Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 2, 337-346.

Multivariate Descartes' Rule

Ilia Itenberg and **Marie-Françoise Roy**

Institut de Recherche Mathématique de Rennes (Laboratory URA 305, CNRS)
Campus de Beaulieu, 35042 Rennes Cedex, France
e-mail: itenberg@univ-rennes1.fr, costeroy@univ-rennes1.fr

1. Introduction

In this paper we formulate a generalization of Descartes' rule to the multivariate case.

Let $P_1, \ldots, P_k$ be real polynomials in k variables, and let $\Delta_1, \ldots, \Delta_k$ be the Newton polytopes of these polynomials. Each polytope Δ_i can be equipped with a distribution δ_i of signs at its integer points: a point gets the sign ("+" or "−") of the coefficient of the corresponding monomial of the polynomial P_i, when this monomial appears in P_i, and gets the sign "0" if the corresponding monomial does not appear in the polynomial P_i. A Newton polytope Δ_i with a distribution of signs δ_i is called a *signed Newton diagram* and is denoted by $\tilde{\Delta}_i$.

We associate in a combinatorial way a number $n(\tilde{\Delta})$ to the collection $\tilde{\Delta}$ of signed Newton diagrams $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k$. We also introduce for any orthant m of $\mathbf{R}^k$, a number $n(\tilde{\Delta}, m)$. In the univariate case, this number for the halfline $X > 0$ is equal to the number of sign changes in the list of coefficients of a polynomial.

We prove that it is always possible to construct real polynomials $Q_1, \ldots, Q_k$ in k variables with the following properties.

(i) The signed Newton diagrams of $Q_1, \ldots, Q_k$ coincide with $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k$.

(ii) The set of common zeroes of $Q_1, \ldots, Q_k$ in $(\mathbf{C}^*)^k$ is finite, and the number of real common zeroes of $Q_1, \ldots, Q_k$ in $(\mathbf{R}^*)^k$ is equal to $n(\tilde{\Delta})$.

Furthermore, for any orthant m of $\mathbf{R}^k$, there exist polynomials $Q'_1, \ldots, Q'_k$ such that

(i′) The signed Newton diagrams of $Q'_1, \ldots, Q'_k$ coincide with $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k$.

(ii′) The set of common zeroes of $Q'_1, \ldots, Q'_k$ in $(\mathbf{C}^*)^k$ is finite, and the number of real common zeroes of $Q'_1, \ldots, Q'_k$ in an (open) orthant m is equal to $n(\tilde{\Delta}, m)$.

The construction uses the Viro method [10], [11] (more precisely, a version proposed by B. Sturmfels [8], which allows one to construct real complete intersections with prescribed topology).

The following questions are natural to ask:

Is the number of real zeroes of $P_1, \ldots, P_k$ in a given (open) orthant m always at most $n(\tilde{\Delta}, m)$?

Is the total number of real zeroes with nonzero coordinates always at most $n(\tilde{\Delta})$?

There are two important cases where the answer to these questions is positive. In the univariate case, this is the famous Descartes' rule (see [4], [2]). If the number $n(\tilde{\Delta})$ is equal to the mixed volume of the polytopes $\Delta_1, \ldots, \Delta_k$, then the answer to the second question is also yes, according to Bernstein's theorem [1].

2. Signed Newton diagrams and real zeroes

We consider k polynomials $P_1, \ldots, P_k$ in k variables with real coefficients whose set of common zeroes is finite. Let $\Delta_1, \ldots, \Delta_k$ and $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k$ be Newton polytopes and signed Newton diagrams of these polynomials, respectively.

We perform the following construction (cf. [8]). Let ω_i be a real-valued function defined on the set A_i of integer points of $\tilde{\Delta}_i$ equipped with "+" or "−" (not with "0"). By taking the lower convex hull in $\mathbf{R}^{k+1}$ of the graph of ω_i and then projecting each facet to $\mathbf{R}^k \times \{0\}$, the function ω_i defines a polyhedral subdivision τ_i of Δ_i. Denote by Δ_M the Minkowski sum of the polytopes $\Delta_1, \ldots, \Delta_k$ and by A the set

$$\{a \in \Delta_M \mid a = a_1 + \ldots + a_k, \text{ where } a_i \in A_i\}.$$

Let us define a function $\omega : A \longrightarrow \mathbf{R}$ as follows :

$$\omega(a) = \min\{\omega_1(a_1) + \ldots + \omega_k(a_k) \mid a_1 + \ldots + a_k = a\}.$$

Such a function ω defines a polyhedral subdivision τ_ω of Δ_M. The vertices of τ_ω belong to A. Each facet V of τ_ω has a unique representation

$$V = v_1 + \ldots + v_k$$

where v_i is a face of τ_i. Suppose that the functions $\omega_1, \ldots, \omega_k$ are *generic*: this means that for any facet V of τ_ω equal to $v_1 + \ldots + v_k$, we have

$$\dim(V) = \dim(v_1) + \ldots + \dim(v_k).$$

A facet $V = v_1 + \ldots + v_k$ of τ_ω is called a *mixed cell* if $\dim(v_1) = \ldots = \dim(v_k) = 1$. Every mixed cell is affinely isomorphic to the regular k-dimensional cube. The *index of parity* $p(V)$ of a mixed cell $V = v_1 + \ldots + v_k$ is defined as follows: it is equal to the corank (over $\mathbf{Z}/2\mathbf{Z}$) of the matrix $\hat{V}$ whose rows are composed of the coordinates modulo 2 of the vectors $v_1, \ldots, v_k$ (it is not difficult to see that this definition coincides with one given in [7]).

Let v_i be one of the edges defining a mixed cell V. We call the edge v_i *alternating*, if the distribution of signs δ_i associates different signs to the endpoints of v_i. We define a new $k \times (k+1)$ matrix $\bar{V}$ by adding to $\hat{V}$ a $(k+1)$-st column, taking the $(i, k+1)$ -th element equal to 0 if the edge v_i is alternating, and equal to 1 otherwise. A mixed cell V is called *contributing* if $\operatorname{rank}(\hat{V}) = \operatorname{rank}(\bar{V})$. Let $n(\tilde{\Delta}, \omega) = \sum_V 2^{p(V)}$, where V ranges over all contributing mixed cells of τ_ω.

Example. Consider two triangles T_1, T_2 with vertices $(2,0)$, $(1,2)$, $(0,1)$ and $(2,0)$, $(0,2)$, $(0,1)$, respectively (see Figure 1).

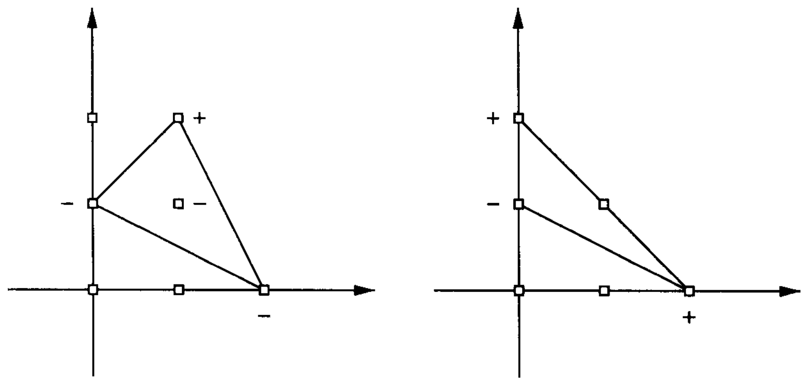

Figure 1

Let $A_1 = \{(2,0), (1,2), (0,1), (1,1)\}$; $A_2 = \{(2,0), (0,2), (0,1)\}$. Suppose that the points of the sets A_1 and A_2 are equipped with signs ("+" or "−") as it is shown in Figure 1.

Consider the two functions $\omega_1 : A_1 \to \mathbf{R}$ and $\omega_2 : A_2 \to \mathbf{R}$ such that $\omega_1((2,0)) = 0$, $\omega_1((1,2)) = 3$, $\omega_1((0,1)) = 3$, $\omega_1((1,1)) = 1$, $\omega_2((2,0)) = 3$, $\omega_2((0,2)) = 0$, $\omega_2((0,1)) = 1$. The corresponding mixed subdivision of the Minkowski sum of T_1 and T_2 is shown in Figure 2. It is easy to calculate the number $n(\tilde{\Delta}, \omega)$: in this case it is equal to 0 (there are two mixed cells with index of parity 1, but they are not contributing).

Consider now another pair of functions $\omega_1' : A_1 \to \mathbf{R}$ and $\omega_2' : A_2 \to \mathbf{R}$ such that $\omega_1'((2,0)) = 3$, $\omega_1'((1,2)) = 0$, $\omega_1'((0,1)) = 3$, $\omega_1'((1,1)) = 1$, $\omega_2'((2,0)) = 3$, $\omega_2'((0,2)) = 1$, $\omega_2'((0,1)) = 0$. The corresponding mixed subdivision of the Minkowski sum of T_1 and T_2 is shown in Figure 3. In this case $n(\tilde{\Delta}, \omega') = 2$ (there are two mixed cells with index of parity 0, and they are automatically contributing).

Theorem 1. *There exist real polynomials $Q_1, \ldots, Q_k$ in k variables with respective signed Newton diagrams $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k$, such that the number of common real zeroes in $(\mathbf{R}^*)^k$ of these polynomials is equal to $n(\tilde{\Delta}, \omega)$.*

Moreover, the polynomials $Q_1, \ldots, Q_k$ can be written down explicitly. Namely, let

$$Q_i^{(t)}(X) = \sum_{a_i^+} X^{a_i^+} t^{\omega_i(a_i^+)} - \sum_{a_i^-} X^{a_i^-} t^{\omega_i(a_i^-)}$$

where a_i^+ (resp. a_i^-) ranges over all points of A_i equipped with the sign "+" (resp. with the sign "−"). Then for t positive and sufficiently small, the number of common zeroes of the polynomials $Q_1^{(t)}, \ldots, Q_k^{(t)}$ in $(\mathbf{R}^*)^k$ is equal to $n(\tilde{\Delta}, \omega)$.

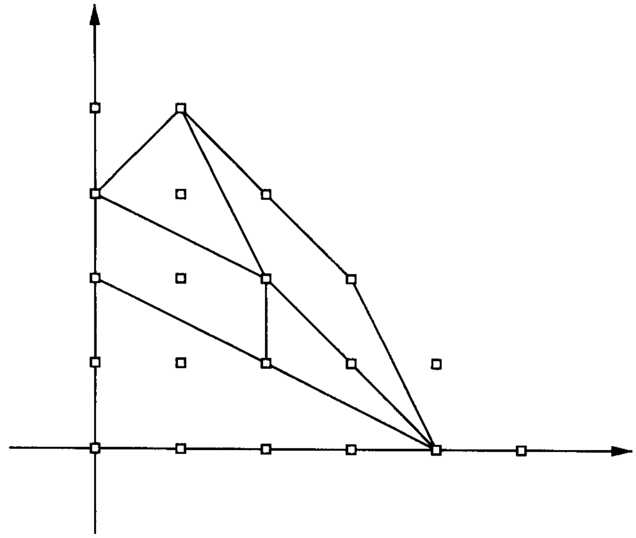

Figure 2

Theorem 1 is a corollary of Viro's theorem (see [10], [11]) adapted for the construction of complete intersections by B. Sturmfels [8]. However, we give here a proof of Theorem 1, which is rather easier than the proof of the general statement valid for complete intersections of an arbitrary dimension. Theorem 1 is a slight strengthening of [7, Proposition 4.1.].

Before starting the proof of Theorem 1, let us make some definitions. To each open orthant m of $\mathbf{R}^k$ we associate a vector $\vec{m}$ of $(\mathbf{Z}/2\mathbf{Z})^k$ in the following way: let $(x_1, \ldots, x_k)$ belong to m; then we put the i-th coordinate of $\vec{m}$ equal to 0 if $x_i > 0$, and equal to 1 otherwise. We also associate to any integer point $a_i \in A_i$ a vector $\vec{a}_i$ of $(\mathbf{Z}/2\mathbf{Z})^k$ by replacing each coordinate of a_i by its parity (0 or 1).

Consider a composition of symmetries with respect to coordinate hyperplanes sending the points of the positive orthant to the points of an orthant m. Let $a_i^{(m)}$ be the image of a point a_i under this composition. Now define the sign of $a_i^{(m)}$ as follows :

$$\mathrm{sign}(a_i^{(m)}) = (-1)^{\vec{a}_i \cdot \vec{m}} \mathrm{sign}(a_i),$$

where $\vec{a} \cdot \vec{m}$ stands for the scalar product (over $\mathbf{Z}/2\mathbf{Z}$) of the vectors $\vec{a}$ and $\vec{m}$ and $\mathrm{sign}(a_i)$ stands for the sign associated by δ_i to a_i.

Let $V = v_1 + \ldots + v_k$ be a mixed cell of the subdivision τ_ω, and let $V^{(m)}$ be the symmetric copy of V in the orthant m. Denote by a_i and b_i the endpoints of v_i $(i = 1, \ldots, k)$. The cell $V^{(m)}$ is called *alternating*, if for any $i = 1, \ldots, k$ the signs $sign(a_i^{(m)})$ and $sign(b_i^{(m)})$ are opposite.

Lemma 1. *The number of alternating symmetric copies of a mixed cell V is equal to* 0 *if V is not contributing, and is equal to* $2^{p(V)}$ *if V is contributing.*

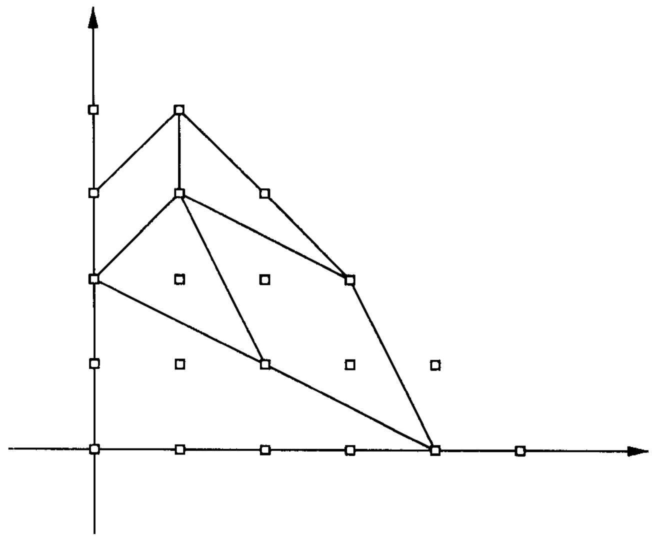

Figure 3

Proof. Let $V = v_1 + \ldots + v_k$. We can assume that every edge v_i has the origin as an endpoint, and that the sign of every v_i at the origin is "+" (translation of an edge v_i or change the sign at both the endpoints of v_i preserves the property of being alternating for any symmetric copy of V). Let us denote the endpoint of v_i different from the origin by e_i. The condition that the symmetric copy $V^{(m)}$ in an orthant m is alternating means now that

$$\vec{e}_1 \cdot \vec{m} = \varepsilon_1, \ldots, \vec{e}_k \cdot \vec{m} = \varepsilon_k,$$

where ε_i is equal to 0 if e_i has sign "−", and is equal to 1 if e_i has sign "+". Thus, we obtain a system of k linear equations in k variables (the coordinates of $\vec{m}$) over $\mathbf{Z}/2\mathbf{Z}$. A solution of this system exists iff $\mathrm{rank}(\hat{V}) = \mathrm{rank}(\bar{V})$, in other words, iff V is contributing. The number of solutions is equal to $2^{p(V)}$, where $p(V)$ is the corank of the system. □

Let $V = v_1 + \ldots + v_k$ be a mixed cell of the subdivision τ_ω, and let a_ℓ and b_ℓ be the endpoints of v_ℓ. A binomial system associated to the cell V is any system of k binomials of the form

$$\alpha_\ell X^{a_\ell} + \beta_\ell X^{b_\ell} = 0, \ \ell = 1, \ldots, k$$

where α_l and β_l are any real numbers such that $\mathrm{sign}(\alpha_\ell) = \mathrm{sign}(a_\ell)$, $\mathrm{sign}(\beta_\ell) = \mathrm{sign}(b_\ell)$. Denote by $\mathrm{Vol}(V)$ the volume of the cell V.

Lemma 2. *A binomial system associated to a mixed cell V has* $\mathrm{Vol}(V)$ *distinct zeroes in* $(\mathbf{C}^*)^k$ (cf. [5]) *and exactly one real zero in each open orthant of* $\mathbf{R}^k$ *where the copy of the cell is alternating.*

Proof. Consider a system

$$X^{a_\ell - b_\ell} = (-1)^{\varepsilon_\ell}\gamma_\ell, \ \ell = 1, \ldots, k$$

where each γ_ℓ is a positive real number, and ε_ℓ is equal to 0 if the edge (a_ℓ, b_ℓ) is alternating, and is equal to 1 otherwise.

Let

$$log(X) = (log(x_1), \ldots, log(x_k)),\ arg(X) = (arg(x_1), \ldots, arg(x_k))$$
$$log(\mid X \mid) = (log(\mid x_1 \mid), \ldots, log(\mid x_k \mid)).$$

We have

$$< a_\ell - b_\ell, log(X) > \ = \ log(\gamma_\ell) + \varepsilon_\ell \pi i + 2\pi N_\ell i,$$

where $< a_\ell - b_\ell, log(X) >$ stands for the scalar product over $\mathbf{C}$ of $a_\ell - b_\ell$ and $log(X)$; each N_ℓ is an integer.

Remark that

$$< a_\ell - b_\ell, log(\mid X \mid) > \ = \ log(\gamma_\ell)$$

and thus

$$< a_\ell - b_\ell, arg(X) > \ = \ \varepsilon_\ell \pi + 2\pi N_\ell.$$

All the solutions in $(\mathbf{C}^*)^k$ of the system

$$X^{a_\ell - b_\ell} = (-1)^{\varepsilon_\ell}\gamma_\ell,\ \ell = 1, \ldots, k$$

can be obtained by fixing a generic translation of V, choosing an integer point $N = (N_1, ..., N_k)$ in the image of V under the fixed translation, and then finding the corresponding $arg(X)$.

It is also clear now that the binomial system has one real zero in each orthant where the copy of V is alternating. □

Let

$$V_1 = v_{1,1} + \ldots + v_{k,1}\ , \ldots ,\ V_s = v_{1,s} + \ldots + v_{k,s}$$

be all the mixed cells of the subdivision τ_ω and let $a_{i,j}$ and $b_{i,j}$ be the endpoints of $v_{i,j}$. Denote by B_j the binomial system associated to the mixed cell V_j with the coefficients of the binomials coinciding with the corresponding coefficients of the initial polynomials P_i. More precisely, let B_j be the system

$$\alpha_{i,j} x^{a_{i,j}} + \beta_{i,j} x^{b_{i,j}} = 0,\ i = 1, \ldots, k$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are the coefficients of the monomials $X^{a_{i,j}}$ and $X^{b_{i,j}}$ of P_i. Remark that $\mathrm{sign}(\alpha_{i,j}) = \mathrm{sign}(a_{i,j})$, $\mathrm{sign}(\beta_{i,j}) = \mathrm{sign}(b_{i,j})$. Denote by $c_j^{(m)}$ the number of solutions of B_j in an orthant m. By the previous lemma, the number $c_j^{(m)}$ is equal to 1 if the copy of the mixed cell V_j is alternating in the orthant m, and is equal to 0 otherwise.

Proposition 1. *For t positive and sufficiently small, the number of real zeroes of the system $Q^{(t)}(X)$ of polynomials*

$$Q_i^{(t)}(X) = \sum_{a_i \in \Delta_i} \alpha_i X^{a_i} t^{\omega_i(a_i)}$$

(where α_i is the coefficient of monomial X^{a_i} in the polynomial P_i) in an open orthant m is equal to $c_1^{(m)} + \ldots + c_s^{(m)}$;
the number of complex zeroes of $Q^{(t)}(X)$ in $(\mathbf{C}^*)^k$ is equal to the mixed volume of the polytopes $\Delta_1, \ldots, \Delta_k$.

Proof. Let $\lambda_j : \mathbf{R}^k \to \mathbf{R}$ be the linear function coinciding with ω on $V_j \cap A$ and let $\nu_{i,j}(a_i) = \omega_i(a_i) - \lambda_j(a_i)$ for any point a_i of A_i. The substitution of ω_i for $\nu_{i,j}$ in the polynomial $Q_i^{(t)}(X) = \sum_{a_i \in \Delta_i} \alpha_i X^{a_i} t^{\omega_i(a_i)}$ is the composition of the linear coordinate change

$$L_j(x_1, ..., x_k) = (x_1 t^{\lambda_{1j}}, ..., x_k t^{\lambda_{kj}})$$

(where $-\lambda_{1j}, \ldots, -\lambda_{kj}$ are the coefficients of the linear function λ_j), with the multiplication of the polynomial by a power of t. This operation does not change the number of common zeroes of the polynomials in $(\mathbf{C}^*)^k$ or in any orthant of $(\mathbf{R}^*)^k$.

Let $K \subset (\mathbf{C}^*)^k$ be a compact set, whose interior contains all the zeroes of the binomial systems $V_1, ..., V_s$ in $(\mathbf{C}^*)^k$. There exists $t_1 > 0$ such that, for any $t \in (0, t_1)$, the compact sets $L_1(K), ..., L_s(K)$ are disjoint. It is easy to see that for any $j = 1, \ldots, s$,

$$Q_i^{(t)}\left(L_j(x_1, ..., x_k)\right) = \xi_{ij}\left(B_{ij}(x_1, ..., x_k) + C_{ij}^{(t)}(x_1, ..., x_k)\right),$$

where B_{ij} is the i-th polynomial of the binomial system B_j, each coefficient of the polynomial $C_{ij}^{(t)}$ contains t to a positive power ; each ξ_{ij} is a power of t. Then there exists $t_2 > 0$ such that for any $t \in (0, t_2)$ and for any $j = 1, \ldots, s$, the polynomials

$$Q_i^{(t)}\left(L_j(x_1, ..., x_k)\right)$$

have exactly $\mathrm{Vol}(V_j)$ common zeroes in K and exactly $c_j^{(m)}$ common zeroes in the intersection of K with an orthant m.

We have shown that the polynomials $Q_i^{(t)}(x_1, ..., x_k)$ have at least $\mathrm{Vol}(V_1) + \ldots + \mathrm{Vol}(V_s)$ zeroes in $(\mathbf{C}^*)^k$, and that exactly $c_1^m + \ldots + c_s^m$ of these $\mathrm{Vol}(V_1) + \ldots + \mathrm{Vol}(V_s)$ zeroes are contained in an orthant m.

To complete the proof, it suffices to note that according to Bernstein's theorem [1], the polynomials $Q_i^{(t)}(x_1, ..., x_k)$ cannot have more than $\mathrm{Vol}(V_1) + \ldots + \mathrm{Vol}(V_s)$ zeroes in $(\mathbf{C}^*)^k$.□

Proposition 1, Lemmas 1 and 2 immediately give the statement of Theorem 1.

Remark. It is interesting to notice that we do not really need to apply Bernstein's theorem at the end of the proof of Proposition 1. Moreover, as it was shown by B. Huber and B. Sturmfels [5], Bernstein's theorem can be proved using the combinatorial construction described above.

The idea is to show first that for sufficiently small positive t, all zeroes in $(\mathbf{C}^*)^k$ of the polynomials $Q_i^{(t)}(x_1, ..., x_k)$ are close to the zeroes of the systems associated to the cells of the subdivision τ_ω. If a cell $V = v_1 + \ldots + v_k$ is not mixed, than the corresponding system does not have zeroes in $(\mathbf{C}^*)^k$. Thus the number of common zeroes in $(\mathbf{C}^*)^k$ of the polynomials $Q_i^{(t)}(x_1, ..., x_k)$ for sufficiently small positive t is equal to the sum of volumes of the mixed cells. It is not difficult to see that the same property holds true for t sufficiently close to 1, and to conclude that for $t = 1$ the number of zeroes in $(\mathbf{C}^*)^k$ of the polynomials $Q_i^{(t)}(x_1, ..., x_k)$ is not greater than the sum of volumes of the mixed cells.

Going back to Theorem 1, let us remark that we have also proved the following statement (which is a particular case of [8, Theorem 5]).

Theorem 2. *If t is positive and sufficiently small, then the number of real zeroes of the polynomials*

$$Q_i^{(t)}(X) = \sum_{a_i \in \Delta_i} \alpha_i X^{a_i} t^{\omega_i(a_i)}, \quad i = 1, \dots, k$$

(where α_i is the coefficient of monomial X^{a_i} in the polynomial P_i) in an orthant m is equal to the total number of alternating copies in m of all mixed cells of the subdivision τ_ω.

We now define $n(\tilde{\Delta}, \omega; m)$ as the number of alternating copies in an orthant m of all mixed cells of τ_ω. Let $n(\tilde{\Delta}; m)$ (resp. $n(\tilde{\Delta})$) be the maximal number $n(\tilde{\Delta}, \omega; m)$ (resp. $n(\tilde{\Delta}, \omega)$) for all possible choices of generic functions $\omega_1, \dots, \omega_k$. In fact, these choices are grouped into finitely many equivalence classes: we say that two choices of generic functions $\omega_1, \dots, \omega_k$ are equivalent if they give the same subdivision τ_ω of Δ_M. There exists a one-to-one corresponence between the subdivisions τ_ω and convex (or regular) triangulations of the Cayley polytope associated with $\tilde{\Delta}_1, \dots, \tilde{\Delta}_k$ (see [9]). Thus, the calculation of the numbers $n(\tilde{\Delta}; m)$ and $n(\tilde{\Delta})$ can be carried out once we have a list of all convex triangulations of the Cayley polytope. To get such a list, one can use an algorithm due to J. De Loera [3] which enumerates all convex triangulations of a point configuration in an affine space.

In the case of a univariate polynomial there are two orthants, numbered 0 (corresponding to $X > 0$) and 1 (corresponding to $X < 0$). A signed Newton diagram is simply a list of pairs of integers and associated signs, listed by increasing order of the integers. A mixed cell is given by two consecutive integers in this list. If the difference of two consecutive integers is even (resp. odd), the index of parity of the cell is 1 (resp. 0). The cell is contributing if its index of parity is 0, or if its index of parity is 1 and the two corresponding integers have different associated signs. The number $n(\tilde{\Delta}; 0)$ is the number of sign changes in the list of associated signs.

The famous Descartes' rule [4] states that the number of positive real zeroes of a polynomial is not greater than the number of sign changes in the list of its coefficients.

The multivariate Descartes' rule we propose is the following.

Conjecture. *Suppose that k real polynomials in k variables with signed Newton diagrams $\tilde{\Delta}_1, \dots, \tilde{\Delta}_k$ have a finite number of common real zeroes in $(\mathbf{R}^*)^k$. Then*

(1) *the number of common zeroes of these polynomials in an open orthant m of $\mathbf{R}^k$ is not greater than $n(\tilde{\Delta}; m)$;*

(2) *the number of common zeroes of these polynomials in $(\mathbf{R}^*)^k$ is not greater than $n(\tilde{\Delta})$.*

A positive answer to the conjecture would imply the following result: if the maximal number of monomials appearing in each of the polynomials is bounded by S, then the number of zeroes of these polynomials in an open orthant is not greater than $(S(S-1)/2)^k$, and the number of zeroes in $(\mathbf{R}^*)^k$ is not greater than $(S(S-1))^k$. A. Kushnirenko [6] conjectured that the number of real zeroes of polynomials $P_1, \dots, P_k$ in an open orthant is not greater than $(S_1 - 1)\dots(S_k - 1)$, where S_i is the number of monomials of P_i. A. Khovanskii [6] proved that the number of real zeroes in an open orthant does not exceed $(k+2)^{S'} 2^{S'(S'+1)/2}$, where S' is the number of different monomials which occur in polynomials P_i, $i = 1, \dots, k$.

There is an important case when the second part of the conjecture stated above is true. Namely, if one can find such functions $\omega_1, \dots, \omega_k$ that all the mixed cells of the subdivision

τ_ω of Δ_M are of volume 1, then the number $n(\tilde{\Delta}, \omega)$ is equal to the mixed volume of the polytopes $\Delta_1, \ldots, \Delta_k$. Thus, the second statement of the conjecture in this case follows from Bernstein's theorem.

For example, let P_1 and P_2 be two polynomials in two variables, and let Δ_1 and Δ_2 be the Newton polygons of these polynomials. Suppose that each side of Δ_1 and Δ_2 is parallel to one of the lines $x = 0$, $y = 0$, $x + y = 0$, $x - y = 0$. Suppose also that each integer point of Δ_1 (resp. Δ_2) corresponds to a nonzero monomial of P_1 (resp. P_2). Then, clearly, there exist generic functions ω_1 and ω_2 such that all the mixed cells of τ_ω are parallelograms of area 1. Thus, the second statement of the conjecture is true for the polynomials P_1 and P_2.

On the other hand, it is easy to find a pair of polygons Δ_1 and Δ_2 in $\mathbf{R}^2$ such that for any distributions of signs δ_1 and δ_2 at the integer points of Δ_1 and Δ_2, the number $n(\tilde{\Delta})$ is less than the mixed area of Δ_1 and Δ_2. For example, one can take as polygons Δ_1 and Δ_2 the triangles shown in Figure 4. The mixed area of these triangles is equal to 5, however $n(\tilde{\Delta}) \leq 3$ for any distributions of signs δ_1 and δ_2.

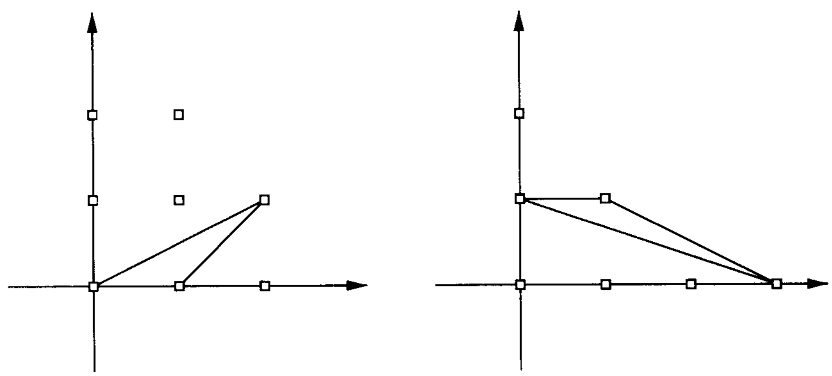

Figure 4

References

[1] Bernstein, D.: *The number of roots of a system of equations.* Functional Analysis and its Applications **9** (1975), 1–4.

[2] Bochnak, J.; Coste, M.; Roy, M.-F.: *Géométrie algébrique réelle.* A Series of Modern Surveys in Mathematics. Springer-Verlag 1987.

[3] De Loera, J.: *Triangulations of Polytopes and Computational Algebra.* Ph. D. Dissertation. Cornell University.

[4] Descartes, R.: *Géométrie.* (1636) In: A source book in Mathematics. Massachussetts, Harvard University Press 1969, 90–93.

[5] Huber, B.; Sturmfels, B.: *A polyhedral method for solving sparse polynomial systems.* Mathematics of Computation **64** (1995), 1541–1555.

[6] Khovanskii, A.: *On a class of systems of transcendental equations.* Soviet Math. Dokl. **22** (1980), 762–765.

[7] Pedersen, P.; Sturmfels, B.: *Mixed monomial bases.* To appear in the Proceedings of MEGA 94.

[8] Sturmfels, B.: *Viro's theorem for complete intersections.* Annali della Scuola Normale Superiore di Pisa (4), **21** (1994), no. 3, 377–386.

[9] Sturmfels, B.: *On the Newton polytope of the resultant.* Journal of Algebraic Combinatorics **3** (1994), 207–236.

[10] Viro, O.: *Gluing of plane real algebraic curves and constructions of curves of degrees* 6 *and* 7. Lecture Notes in Math. **1060** (1984), Springer-Verlag, 187–200.

[11] Viro, O.: *Plane real algebraic curves: constructions with controlled topology.* Algebra i analiz **1** (1989), 1–73 (Russian), English translation in: Leningrad Math. J. **1:5** (1990), 1059–1134.

Received July 20, 1995