outside of $\widetilde{S}$ choose again the Harnack distribution of signs at the vertices of the triangulation.

One can calculate that for the corresponding piecewise-linear curve L we have

$$p = \frac{3k^2 - 3k + 2}{2} + a$$

where a is the number of the marked hexagons, and $a = \left[\frac{(k-3)^2+4}{8}\right]$.

Improved Petrovsky inequalities for M-curves of degree $4l+2$ with one non-empty oval can be rewritten (using the Gudkov - Rokhlin congruence and the fact that $p_- \leq 1,\ n_- = 0$ in this case) as follows :

$$p \leq \frac{3k^2 - 3k + 2}{2}, \quad n \leq \frac{3k^2 - 3k}{2}$$

It turns out that the Gudkov - Rokhlin congruence and the improved Petrovsky inequalities are the only restrictions for the topology of M-curves of degree $4l+2$ with one non-empty oval.

THEOREM 5.1. *Let $m = 2k = 4l + 2$, where l is a positive integer number. Then for any positive integer numbers p, n such that*

$$p + n = \frac{(m-1)(m-2)}{2} + 1$$

satisfying the Gudkov - Rokhlin congruence and the improved Petrovsky inequalities there exists a real algebraic plane projective M-curve of degree m with the real scheme

$$< (p-1) \cup 1 < n >>$$

PROOF. Let $\varepsilon, \delta \in \{0,1\}$. Denote by $H^+_{\varepsilon,\delta}$ (resp. $H^-_{\varepsilon,\delta}$) the following distribution of signs at integer points :

a vertex (i,j) gets sign "+" (resp. "-"), if $i \equiv \varepsilon \bmod 2,\quad j \equiv \delta \bmod 2$, and sign "-" (resp. "+") otherwise.

So, the Harnack distribution of signs, described in the section 3, is denoted now by $H^-_{0,0}$.

Remark that any distribution $H^\pm_{\varepsilon,\delta}$ can be formulated as the Harnack one for the appropriate quadrant of the plane (exchanging, if necessary, "+" and "-"). Thus, Proposition 3.1 also holds true for any distribution $H^\pm_{\varepsilon,\delta}$.

Let us divide the triangle T in two polygons T_1 and T_2 (where T_1 is a quadrangle, T_2 is a triangle) by a segment with the following properties :

(i) the ends of the segment are odd points lying on the boundary of T,
(ii) the segment does not contain integer points except the ends.

Consider an arbitrary convex primitive triangulation in each polygon T_1, T_2 (the union of these triangulations is a convex primitive triangulation of T, because the chosen segment does not contain vertices of the triangulations except the ends). Let us take the Harnack distribution of signs in T_1 and choose in T_2 the only distribution $H^-_{\varepsilon,\delta}$ different from the Harnack one and compatible on the common boundary of the polygons with the chosen distribution in T_1.

The arguments of the proof of Proposition 3.1 show again that the triangulation and the distribution of signs described above give an M-curve with one non-empty oval.

Let P_1, P_2 be the numbers of interior even points of T_1 and T_2, and N_1, N_2 be the numbers of interior odd points of these polygons.

The constructed curve has the following real scheme

$$< \frac{3k^2-3k-2a}{2} \cup 1 < \frac{(k-1)(k-2)-8a}{2} \cup a \times 1 < 2 >>>$$

To prove the part b) of the theorem, let us fix in addition to the triangulation of $\widetilde{S}$ a triangulation of some part P of a neighbourhood of the axe OY and the signs at the vertices of this triangulation as it is shown in Figure 3 (more precisely, only the case $k \equiv 1 \ mod\ 4$ is presented in this figure, if $k \not\equiv 1 \ mod\ 4$ one should change the triangulation near the point $(0, m)$). The chosen primitive convex triangulation of $\widetilde{S} \cup P$ can be extended to a primitive convex triangulation of the triangle T. Outside of $\widetilde{S} \cup P$, let us take the Harnack distribution of signs at the vertices of the triangulation.

For the corresponding piecewise-linear curve L

$$n = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right]$$

(the case $k \equiv 1 \ mod\ 4$) or

$$n = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right] - 1$$

(the case $k \not\equiv 1 \ mod\ 4$). □

Recently, B. Haas [1] modified the presented construction and obtained examples of T-curves of degree $2k$ with

$$p = \frac{3k^2-3k+2}{2} + \left[\frac{k^2-7k-10}{6}\right]$$

5. M-curves with one non-empty oval

In this subsection we discuss a classification of the real schemes of M-curves of degree $4l+2$ with one non-empty oval.

Let us start with two well-known restrictions for the topology of real plane projective curves (see, for example, [14], [16]).

Gudkov - Rokhlin congruence

$p - n \equiv k^2 \ mod\ 8$ *for an M-curve of degree* $2k$

Improved Petrovsky inequalities

Let A be a curve of degree $2k$. Denote by p_- (resp. by n_-) the number of even (resp. odd) ovals of $\mathbf{R}A$ bounding from the exterior the components of $\mathbf{R}P^2 \setminus \mathbf{R}A$ with negative Euler characteristic. Then

$$p - n_- \le \frac{3k^2-3k+2}{2}, \quad n - p_- + 1 \le \frac{3k^2-3k+2}{2}$$

One can see that for the curve obtained

$$p = N_1 + P_2 + 1, \quad n = P_1 + N_2$$

To prove Theorem 5.1, we divide the triangle T by segments with the properties (i), (ii) as it shown in Figure 4 (the vertices on the axe OX have the coordinates $(k + (4i + 2),\ 0)$, the vertices on the axe OY have the coordinates $(0,\ k + (4i + 2))$, and the vertices on the line $x + y = m$ have the coordinates $(k \pm 4i,\ k \mp 4i)$ with appropriate values of a non-negative integer i).

Let us take in the quadrangle $OABC$ the Harnack distribution of signs. Now we choose one of the distributions $H^-_{0,0}$, $H^-_{0,1}$ in each triangle part of the subdivision of T lying under the line $x = y$. Also, we choose one of the distributions $H^-_{0,0}$, $H^-_{1,0}$ in each triangle part of the subdivision of T lying over the line $x = y$.

Finally, we choose in each part an arbitrary convex primitive triangulation (actually, the real scheme of the resulting T-curve does not depend of this choice of primitive triangulations). The chosen triangulation and distribution of signs produce an M-curve with one non-empty oval. It is easy to verify that all possible (in the sense of the statement of Theorem 5.1) pairs p, n can be realized using the described construction. For example, to realize two extremal cases one can take the Harnack distribution of signs in T (the case $p = \frac{3k^2-3k+2}{2}$, $n = \frac{(k-1)(k-2)}{2}$, a Harnack curve) or the Harnack distribution in the quadrangle $OABC$ and the distributions $H^-_{0,1}$ and $H^-_{1,0}$ in $T \setminus OABC$ (for the opposite extremal case $p = \frac{(k-1)(k-2)}{2} + 1$, $n = \frac{3k^2-3k}{2}$). □

6. Sharpness of the second Petrovsky inequality

It is well-known that the first Petrovsky inequality

$$p - n \leq \frac{3k^2 - 3k + 2}{2}$$

is sharp for any positive integer k. One can take, for example, a Harnack curve of degree $2k$ with the real scheme

$$< \frac{3k^2 - 3k}{2} \cup 1 < \frac{(k-1)(k-2)}{2} >>$$

and contract all odd ovals. Let us explain how this contraction can be done for T-curves described in the section 3. To contract an odd oval α in this case, one can choose such a triangulation of the triangle T that the corresponding to α interior even vertex would have exactly three neighbours, and then change the sign of this vertex.

The sharpness of the second Petrovsky inequality

$$n - p + 1 \leq \frac{3k^2 - 3k + 2}{2}$$

is more difficult to show. A curve with $n - p + 1 = \frac{3k^2-3k+2}{2}$ is, evidently, a counter-example to Ragsdale conjecture.

PROPOSITION 6.1. *The second Petrovsky inequality*

$$n - p + 1 \leq \frac{3k^2 - 3k + 2}{2}$$

is sharp for any integer $k \geq 4$.

PROOF. To obtain the curves with

$$n - p + 1 = \frac{3k^2 - 3k + 2}{2}$$

for an even $k \geq 4$, one can take a curve constructed by Viro (see [11]) with the real scheme

$$< \frac{k^2 - 3k}{2} \cup 1 < \frac{3k^2 - 3k + 2}{2} >>$$

and contract all even empty ovals (it is possible in Viro's construction).

Suppose now, that k is odd and $k \geq 5$, $k \neq 7$ (the case $k = 7$ we will consider later). We can choose a positive number a and a non-negative number b such that $6a + 8b + 4 = 2k$. Let us consider a partition of the triangle T shown in Figure 5, where the centers of the hexagons $S_1, \ldots, S_a, S_{a+1}, \ldots, S_{a+b}$ have the second coordinates

$$3, \ldots,\ 6(a-1) + 3,\ 6(a-1) + 11, \ldots,\ 6(a-1) + 8b + 3$$

respectively. Suppose that the triangulation and the signs of each marked hexagon coincide with the triangulation and the signs of the hexagon S (shown in Figure 1). The chosen primitive convex triangulation of the union $\widetilde{S}$ of the marked hexagons and of the part P' of a neighbourhood of the axe OY can be extended to a primitive convex triangulation of the triangle T. Moreover, this extension can be chosen in such a way that any interior even vertex of T lying in $T \setminus (\widetilde{S} \cup P')$ would have exactly three neighbours. Now, let us put the sign "+" at all integer points of $T \setminus (\widetilde{S} \cup P')$ except the even points lying on the boundary of T, where we put the sign "-".

The corresponding T-curve has the real scheme

$$< (a+b) \times 1 < 2 > \cup\ 1 < \frac{3k^2 - 3k + 2}{2} - (a+b) >>$$

(in the case $k = 9$ the oval of the T-curve intersecting the coordinate axes is shown in Figure 6).

Consider now the case $k = 7$. Take the partition of T shown in Figure 7. We choose inside of the hexagon S_1 the triangulation and the signs of S. The triangulation of S_1 and of the polygon P'' with the vertices $(0,5)$, $(7,7)$, $(8,6)$, $(6,5)$, $(5,6)$ can be extended to a convex primitive triangulation of T in such a way that any interior even vertex lying outside of S_1 and P'' would have three neighbours.

Let us take the Harnack distribution of signs at the integer points of the quadrangle $\{(0,0),(0,5),(7,7),(14,0)\}$ except the points of S_1 and P''. In the triangle $\{(0,5),(0,14),(7,7)\}$ we choose the distribution $H^+_{0,1}$ (see section 5).

The corresponding T-curve has the real scheme $< 1 < 2 > \cup 1 < 63 >>$. It rests to contract even empty ovals. It can be done changing "-" for "+" at the interior even points of $T \setminus (S_1 \cup P'')$. $\square$

7. How large is the class of T-curves ?

It is natural to pose the following question : can the real scheme of an arbitrary non-singular real algebraic plane projective curve be realized by a T-curve of the same degree ?

One can immediately find a trivial restriction : evidently, the empty real scheme of a curve of an even degree cannot be realized by T-curves. We will formulate another restriction.

Let us, first, give a necessary definition. A pair of ovals is called *injective* if one oval of this pair lies inside of the other one. Denote by J the number of ovals of a curve containing inside of them at least one injective pair.

THEOREM 7.1. *For a T-curve of degree m being an M-curve the following inequality holds*

$$J \leq 3m/2$$

To prepare the proof of Theorem 7.1 we will prove the Harnack inequality (the number of components of the real point set of a curve of degree m is not greater than $\frac{(m-1)(m-2)}{2} + 1$) for T-curves in a combinatorial way.

Take an arbitrary primitive triangulation of the triangle T with some distribution of signs at the vertices. We will show that the number of connected components of $T_* \setminus L$ is at most $\frac{(m-1)(m-2)}{2} + 2$.

Consider a sequence $U_1, U_2, \dots, U_i, \dots$, where each U_i is a union of some triangles of the triangulation of T, the element U_1 consists of one triangle with the side $[(0;0),(1;0)]$, and U_i is defined by induction as follows. Denote by γ_i an edge of the triangulation lying on the boundary of U_{i-1} but not on the boundary of the triangle T (if such an edge does not exist, then U_{i-1} coincides with T and the construction of the sequence is finished). Let Γ_i be the triangle of the triangulation which does not belong to U_{i-1} and has the side γ_i. Let v_i be the vertex of Γ_i which does not belong to γ_i. Then we take U_i equal to the union of all triangles of U_{i-1} and of all triangles Γ such that v_i is a vertex of Γ, and two other vertices of Γ belong to U_{i-1}.

Now, consider the symmetric copies

$$U_{i,x} = s_x(U_i), \;\; U_{i,y} = s_y(U_i), \;\; U_{i,xy} = s(U_i)$$

of U_i, where $s = s_x \circ s_y$ and s_x, s_y again are reflections with respect to the coordinate axes. Let $V_i = U_i \cup U_{i,x} \cup U_{i,y} \cup U_{i,xy}$. Denote by $C(i)$ the number of connected components of $T_* \setminus L$ intersecting V_i.

The following statements are easy to verify :

i) $C(1) \leq 3$, and $C(1) \leq 2$ if all vertices of U_1 belong to the boundary of T,
ii) $C(i) \leq C(i-1) + 1$,
iii) $C(i) \leq C(i-1)$, if the vertex v_i belongs to the boundary of T.

The number of interior integer points of the triangle T is equal to $\frac{(m-1)(m-2)}{2}$. So, the number of connected components of $T_* \setminus L$ is at most $\frac{(m-1)(m-2)}{2} + 2$.

One can obtain the statement below using the described combinatorial proof of the Harnack inequality for T-curves.

PROPOSITION 7.2. *Let a convex primitive triangulation of the triangle T and a distribution of signs at the vertices of this triangulation be chosen in such a way that the resulting T-curve is an M-curve. If the star $St(v)$ of an interior vertex v does not intersect the boundary of T, then $St(v)$ or one of its symmetric copies contains an oval of the curve L.*

PROOF. Let us construct a sequence $U_1, \dots, U_r$ (where $r = \frac{(m+1)(m+2)}{2} - 2$) as it was described in the proof of the Harnack inequality but verifying the additional condition, that the last vertex v_r coincides with v (it is possible, because $St(v)$ does not intersect the boundary of T).

Our T-curve is an M-curve, thus $C(i)$ should be equal to $C(i-1)+1$ if v_i is an interior vertex. In particular, we have $C(r) = C(r-1)+1$, which means that the star $St(v)$ or one of its symmetric copies contains an oval of L. □

The statement of Proposition 7.2 was, first, proved by V. Kharlamov [5] in a different (non-combinatorial) way.

PROOF OF THEOREM 7.1. We will show that for a T-curve being an M-curve any oval of the curve L surrounding an injective pair should intersect the boundary of one of the triangles T, $s_x(T)$, $s_y(T)$, $s(T)$. It will give the proof of the theorem, because the number of the points of intersection of L and the boundary of these triangles is equal to $3m$.

Suppose that β is an oval of L containing inside an injective pair and non-intersecting the boundary of T, $s_x(T)$, $s_y(T)$, $s(T)$. Let β lie, for example, inside of T. Denote by β_1 the exterior oval of the injective pair. All vertices inside of β_1 have stars non-intersecting the boundary of T. Let W be the union of the stars of all vertices lying inside of β_1 and let w be the number of interior points of W. Then W and its symmetric copies contain at least $w+1$ ovals of L (one for each interior point of W as it was proved in Proposition 7.2, and also the oval β_1).

Consider a sequence $U_1, U_2, \dots, U_r$ described in the proof of the Harnack inequality. Let i be the minimal number such that v_i lies inside of W. It is easy to see that $C(i)$ should be greater than $C(i-1)+1$. This contradiction proves the theorem. □

Remark that the Theorem 7.1 gives a strong restriction on the topology of T-curves being M-curves. One can easily construct such a family of M-curves of increasing degrees that the numbers J of the curves of this family would depend quadratically in the degree.

The following question is open : is it true that a T-curve of degree m being an M-curve has no more than $O(m)$ non-empty ovals ?

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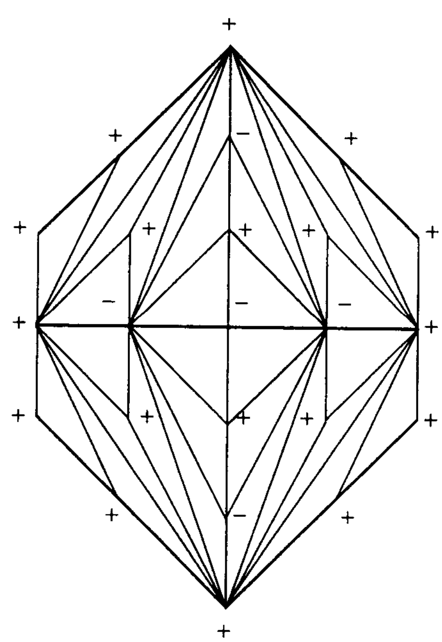

Figure 1. Hexagon S

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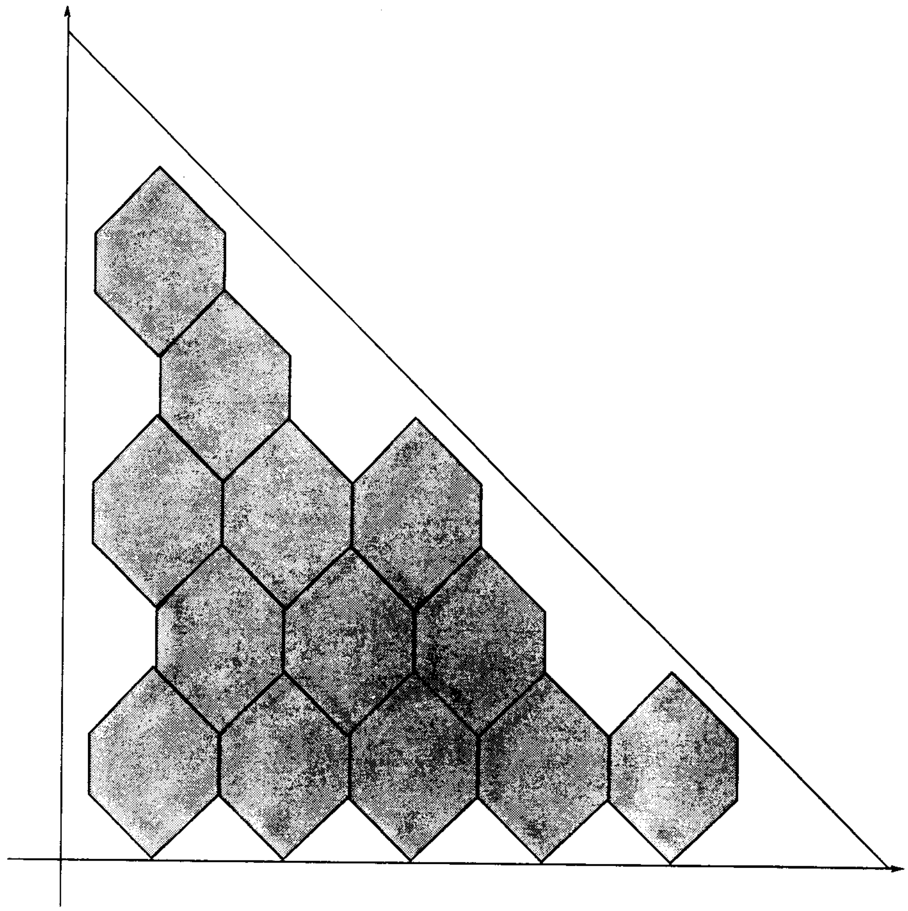

Figure 2. Partition for part a) of Theorem 1.1

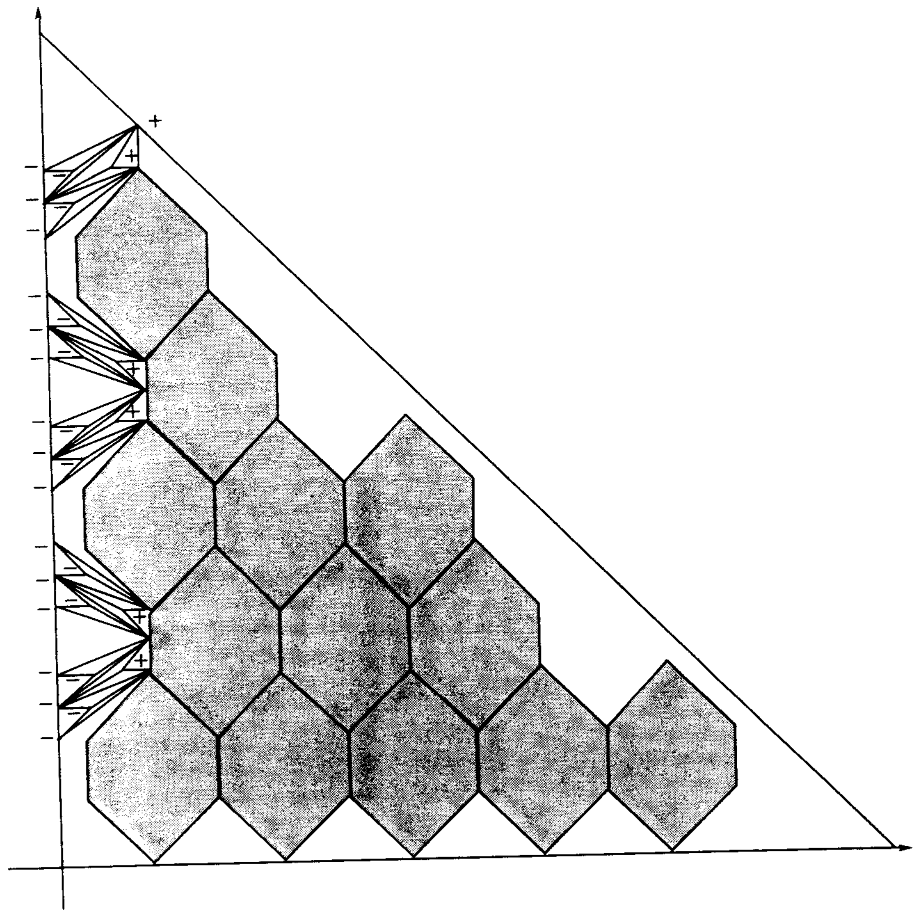

Figure 3. Partition for part b) of Theorem 1.1

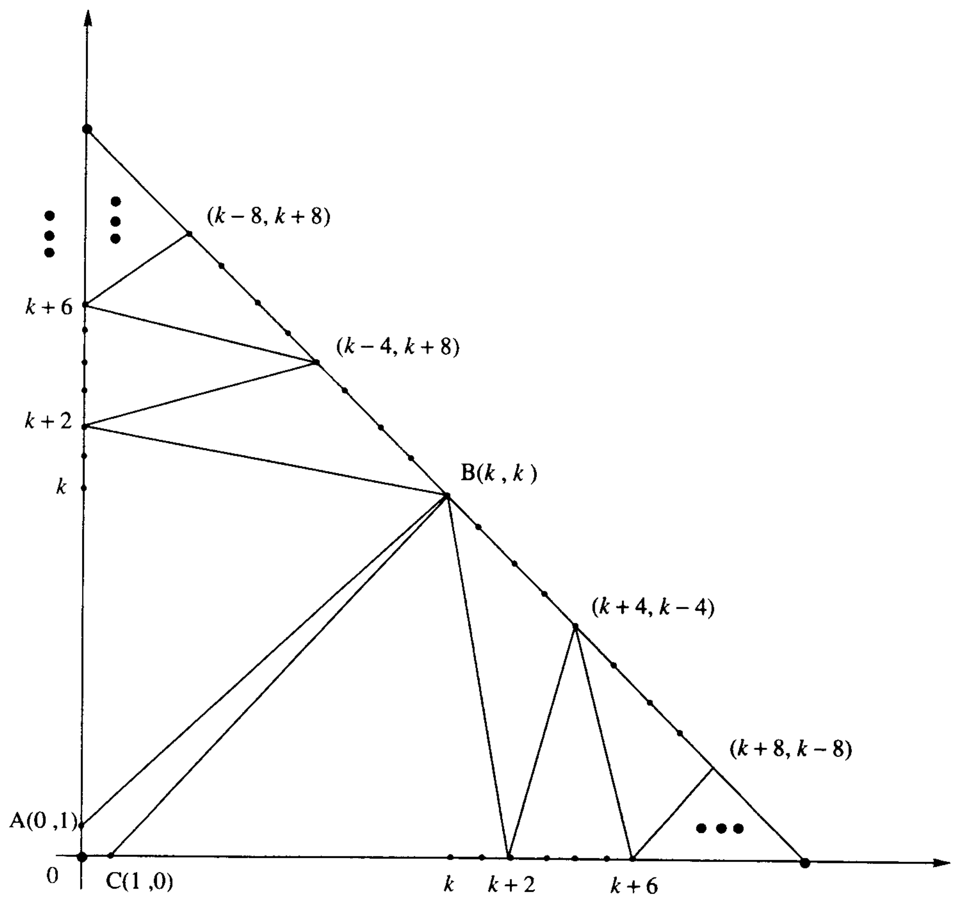

Figure 4. Partition for Theorem 5.1

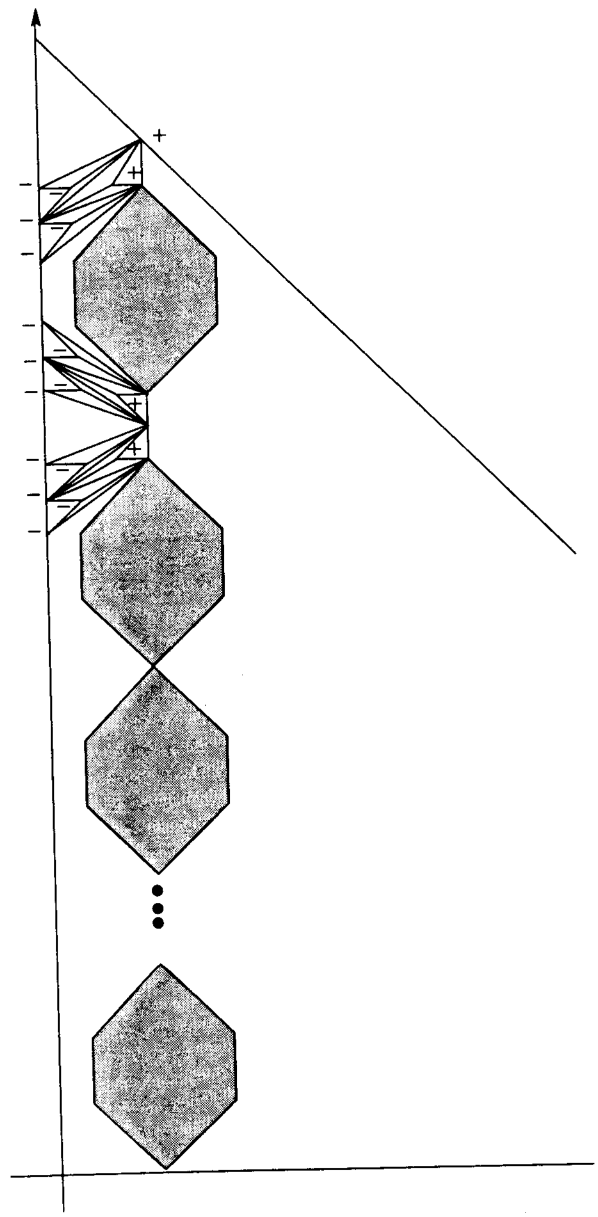

Figure 5. Partition for Proposition 6.1

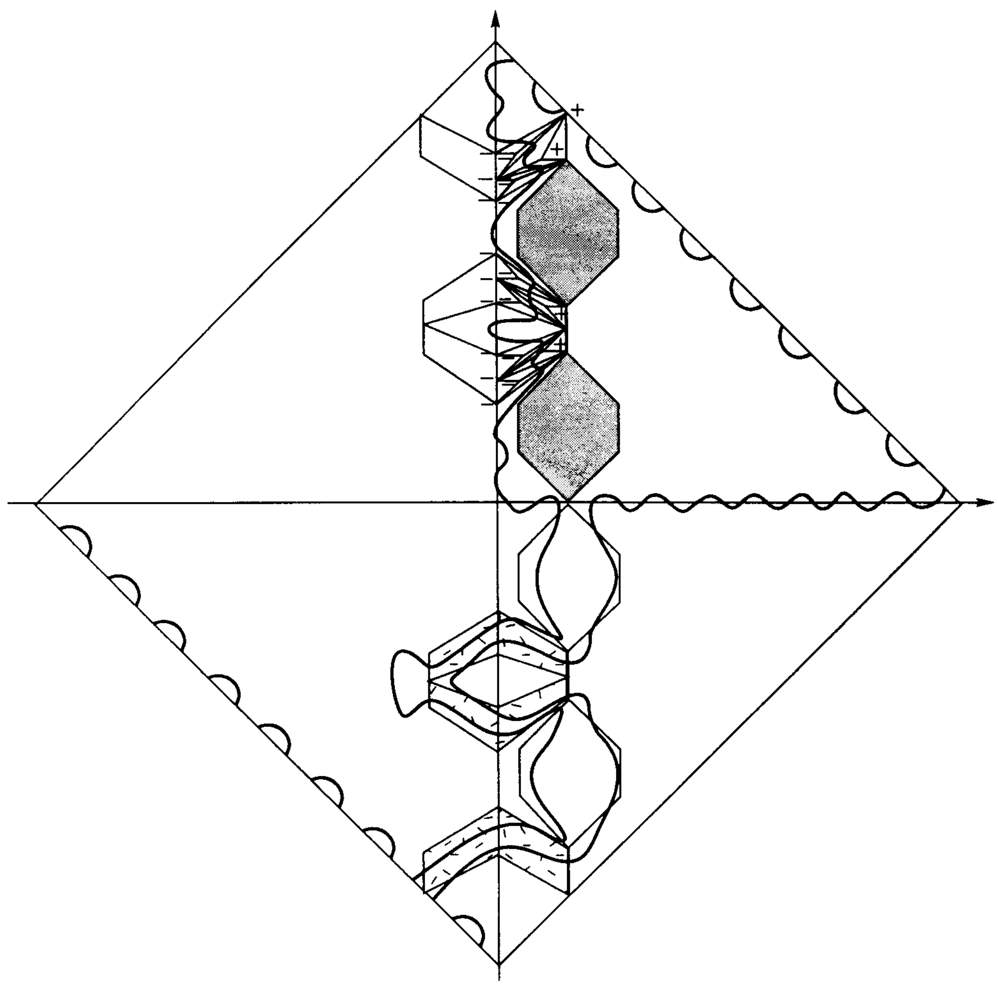

Figure 6. Case $k = 9$

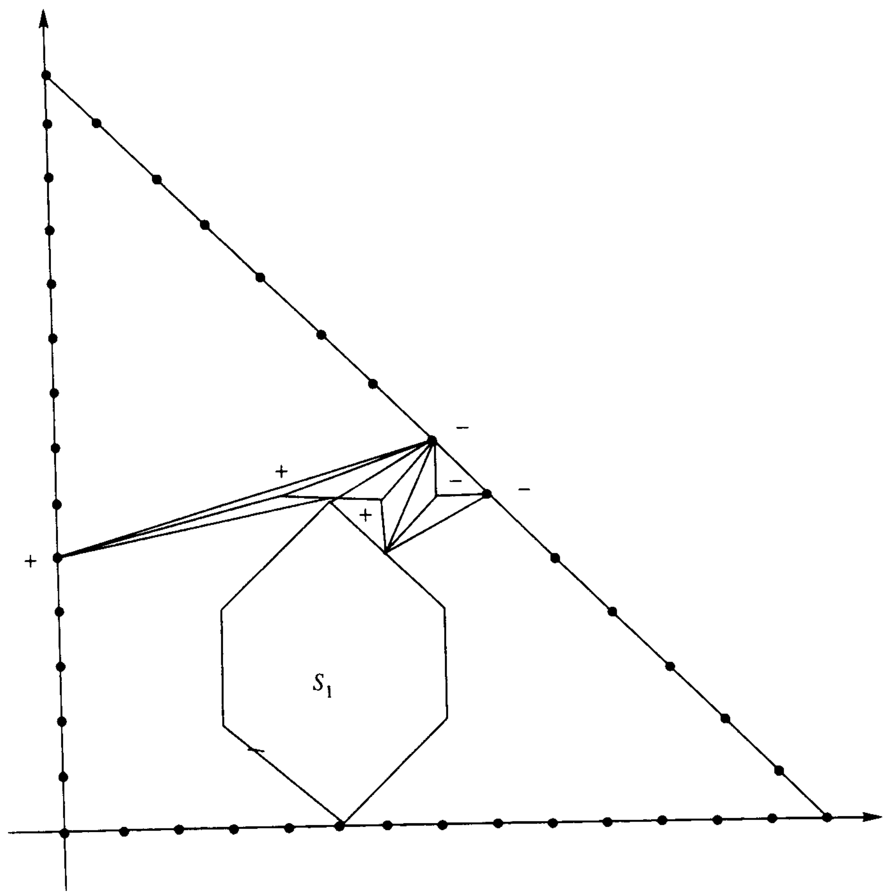

Figure 7. Case $k = 7$