FLEXIBLE COMBINATORIAL HYPERSURFACES (draft)

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Abstract

We generalize the Viro method of constructing real algebraic hypersurfaces, based on the use of convex subdivisions of Newton polyhedra, to arbitrary subdivisions and triangulations and show that the combinatorial hypersurfaces appearing from arbitrary subdivisions satisfy the same topological restrictions (congruences, inequalities etc.) as algebraic varieties.

Introduction

The Viro method of gluing polynomials appeared to be the most powerful construction of real algebraic varieties with prescribed topology [18], [19], [22] (see also [11], [7], 11.5, [9]). It provides a nice interaction of real algebraic geometry, toric geometry and combinatorics, and gives rise to various generalizations and applications [17], [14], [15], [16], [8], [3].

Consider the simplest example of Viro's construction. Let $T_d \subset \mathbb{R}^2$, $d \in \mathbb{N}$, be the triangle with vertices $(0,0)$, $(0,d)$, $(d,0)$,

$$\tau : \quad T_d = \Delta_1 \cup \ldots \cup \Delta_N$$

be a triangulation with the set of vertices $V \subset \mathbb{Z}^2$, and let $\sigma : V \to \{\pm 1\}$ be any function. Out of this combinatorial data we will construct continuous plane curves. Denote by $T_d^{(1)}, T_d^{(2)}, T_d^{(3)}$ the copies of T_d under reflections with respect to

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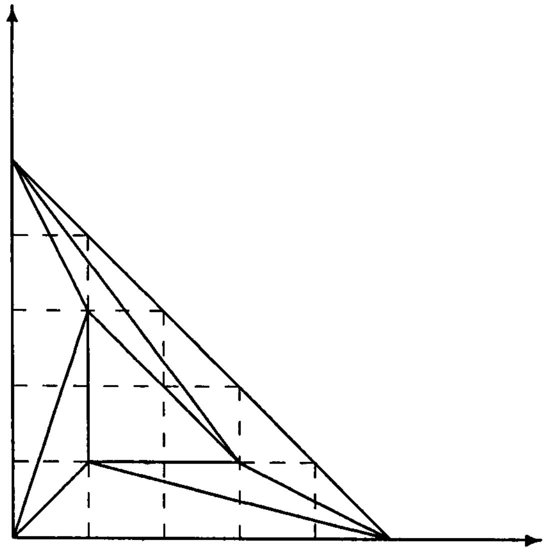

Figure 1:

the coordinate axes with corresponding triangulations, and define σ at the vertices of new triangulations by

$$\sigma(\varepsilon_1 i, \varepsilon_2 j) = \varepsilon_1^i \varepsilon_2^j \sigma(i,j), \quad (i,j) \in V, \quad \varepsilon_1, \varepsilon_2 = \pm 1.$$

Now in any triangle of the triangulation of $T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)}$, having vertices with different values of σ, we draw the mean line separating the vertices with different signs. The union $C(\tau,\sigma)$ of all these mean lines is a broken line homeomorphic to a disjoint union of circles and segments. Introduce natural maps:

$$\Phi : T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)} \to \mathbb{R}P^2, \quad \Psi : \mathrm{Int}(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)}) \to \mathbb{R}^2,$$

where Φ is continuous onto, identifying antipodal points on $\partial(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)})$, and Ψ is a homeomorphism. We call the curves $\Phi(C(\tau,\sigma)) \subset \mathbb{R}P^2$, $\Psi(C(\tau,\sigma)) \subset \mathbb{R}^2$ as *projective and affine T-curves of degree d.*

The Viro theorem states that the projective (affine) T-curve of degree d is isotopic in $\mathbb{R}P^2$ (resp. $\mathbb{R}^2$) to a non-singular algebraic projective (resp. affine) curve of degree d, providing the triangulation τ is *convex* (or *coherent* as in [7]), i.e. there exists a convex piece-wise linear function $\nu : T_d \to \mathbb{R}$, whose linearity domains are just $\Delta_1, ..., \Delta_N$. The Viro theorem, in fact, endows the combinatorial broken line $C(\tau,\sigma)$ with a rich structure, which implies a number of restrictions to the topology of $C(\tau,\sigma)$ (see an account of known results in [23], [12], [21], [20]).

On the other hand, there are *non-convex* triangulations (see the simplest classical example in Fig.1). There are examples of T-curves beyond the range of known algebraic curves [13], and there is some similarity T-curves and algebraic curves: up to degree 6 all T-curves are isotopic to algebraic ones [4], T-curves satisfy some consequences of Bezout's theorem [5] and the Harnack inequality [6]. The natural question arising is if any T-curve is isotopic to an algebraic curve of the same degree, and if not, how far T-curves may differ from algebraic curves.

In the present paper, for any T-curve, we construct a complexification, i.e. an equivariant surface in $\mathbb{C}P^2$ or $\mathbb{C}^2$, whose real part coincides with the given T-curve (*flexible curve* in definitions of [21]). Then we show that the topology of such complexifications is similar to the topology of the complexifications of real algebraic curves and deduce that arbitrary T-curves satisfy all *topological* restrictions known for real algebraic curves. Similar construction and results we obtain for higher-dimensional T-hypersurfaces in the real projective and affine spaces.

The material is organized as follows: in section 1 we describe the general construction of complexification, in section 2 we study topology of T-curves, section 3 is devoted to topology of T-hypersurfaces.

1 Complexification of T-hypersurfaces

1.1 Notations and definitions

Further on the term *polyhedron* (*polygon*) means a convex polyhedron (polygon) in $\mathbb{R}^n$, $n \geq 2$, whose vertices have non-negative integral coordinates.

Given a polynomial

$$F = \sum_{i_1,...,i_n} A_{i_1...i_n} z_1^{i_1} \cdot ... \cdot z_n^{i_n} \ ,$$

by $\Delta(F)$ we denote its Newton polyhedron, the convex hall of the set

$$\{(i_1, ..., i_n) \in \mathbb{R}^n \ : \ A_{i_1...i_n} \neq 0\} \ .$$

The truncation of F on a face δ of $\Delta(F)$ is the polynomial

$$F^\delta = \sum_{(i_1,...,i_n)\in\delta} A_{i_1...i_n} z_1^{i_1} \cdot ... \cdot z_n^{i_n} \ .$$

A polynomial $F \in \mathbb{C}[z_1, ..., z_n]$ is called *non-degenerate*, if F and any truncation F^δ on a proper face δ of $\Delta(F)$ have a non-singular zero set in $(\mathbb{C}^*)^n$ (cf. [18]).

1.2 Complexification of the moment map

Let $\Delta \subset \mathbb{R}^n$ be a polyhedron, and let $\mu_\Delta : (\mathbb{C}^*)^n \to I(\Delta)$ be the moment map (see [1], [2], [7], 6.1, [10]), where $I(\Delta)$ is the complement in Δ of the union of all its proper faces,

$$\mu_\Delta(z_1, ..., z_n) = \frac{\sum_{(i_1,...,i_n)\in\Delta} |z_1|^{i_1} ... |z_n|^{i_n} \cdot (i_1, ..., i_n)}{\sum_{(i_1,...,i_n)\in\Delta} |z_1|^{i_1} ... |z_n|^{i_n}} = (\mu_\Delta^{(1)}, ..., \mu_\Delta^{(n)}). \qquad (1)$$

Put

$$\mathbb{C}(I(\Delta)) = \{(w_1, ..., w_n) \in \mathbb{C}^n \ : \ (|w_1|, ..., |w_n|) \in I(\Delta)\},$$

and define *the complexification* $\mathbb{C}\Delta$ *of* Δ to be the closure of $\mathbb{C}I(\Delta)$ in $\mathbb{C}^n$.

Proposition 2 *$\mathbb{C}\Delta$ is a PL-manifold with boundary and the singular set*

$$\mathrm{Sing}(\mathbb{C}\Delta) = \bigcup_{k=1}^{n} \bigcup_{\substack{\delta=\Delta\cap\{w_k=0\}\\ \dim\delta\le\dim\Delta-2}} \mathbb{C}\Delta\ .$$

The real part $\mathbb{R}\Delta$ of $\mathbb{C}\Delta$ is the union of Δ with all its symmetric copies with respect to the coordinate hyperplanes.

Proof. Straightforward. □

Define *the complex moment map* by

$$\mathbb{C}\mu_\Delta : (\mathbb{C}^*)^n \to \mathbb{C}(I(\Delta)), \quad \mathbb{C}\mu_\Delta(z_1,...,z_n) = (\mu_\Delta^{(1)}(\overline{x})v_1, ..., \mu_\Delta^{(n)}(\overline{x})v_n)\ , \tag{3}$$

where

$$x_i = |z_i|,\ v_i = \frac{z_i}{|z_i|},\ i = 1,...,n, \quad \overline{x} = (x_1,...,x_n).$$

As an easy consequence of classical results we state

Proposition 4 *The map $\mathbb{C}\mu_\Delta$ is surjective and commutes with the complex conjugation* Conj. *It is a diffeomorphism when* $\dim\Delta = n$. *The real part of $\mathbb{C}(I(\Delta))$ is the image of $(\mathbb{R}^*)^n$.*

1.3 Complex chart of a real polynomial

Let $F \in \mathbb{R}[z_1,...,z_n]$ be a non-degenerate polynomial with Newton polyhedron $\Delta(F) = \Delta$. The closure $\mathbb{C}Ch(F) \subset \mathbb{C}\Delta$ of the set $\mathbb{C}\mu_\Delta(\{F = 0\} \cap (\mathbb{C}^*)^n)$ is called *the (complex) chart of the polynomial* F, and $\mathbb{R}Ch(F) = \mathbb{C}Ch(F) \cap \mathbb{R}\Delta$ is called *the real chart* of F.

Lemma 1 *The set $\mathbb{C}Ch(F)\backslash\mathrm{Sing}(\mathbb{C}\Delta)$ is a PL-submanifold in $\mathbb{C}\Delta\backslash\mathrm{Sing}(\mathbb{C}\Delta)$ of codimension 2 with boundary on $\partial\mathbb{C}\Delta$. It is smooth in $\mathbb{C}I(\Delta)$ and invariant with respect to* Conj. *For any proper face δ of Δ of positive dimension,*

$$\mathbb{C}Ch(F) \cap \mathbb{C}\delta = \mathbb{C}Ch(F^\delta)\ .$$

Proof. Since the smoothness of $\mathbb{C}Ch(F)$ in $\mathbb{C}I(\Delta)$ is obvious, we have to consider only the limit behavior of $\mathbb{C}Ch(F)$ on $\partial\mathbb{C}I(\Delta)$.

Step 1. We start with study of transformations simplifying the original problem.

First note that the multiplication of F by z_k^a, $a > 0$, translates Δ into Δ', so one has the commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\mathbb{C}\mu_\Delta} & \mathbb{C}I(\Delta) \\ \| & & \uparrow \\ (\mathbb{C}^*)^n & \xrightarrow{\mathbb{C}\mu_{\Delta'}} & \mathbb{C}I(\Delta') \end{array} \tag{5}$$

where the vertical arrow means the subtraction of $az_k/|z_k|$ from the k-th coordinate. Hence this does not influence on the statement of Lemma.

Assume that Δ is contained in a hyperplane $H = \{i_n = a\} \subset \mathbb{R}^n = \{(i_1, ..., i_n)\}$. Denote by Δ' the projection of Δ into $\mathbb{R}^{n-1} = \{i_n = 0\} \subset \mathbb{R}^n$. Then one has the commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\mathbb{C}\mu_\Delta} & \mathbb{C}I(\Delta) \\ \| & & \| \\ (\mathbb{C}^*)^{n-1} \times \mathbb{C}^* & \xrightarrow{\mathbb{C}\mu_{\Delta'} \times az/|z|} & \mathbb{C}I(\Delta') \times \{|z| = 1\} \end{array} \tag{6}$$

Hence the statement of Lemma for Δ is reduced to that for Δ'.

Assume that Δ is contained in a hyperplane $H \subset \mathbb{R}^n = \{(i_1, ..., i_n)\}$. There is a transformation $\alpha = (a_{ij}) \in SL(\mathbb{Z}, n)$ which takes H into a hyperplane $i_n = \text{const}$. Then the coordinate change

$$z_k = \prod_{j=1}^{n} {z'_j}^{a_{jk}}, \quad k = 1, ..., n,$$

with multiplication by $(z'_1 \cdot ... \cdot z'_n)^b$, $b >> 0$, takes F into a polynomial G and moves Δ to some polyhedron Δ'. As said above, this operation does not influence on the statement of Lemma.

Step 2. The remarks in the first step reduce the problem on the intersection of $\mathbb{C}Ch(F)$ with $\partial\mathbb{C}I(\Delta)$ to the following case: $\dim \Delta = n$, Δ lies in the halfspace $i_n \geq a$, and $\delta \subset \Delta$ is a facet (face of codimension 1) lying on the hyperplane $i_n = a$. Let $p \in \mathbb{C}Ch(F) \cap \mathbb{C}I(\delta)$. Then

$$p = \lim_{t \to 0} \mathbb{C}\mu_\Delta(\gamma(t)) \ ,$$

where $\gamma : (0, \varepsilon) \to \{F = 0\} \cap (\mathbb{C}^*)^n$ is an analytic curve with $\gamma(t)$ tending to the hypersurface $x_n = 0$ as $t \to 0$:

$$\gamma(t) = (z_1(t), ..., z_n(t)), \quad z_n(t) = v_n t + O(t^2), \ z_k = \alpha_k v_k + O(t), \ k = 1, ..., n-1, \tag{7}$$

$$\alpha_1, ..., \alpha_{n-1} > 0, \quad |v_1| = ... = |v_n| = 1,$$

$$\mathbb{C}\mu_\Delta(\gamma(t)) = \frac{t^a \sum \alpha_1^{i_1} \cdot ... \cdot \alpha_{n-1}^{i_{n-1}} (i_1 v_1, ..., i_{n-1} v_{n-1}, a v_n) + O(t^{a+1})}{t^a \sum \alpha_1^{i_1} \cdot ... \cdot \alpha_{n-1}^{i_{n-1}} + O(t^{a+1})}$$

$$= \mathbb{C}\mu_\delta(\alpha_1 v_1, ..., \alpha_{n-1} v_{n-1}, t v_n) + O(t) \ , \tag{8}$$

where the sum is taken over all points $(i_1, ..., i_{n-1}, a) \in \delta$. Since $0 = F(\gamma(t)) = F^\delta(\alpha_1 v_1, ..., \alpha_{n-1} v_{n-1}, t v_n) + O(t^{a+1}))$ and $F^\delta(\alpha_1 v_1, ..., \alpha_{n-1} v_{n-1}, t v_n)$ is of order a in t, one obtains

$$F^\delta(\alpha_1 v_1, ..., \alpha_{n-1} v_{n-1}, t v_n) = 0,$$

which together with (8) gives $p \in \mathbb{C}Ch(F^\delta)$.

On the other hand, assuming $p \in \mathbb{C}Ch(F^\delta) \cap \mathbb{C}I(\delta)$, one has

$$p = \mathbb{C}\mu_\delta(\alpha_1 v_1, ..., \alpha_{n-1}v_{n-1}, \alpha_n v_n), \quad \alpha_k > 0,\ |v_k| = 1,\ k = 1, ..., n,$$

$$F^\delta(\alpha_1 v_1, ..., \alpha_{n-1}v_{n-1}, \alpha_n v_n) = 0,$$

which means, in particular, that $(\alpha_1 v_1, ..., \alpha_{n-1}v_{n-1}, 0)$ is a non-singular intersection point of $\{F = 0\}$ with the hyperplane $x_n = 0$. Hence there exists a curve $\gamma : (0, \varepsilon) \to \{F = 0\} \cap (\mathbb{C}^*)^n$ as in (7), and we show $p \in \mathbb{C}Ch(F)$ in the previous manner.

Step 3. The problem on the boundary of $\mathbb{C}Ch(F)$ similarly is reduced to the study of the case $\dim \Delta = n$ by induction in n.

Let δ be a facet of Δ, which is not contained in a coordinate hyperplane. Then

$$\dim(\mathbb{C}Ch(F^\delta) \cap \mathbb{C}I(\delta)) = \dim \mathbb{C}Ch(F) - 1,$$

which means $\mathbb{C}Ch(F^\delta) \cap \mathbb{C}I(\delta)$ is a part of the boundary of $\mathbb{C}Ch(F)$. If two facets δ_1, δ_2 have a common face δ_* of dimension $n-2$, which is not contained in a coordinate hyperplane, then $\mathbb{C}Ch(F^{\delta_1})$ and $\mathbb{C}Ch(F^{\delta_2})$ have a common boundary $\mathbb{C}Ch(F^{\delta_*})$ (in $\mathbb{C}\Delta \backslash \mathrm{Sing}(\mathbb{C}\Delta)$). Hence $\bigcup \mathbb{C}Ch(F^\delta) \backslash \mathrm{Sing}(\mathbb{C}\Delta)$, where δ runs through all facets of Δ, not lying in coordinate hyperplanes, is a manifold of dimension $\dim \mathbb{C}Ch(F) - 1$ without boundary, which is the boundary of $\mathbb{C}Ch(F)$ in the complement of $\mathrm{Sing}(\mathbb{C}\Delta)$.

Let δ be a facet of Δ lying in the coordinate plane $w_1 = 0$. Since δ contains a point $(0, i_2, ..., i_n)$, $i_2 \cdot ... \cdot i_n \neq 0$, the formulae (1), (3) define a smooth continuation of $\mathbb{C}\mu_\Delta$ on $(\mathbb{C}^*)^n \cup (\{w_1 = 0\} \backslash \{w_2 \cdot ... \cdot w_m \neq 0\})$. The non-degeneracy of F implies that $\{f = 0\}$ is non-singular in $(\mathbb{C}^*)^n \cup (\{z_1 = 0\} \backslash \{z_2 \cdot ... \cdot z_n \neq 0\})$. Hence $\mathbb{C}CH(F) \cap \mathrm{Int}(\mathbb{C}\Delta)$ is a smooth manifold without boundary, and we are done. □

Corollary 9 *If* $\dim \Delta = 2$, $\Delta \subset \mathbb{R}^2$, *then* $\mathbb{C}Ch(F)$ *is a smooth surface with boundary.*

1.4 Gluing and projectivization of charts

Let us be given

- the simplex T_d^n with the vertices

  $$(0, ..., 0),\ (d, 0, ..., 0),\ (0, d, 0, ..., 0),\ \dots\ ,\ (0, ..., 0, d) \in \mathbb{R}^n\ ,$$

  and its subdivision $\mathcal{S}$:

  $$T_d^n = \Delta_1 \cup ... \cup \Delta_N$$

  into equidimensional polyhedra (i.e. $\Delta_i \cap \Delta_j$ is empty or a common proper face),

- a set of real numbers $\mathcal{A} = \{A_{i_1...i_n},\ (i_1, ..., i_n) \in T_d^n \cap \mathbb{Z}^n\}$, such that $A_{i_1...i_n} \neq 0$ if $(i_1, ..., i_n)$ is a vertex of Δ_i, $1 \le i \le N$.

Assume that the polynomials

$$F_k(z_1,...,z_n) = \sum_{(i_1,...,i_n)\in\Delta_k} A_{i_1...i_n} z_1^{i_1}\cdot ... \cdot z_n^{i_n}\ , \quad k=1,...,N,$$

are non-degenerate. We define *the gluing of the complex and real charts of* $F_1,...,F_N$ by

$$\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \bigcup_{k=1}^N \mathbb{C}Ch(F_k)\ , \quad \mathbb{R}Ch(\mathcal{S},\mathcal{A}) = \bigcup_{k=1}^N \mathbb{R}Ch(F_k)\ .$$

Lemma 2 $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ *is a PL-manifold in* $\mathbb{C}T^n_d$, *which is invariant with respect to* Conj, *and*

$$\partial\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \mathbb{C}Ch(\mathcal{S},\mathcal{A})\cap\subset\partial\mathbb{C}T^n_d\ .$$

Remark 1 Neither $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$, nor $\mathbb{R}Ch(\mathcal{S},\mathcal{A})$ are smooth in general. Indeed, for the polynomials

$$F_1(x,y) = x(y-x-1), \quad F_2(x,y) = xy-x+1$$

with the Newton triangles

$$\Delta_1 = [(1,0),(1,1),(2,0)], \quad \Delta_2 = [(0,0),(1,0),(1,1)]$$

the curves

$$\mu_{\Delta_1}(\{F_1=0\}) = \left\{\left(\frac{2+3x}{2+2x}\, ,\, \frac{x+1}{2+2x}\right)\ :\ x>0\right\},$$

$$\mu_{\Delta_2}(\{F_2=0\}) = \left\{\left(\frac{2x-1}{2x}\, ,\, \frac{x-1}{2x}\right)\ :\ x>0\right\}$$

have the common limit point $(1,1/2)$ on the edge $[(1,0),(1,1)]$, and different tangent lines

$$y=(a-1)x-a+\frac{3}{2}, \quad y=x-\frac{1}{2}$$

at this point.

Proof of Lemma 2. We have to consider only the gluing of charts on the coordinate hyperplanes.

Step 1. Let us translate all the polyhedra by the vector $(1,...,1)$ and multiply all the polynomials by $z_1\cdot ... \cdot z_n$. Denote the corresponding objects by the same symbols with an additional tilde. Introduce the following deformation of the maps $\mathbb{C}\mu_{\Delta_k}$, $k=1,...,N$:

$$\mathbb{C}\mu_{k,t}(z_1,...,z_n) = \mathbb{C}\mu_{\Delta_k}(z_1,...,z_n) + (tv_1,...,tv_n),$$

$$v_1 = \frac{z_1}{|z_1|},\ ...\ ,\ v_n = \frac{z_n}{|z_n|}, \quad t\in[0,1].$$

Clearly, this is a homotopy, connecting $\mathbb{C}\mu_{\Delta_k}$ with $\mathbb{C}\mu_{\widetilde{\Delta}_k}$, for any $k=1,...,N$. Then $\mathrm{Cl}(\mathbb{C}\mu_{k,t}(\{F_k=0\}))$, the closure of $\mathbb{C}\mu_t(\{F_k=0\})$, varies from $\mathbb{C}Ch(F_k)$ to $\mathbb{C}Ch(\widetilde{F}_k)$.

Since $\mathrm{Sing}(\mathbb{C}\widetilde{\Delta}_k) = \emptyset$, $k = 1, ..., N$,

$$\mathcal{K}_1 = \bigcup_{k=1}^{N} \mathbb{C}Ch(\widetilde{F}_k)$$

is a PL-manifold with boundary on $\partial\mathbb{C}\widetilde{T}^n_d$, according to Lemma 1. Denote some faces of $\widetilde{T}^n_d$ as follows:

$$\widetilde{T}^n_d(k) = \widetilde{T}^n_d \cap \{i_k = 1\}, \quad \widetilde{T}^n_d(k_1, ..., k_s) = \bigcap_{j=1}^{s} \widetilde{T}^n_d(k_j) \ .$$

Put $\delta = \widetilde{T}^n_d$. By (5)

$$\begin{aligned} \mathbb{C}\delta = \ & \{(w_1, ..., w_n) \ : \ (w_1, ..., w_{k-1}, w_{k+1}, ..., w_n) \in \mathbb{C}T^{n-1}_d, \ |w_k| = 1\} \\ & \cong \mathbb{C}T^{n-1}_d \times S^1. \end{aligned} \tag{10}$$

Similarly, by (6) $\mathcal{K}_1 \cap \mathbb{C}\delta$ is a PL-manifold of dimension $2n - 3$ with boundary on $\partial\mathbb{C}\delta$, satisfying

$$\mathcal{K}_1 \cap \mathbb{C}\delta \cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}^\delta_l) \right) \times S^1, \tag{11}$$

where the product structure is compatible with that in (10). Using the product representations (10) and (11), we define

$$\begin{aligned} D^n_d(k) = \{(w_1, ..., w_n) \ : \ (w_1, ..., w_{k-1}, w_{k+1}, ..., w_n) \in \mathbb{C}T^{n-1}_d, \ |w_k| \leq 1\} \\ \cong \mathbb{C}T^{n-1}_d \times D^2 \ , \end{aligned}$$

where D^2 is the closed 2-dimensional unit ball, and inside $D^n_d(k)$ the $(2n-2)$-dimensional manifold

$$M(k) = \{(w_1, ..., w_n) \ : \ (w_1, ..., w_{k-1}, 1, w_{k+1}, ..., w_n) \in \bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}^\delta_l), \ |w_k| \leq 1\}$$

$$\cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}^\delta_l) \right) \times D^2$$

with boundary on $\partial D^n_d(k)$ and such that

$$M(k) \cap \mathbb{C}\delta = \bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}^\delta_l) \ .$$

Similarly, given $\delta = T^n_d(k_1, ..., k_s)$, we have by (5), (6)

$$\mathbb{C}\delta \cong \mathbb{C}T^{n-s}_d \times (S^1)^s,$$

$$\mathcal{K}_1 \cap \mathbb{C}\delta \cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}_l^\delta) \right) \times (S^1)^s ,$$

and one defines

$$D_d^n(k_1, ..., k_s) = \{(w_1, ..., w_n) \ : \ (w_j)_{j \neq k_1,...,k_s} \in \mathbb{C}T_d^{n-s}, \ |w_{k_1}|, ..., |w_{k_s}| \leq 1\}$$
$$\cong \mathbb{C}T_d^{n-s} \times (D^2)^s ,$$

and the $(2n-2)$-dimensional manifold

$$M(k_1, ..., k_s) = \{(w_1, ..., w_n) \ : \ (w_j)_{j \neq k_1,...,k_s} \in \bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}_l^\delta), \ |w_{k_1}|, ..., |w_{k_s}| \leq 1\}$$
$$\cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}_l^\delta) \right) \times (D^2)^s$$

with boundary on $\partial D_d^n(k_1, ..., k_s)$ and such that

$$M(k_1, ..., k_s) \cap \mathbb{C}\delta = \bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}_l^\delta) .$$

It can easily be shown that

$$\mathcal{T}_1 = \mathbb{C}\widetilde{T}_d^n \cup \bigcup_{s=1}^{n} \bigcup_{1 \leq k_1 < ... < k_s} D_d^n(k_1, ..., k_s)$$

is the convex hull of $\mathbb{C}\widetilde{T}_d^n$ in $\mathbb{C}^n$, and

$$\mathcal{K}_1 \cup \bigcup_{s=1}^{n} \bigcup_{1 \leq k_1 < ... < k_s} M(k_1, ..., k_s)$$

is a PL-manifold in $\mathcal{T}_1$ of codimension 2 with boundary on $\partial \mathcal{T}_1$.

Step 2. To connect the previous picture with the original one we introduce the following deformation of the maps $\mathbb{C}\mu_\delta$:

$$\mathbb{C}\mu_{\delta,t}(z_1, ..., z_n) = \mathbb{C}\mu_\delta(z_1, ..., z_n) + (tv_1, ..., tv_n),$$

$$v_1 = \frac{z_1}{|z_1|}, \ \dots \ , \ v_n = \frac{z_n}{|z_n|}, \quad t \in [0,1].$$

Clearly, this is a homotopy, connecting $\mathbb{C}\mu_\delta$ with $\mathbb{C}\mu_{\widetilde{\delta}}$. Similarly, $\mathrm{Cl}(\mathbb{C}\mu_{k,t}(\{F_k = 0\}))$, the closure of $\mathbb{C}\mu_{\Delta_k,t}(\{F_k = 0\})$, varies from $\mathbb{C}Ch(F_k)$ to $\mathbb{C}Ch(\widetilde{F}_k)$ as t runs from 0 to 1.

Now, for any $t \in (0,1)$, in the same way one constructs similar objects:

- complexifications of polyhedra

$$\mathbb{C}\Delta_{k,t} = \mathrm{Cl}(\mathbb{C}\mu_{\Delta_k,t}((\mathbb{C}^*)^n)), \quad \mathbb{C}T^n_{d,t} = \bigcup_{k=1}^{N} \mathbb{C}\Delta_{k,t},$$

- gluing of charts in $\mathbb{C}T^n_{d,t}$

$$\mathcal{K}_t = \bigcup_{k=1}^{N} \mathrm{Cl}(\mathbb{C}\mu_{\Delta_k,t}(\{F_k = 0\})),$$

  a PL-manifold of codimension 2 with boundary on $\partial\mathbb{C}T^n_{d,t}$,

- completion of $\mathbb{C}T^n_{d,t}$ up to its convex hull

$$\mathcal{T}_t = \mathbb{C}T^n_{d,t} \cup \bigcup_{s=1}^{n} \bigcup_{1\le k_1<\ldots<k_s} D^n_{d,t}(k_1,\ldots,k_s),$$

  where

$$D^n_{d,t}(k_1,\ldots,k_s) = \{(w_1,\ldots,w_n) \ : \ (w_j)_{j\ne k_1,\ldots,k_s} \in \mathbb{C}T^{n-s}_d, \ |w_{k_1}|,\ldots,|w_{k_s}| \le t\}$$
$$\cong \mathbb{C}T^{n-s}_d \times (D^2(t))^s \ ,$$

  $D^2(t)$ is a disc of radius t,

- a $(2n-2)$-dimensional manifold

$$M_t(k_1,\ldots,k_s) = \{(w_1,\ldots,w_n) \ :$$
$$(w_j)_{j\ne k_1,\ldots,k_s} \in \bigcup_{l=1}^{N} \mathrm{Cl}(\mathbb{C}\mu_{\delta,t}(\{F^\delta_l = 0\})), \ |w_{k_1}|,\ldots,|w_{k_s}| \le t\}$$
$$\cong \left(\bigcup_{l=1}^{N} \mathrm{Cl}(\mathbb{C}\mu_{\delta,t}(\{F^\delta_l = 0\}))\right) \times (D^2(t))^s$$

- a PL-manifold

$$\mathcal{K}_t \cup \bigcup_{s=1}^{n} \bigcup_{1\le k_1<\ldots<k_s} M_t(k_1,\ldots,k_s) \tag{12}$$

  of codimension 2 in $\mathcal{T}_t$ with boundary on $\partial\mathcal{T}_t$.

If t varies from 1 to 0, $\mathcal{T}_t$ contracts from $\mathcal{T}_1$ to $\mathbb{C}T^n_d$ and the manifold (12) naturally contracts into manifold $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ with boundary on $\partial\mathbb{C}T^n_d$. □

Evidently, $\mathbb{C}T^n_d$ is homeomorphic to the closed unit ball in $\mathbb{C}^n$. The group $S^1 = \{|v| = 1\}$ acts freely on $\partial\mathbb{C}T^n_d$ by

$$v \in S^1, \ (z_1,\ldots,z_n) \in \partial\mathbb{C}T^n_d \ \mapsto \ (z_1 v,\ldots,z_n v) \in \partial\mathbb{C}T^n_d \ .$$

Proposition 13 *The quotient $\mathbb{C}T_d^n/S^1$ is equivariantly homeomorphic to $\mathbb{C}P^n$.*

Proof. We shall define this homeomorphism explicitly. The composition of the modified moment map for T_1^n with a homothety

$$\mathbb{C}\mu(z_1, ..., z_n) = \frac{d}{1 + |z_1| + ... + |z_n|} \cdot (z_1, ..., z_n)$$

defines an equivariant homeomorphism $\mathbb{C}^n \to \mathrm{Int}(\mathbb{C}T_d^n)$. Consider the inverse map

$$(\mathbb{C}\mu)^{-1} : \mathrm{Int}(\mathbb{C}T_d^n) \to \mathbb{C}^n, \quad (\mathbb{C}\mu)^{-1}(w_1, ..., w_n) = \frac{(w_1, ..., w_n)}{d - |w_1| - ... - |w_n|} .$$

It naturally extends up to the map

$$\nu_{d,n} : \mathbb{C}T_d^n \to \mathbb{C}P^n, \quad \nu(w_1, ..., w_n) = (d - |w_1| - ... - |w_n|, w_1, ..., w_n) .$$

It commutes with Conj, is surjective, and sends orbits of the S^1-action on $\partial\mathbb{C}T_d^n$ into the points of the infinitely far hyperplane in $\mathbb{C}P^n$. □

Definition 1 In the previous notation, in the case $\Delta = T_d^n$ we define *the projective gluing of charts* as

$$P\mathbb{C}Ch(\mathcal{S}, \mathcal{A}) = \nu_{d,n}(\mathbb{C}Ch(\mathcal{S}, \mathcal{A})) \subset \mathbb{C}P^n .$$

Lemma 3 *In the previous notation, $P\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is a PL-submanifold in $\mathbb{C}P^n$ of codimension 2, invariant with respect to* Conj.

Proof. We have to verify only that the quotient of $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ by the S^1-action on $\partial\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is a closed manifold.

Let us consider the following automorphism of the torus $(\mathbb{C}^*)^n$:

$$(z_1, ..., z_n) \mapsto (z_1^{-1}, z_1^{-1}z_2, ..., z_1^{-1}z_n),$$

which comes from the permutation of projective coordinates in $\mathbb{C}P^n \supset (\mathbb{C}^*)^n$

$$(y_0 : y_1 : ... : y_n) \mapsto (y_1 : y_0 : y_2 : ... : y_n), \quad z_k = \frac{y_k}{y_0}, \ k = 1, ..., n.$$

It generates the automorphism of $\mathbb{Z}^n$

$$(i_1, ..., i_n) \mapsto (d - i_1 - ... - i_n, i_2, ..., i_n),$$

which maps T_d^n into itself interchanging the facet with vertices $(d, 0, ..., 0)$, $(0, d, 0, ..., 0)$,..., $(0, ..., 0, d)$ and the facet, lying in the coordinate hyperplane $i_1 = 0$. The latter automorphism, clearly, extends up to automorphism of $\mathbb{C}T_d^n/S^1$ compatible with the structure of projective space. On the other hand, a neighborhood of $(\partial\mathbb{C}T_d^n)/S^1$ in $\mathbb{C}T_d^n/S^1$ is send by the automorphism defined into a neighborhood of $\mathbb{C}T_d^n(1)$ (in the notation of Step 1 in the proof of Lemma 2), but we have shown above that $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is a PL-manifold there, that completes the proof. □

2 Topology of T-curves

3 Topology of T-hypersurfaces

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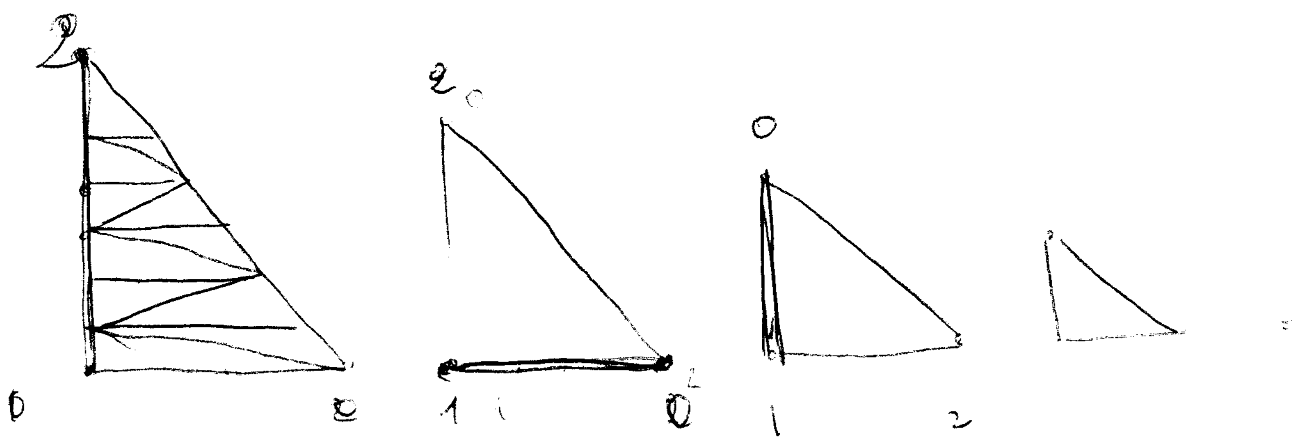

$(0,0,0)$

$\mathbb{R}^{n-1}$

z, x, y $(1,0,0)$

Simplex I

$0 \quad n-1$

The Geometry Center

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SCENE 0: Title
Viro's Patchworking disproves
Ragsdale's Conjecture

Corrections by
Ilía

SCENE I: Hilbert Problem

Voice of Hilbert:(8) "Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science? | Picture of Hilbert

N2:(10) Are any of the 23 questions posed by David Hilbert in his famous address to the International Congress of mathematicians still unsolved? | Title of Compte Rendu

Title of Hilbert's address

N1:(12) Yes, several. For example, Hilbert's 16th problem, which is concerned with the ways that nonsingular level sets of polynomials can be arranged in the projective plane. | Show RP^2 square with ovals. Some nested, others not.

the first part ✓

N2:(7) Nonsingular? The projective plane?

SCENE II: Saddle Slice

N1:(14) Sorry. Let me back up. For a polynomial of two variables, a level set is the set of points in the domain that is mapped up to a certain height. Usually, these points form a curve. | Dissolve initial scene. Show saddle surface with intersecting plane and corresponding level set in domain.

N2:(7) I see. Like representing a hyperbola as the set of points on which a quadratic polynomial is constant.

SCENE III: Complicated Slice

N1:(20) Exactly. And we can consider more complicated polynomials as well. | Dissolve previous graph and level se Replace with complicated graph. Begin slicing.

As we change the height or LEVEL, then typically the level set also changes. But the level set can change its topology only by passing through singular states in which some point on the curve does not have a well-defined tangent line. | Animation of changing levels.

N2:(7) So "nonsingular" means that we don't consider these transitional states.

N1:(1) That's right.

SCENE IV: Projective Plane

N2:(5) And what about this "projective | Fade out surface plane"....

N1:(18) Algebraic geometers discovered that it is easier to study level sets in the projective plane. Geometrically, think of shrinking down the entire affine plane into a square, and then gluing together opposite edges identified with a twist.

Transform previous level set into square model of RP^2

Highlight opposite edges.

or zero sets ✓

N2:(12) I can see how this simplifies things! By gluing together the opposite edges, the two hyperbolas have joined to become a single topological circle!

Trace point on hyperbola around curv It vanishes through the line at infinity then reappears on the other side.

SCENE V: Highlight

N1:(9) Exactly! And notice that sometimes the so-called ovals are nested, (PAUSE) but sometimes they are disjoint.

Show examples of each.

N2:(2) So Hilbert's 16th problem is...

N1:(10) ...to determine all possible arrangements of ovals that arise as the nonsingular level set of a polynomial...

Animation of changing levels of interesting example.

nonsingular zero set of a polynomial of a given degree ✓

N2:(4) ...when we think of the ovals as sitting inside the projective plane!

SCENE VI: Mobius-Disk

N1:(12) Very good. You're Right again. Note that an oval divides the projective plane into two pieces: the outside of the oval is a Mobius strip...

Show two pieces.
Send painted arrow through edge of Mobius to show that it reenters RP^2 with a twist of orientation.

N2:(NA) Oh, I see. Because we;ve glued the opposite edges together with a twist!

SCENE VII: Odd/Even Ovals

N1:(10) ...whereas the inside of the oval is a disk.
We call an oval that lies inside an even number of other ovals, an "even oval."

Highlight even ovals.

N2:(9) Let me guess: ovals that lie inside an ODD number of other ovals are called "odd ovals"?

Highlight odd ovals.

N1:(1) Naturally.

N2:(5) Why would anyone care about the numbers of even and odd ovals?

N1:(17) (Laughs) Mathematical insight, I guess! The difference between these quantities is the Euler characteristic of some region. So these numbers are tied to the topology of real and complex level sets in subtle ways.

zero ?

Highlight region bounded by positive and negative ovals.

these numbers are the numbers of some regions

what do you mean ?

SCENE VIII: Ragsdale

N2:(5) Wow! And the insightful person who thought of this was...?

N1:(23) Virginia Ragsdale. She graduated from Guilford College in 1892, pursued her PhD at Bryn Mawr, and visited Germany to study with Felix Klein and David Hilbert.

Picture of Ragsdale

She conjectured a quadratic bound on the number of positive and negative ovals that can exist for polynomials of even degree.

In general,
p <= (3d^2 -6d +8)/8

even and odd !

N2:(4) So what happened with the Ragsdale Conjecture?

N1:(11) After almost 90 years, a counter-example was recently found using a new method of constructing algebraic curves!

d=10 ==> Max Ovals=37
Conjectured Max P=32

N2:(1) A new method?

SCENE IX: Four Copies

N1:(14) Oleg Viro invented a combinatorial method to build a MODEL of a nonsingular level set. Start with the integer lattice points in a right triangle with legs of length d.

1st quadrant lattice points

N2:(5) Is d still the degree of the polynomial that we are studying?

N1:(10) Correct. Now triangulate this region, and then reflect it four times to form a diamond.

Triangulate and replicate

N2:(6) This wouldn't have anything to do with the projective plane, would it?

N1:(12) Very good! You've anticipated the next step! Glue together antipodal segments on the edge of the diamond to form a projective plane.

Highlight glued edges

N2:(8) Okay, so we have a model of the projective plane, but how do we use this to create a curve?

SCENE X: Add Signs

N1:(16) Specify either a plus sign or a minus sign for all of the vertices in the first quadrant. Then extend these signs to other quadrants by flipping the sign of odd columns and odd rows.

Show signs in 1st quadrant.

Flip into 2nd quadrant.
Flip into 3rd and 4th.

SCENE XI: T-Curve

N1 (cont):(11) Now, if these signs represented the value of a polynomial at selected points in the projective plane, how would you find the level set of ZERO for this polynomial?

the sign of a polynomial (of an even degree) ✓?

N2:(5) Well, I'd separate the positive regions from the negative regions!

Divide one triangle.

N1:(11) Right! So whenever a triangle has vertices of different signs, draw a segment separating the positive vertices from the negative.

Divide all triangles.

N2:(13) If we do this for all of the triangles, then we get a collection of topological ovals! But is this connected to the level sets we looked at before? Can we find a polynomial corresponding to this curve?

N1:(17) A powerful theory started by Viro says that if the triangulation has a property called "convexity", then we can produce a polynomial whose level set has the same topology as the piecewise linear curve that we constructed!

Fade out discrete model;
Fade in continuous curve with same topology

SCENE XII: Counter-example

N2:(6) Incredible! And how did this resolve the Ragsdale Conjecture?

N1:(14) Using this construction, sometimes called PATCHWORKING, Ilya Itenberg found a triangulation and sign distribution that provided a counterexample to Ragsdale's conjecture.

Display Itenberg's triangulation and curve.

N2:(2) You're right! There are are 32 positive ovals there! Then the story is over.

even ! ✓

N1:(24) Far from it! Hilbert's 16th problem is still unsolved,
But now we have an algorithm that can be used to construct examples of algebraic curves,

Construct octahedron...
Triangulate interior...
add signs to vertices..
add surface!

or even a surface inside a model
of projective three-space!

SCENE XIII: Surface

N2:(8) So any time I use Viro's combinatorial algorithm to produce a curve or a surface or even a higher-dimensional object...

Rotate surface.

N1:(8) ...you are guaranteed that you found the topological type of a level set of some polynomial!

N2:(4) Wow. It sounds like magic.

Delete coordinate planes.
Rotate till end.

N1:(5) It's better than magic, it's mathematics!

SCENE XIV: Credits (15 seconds MAX)

Written by DeLoera/Wicklin
Voices, Music.
Software: Pisces, Geomview, StageManager
Thanks to Springer-Verlag and Guilford College
The Geometry Center