Viro's Method and T-curves

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1 Introduction

Let A be a real algebraic plane projective curve of degree m, i.e. a real homogeneous polynomial in three variables of degree m considered up to multiplication by a non-zero real number. We suppose the curve to be non-singular, which means that the polynomial does not have singular points in $\mathbf{R}^3 \setminus 0$.

This polynomial has a well defined zero locus $\mathbf{R}A$ in the real projective plane $\mathbf{R}P^2$. The set $\mathbf{R}A$ is a union of non-intersecting circles embedded in $\mathbf{R}P^2$. The topological type of the pair $(\mathbf{R}P^2,\ \mathbf{R}A)$ is defined by the scheme of disposition of the components of $\mathbf{R}A$. This scheme is called *the real scheme of curve A*.

In 1900 D. Hilbert [Hi] included the following question in the 16-th problem of his famous list : what kind of real schemes can be realized by non-singular curves of a given degree ? The complete answer is known now only for curves of degree not greater than 7.

To solve the problem it is necessary to work in two main directions : first, to find the restrictions for the topological types of pairs $(\mathbf{R}P^2,\ \mathbf{R}A)$, and, second, to give the constructions of curves for realizable real schemes. Many deep and important results were obtained in the first direction using the modern machinary of algebraic and differential topology (see, for example, the survey papers [Vi 5], [Wi]). However, the methods of constructions had not been seriously changed since the XIX century until 1980, when O. Viro proposed a principaly new method to construct the curves (see [Vi 3], [Vi 4], [Vi 6], [Ri]).

In the present paper we discuss a special case of the Viro's method, which is proved to be useful and has some fruitful applications. In this case the Viro's method gives a possibility to construct curves using a simple combinatorial procedure. Slight modifications of this method allow to construct different objects like, for example, real polynomials in two variables with prescribed collections of critical points (see [Sh]) and real polynomial vector fields in $\mathbf{R}^2$ with prescribed collections of non-degenerated singular points (see [It-Sh]).

I would like to thank V. Kharlamov and O. Viro for the useful comments and discussions.

2 T-curves

2.1 Construction

Let m be a positive integer number and T be the triangle in $\mathbf{R}^2$

$$\{x \geq 0, \;\; y \geq 0, \;\; x + y \leq m\}.$$

Suppose that T is triangulated in such a way that the vertices of the triangles are integer, and that some distribution of signs, $a_{i,j} = \pm$ at the vertices of the triangulation, is given. Then there arises a naturally associated piecewise-linear curve L in $\mathbf{R}P^2$.

The construction of L is the following.

Take copies

$$T_x = s_x(T), \;\; T_y = s_y(T), \;\; T_{xy} = s(T)$$

of T, where $s = s_x \circ s_y$ and $s_x, \; s_y$ are reflections with respect to the coordinate axes. Extend the triangulation of T to a symmetric triangulation of $T \cup T_x \cup T_y \cup T_{xy}$ and extend the distribution of signs to a distribution on the vertices of the extended triangulation which verifies the modular property: $g^*(a_{i,j}x^iy^j) = a_{g(i,j)}x^iy^j$ for $g = s_x, \; s_y$ and s (other words, the sign at a vertex is the sign of the corresponding monomial in the quadrant containing the vertex).

If a triangle of the triangulation has vertices of different signs, select a midline separating them. Denote by L' the union of the selected midlines (see, for example, Figure 1). It is contained in $T \cup T_x \cup T_y \cup T_{xy}$. Glue by s the sides of $T \cup T_x \cup T_y \cup T_{xy}$. The resulting space T_* is homeomorphic to $\mathbf{R}P^2$. Take the curve L to be the image of L' in T_*.

A pair (T_*, L) is called *a chart* of a real algebraic plane projective curve A, if there exists a homeomorphism of pairs $(T_*, L) \longrightarrow (\mathbf{R}P^2, \mathbf{R}A)$.

Let us introduce two additional assumptions : the considered triangulation of T is *primitive* and *convex*. The first condition means that all triangles are of area $1/2$ (or, equivalently, that all integer points of T are vertices of the triangulation). The second one means that there exists a convex piecewise-linear function $T \longrightarrow \mathbf{R}$ which is linear on each triangle of the triangulation and not linear on the union of two triangles.

Theorem 2.1 (O. Viro) *Under the assumptions made above on the triangulation of the triangle T, there exists a non-singular real algebraic plane projective curve A of degree m with the chart (T_*, L).*

This statement is the special case of Viro's theorem [Vi 4, Th. 1.4]. We will not discuss a proof in the present paper. Let us just mention that the main idea is to consider a polynomial

$$Q_t(x,y) = \sum_{(i,j) \in T} a_{i,j}x^iy^jt^{\nu(i,j)}$$

(where i, j are integer numbers, t is a parameter, $a_{i,j}$ is the sign of the integer point (i,j), and ν is a convex function defining the triangulation of T) and to remark that the projectivisation of the polynomial $Q_t(x,y)$ for sufficiently small positive values of t defines a curve with the required chart.

A curve having the chart (T_*, L) is called *a T-curve.* This notion was introduced by S. Orevkov [Or].

Theorem 2.1 gives a combinatorial way to construct the curves. One should choose a primitive convex triangulation of the triangle T and signs at the vertices of the triangulation, and then, using the procedure described above, draw the curve L.

Example The construction of a T-curve of degree 3 with two connected components of the real point set is shown in Figure 1.

Let us write down a polynomial defining this curve. Take the convex piecewise-linear function $T \longrightarrow \mathbf{R}$ with the values at the integer points presented in Figure 2. Then the polynomial is as follows :

$$t^3 + tx + ty + tx^2 - xy + ty^2 + t^3x^3 + tx^2y + txy^2 + t^3y^3$$

We cannot precise an acceptable value of t. We are just able to say that for a sufficiently small positive value of t this polynomial defines our curve.

2.2 T-curves among all curves

It is natural to pose the following question : can the real scheme of an arbitrary non-singular real algebraic plane projective curve be realized by a T-curve of the same degree ?

One can immediately find a trivial restriction : evidently, the empty real scheme of a curve of an even degree cannot be realized by T-curves. We will formulate another, more serious restriction. <- !!!

Let us, first, give the necessary definitions. *An M-curve* is a curve having the maximal possible number of connected components of the real point set for a given degree. It was proved by Harnack [Har] that this maximal number is equal to $\frac{(m-1)(m-2)}{2} + 1$ for the degree m.

Each connected component of the real point set $\mathbf{R}A$ of a curve of even degree is called *an oval.* It is embedded in $\mathbf{R}P^2$ two-sidedly and divides $\mathbf{R}P^2$ in two parts. We call the part homeomorphic to a disk *the interior* of the oval.

A pair of ovals is called *injective* if one oval of this pair lies inside of the other one. Let us denote by J the number of ovals of a curve containing inside of them at least one injective pair.

Proposition 2.2 *For a T-curve of degree m being an M-curve the following inequality holds*

$$J \leq 3m$$

Prove this!!!!

This proposition has a purely combinatorial proof.

Remark that the Proposition 2.2 gives a strong restriction on the topology of T-curves being M-curves. One can easily construct such a family of M-curves of increasing degrees that the numbers J of the curves of this family would depend quadratically in the degree.

There are many open questions in the subject under discussion. For example,

(i) how large is the class of T-curves (is it true that in a sense almost all curves are T-curves) ?

(ii) is it true that a T-curve of degree m being an M-curve has no more than $O(m)$ non-empty ovals ?

A statement similar to the statement of the Theorem 2.1 can be formulated and be proved in any dimension. Shustin [Sh] proved that the number of connected components of T-surfaces of degree m is not greater than $m^3/6 + O(m^2)$. However, Viro [Vi 1] constructed the surfaces of degree m with $(7m^3 - 24m^2 + 32m)/24$ connected components for any $m = 4l + 2$ (l is a positive integer number). That means, in particular, that these surfaces are not T-surfaces.

Question settle!

We will give some constructions of curves using the Theorem 2.1 in the following section. These examples show that the class of T-curves is sufficiently rich. Subsection 3.2 is devoted to the counter-examples to Ragsdale conjecture, subsection 3.3 - to the classification of M-curves with one non-empty oval.

3 Examples of T-curves

3.1 Construction of Harnack curves

In this subsection we will describe, using the Theorem 2.1, the construction of some M-curves (a special case of Harnack curves). This construction will play an important role in the subsections 3.2 and 3.3.

Let $m = 2k$ be a positive even number, and T again be the triangle in $\mathbf{R}^2$

$$\{x \geq 0, \ \ y \geq 0, \ \ x + y \leq m\}.$$

An integer point of T is called *even*, if i, j are both even, and *odd* if not.

Let us consider the following distribution of signs at the integer points of the triangle T :

a point has the sign "-", if it is even, and has the sign "+", if it is odd.

We will call this rule *a Harnack distribution of signs.*

We use the system of notations for the real schemes of non-singular curves suggested by Viro [Vi 2]. The scheme consisting of a single oval is denoted by the symbol $< 1 >$, the empty scheme - by the symbol $< 0 >$. If a symbol $< A >$ stands for some set of ovals, then the set of ovals obtained by addition of an oval surrounding all old ovals is denoted by $< 1 < A >>$. If a scheme is the union of two non-intersecting sets of ovals denoted by $< A >$ and $< B >$ respectively with no oval of one set surrounding an oval of the other set, then this scheme is denoted by the symbol $< A \cup B >$. Besides, if A is the notation for some set of ovals then a part $A \cup \ldots \cup A$ of another notation where A repeats n times is denoted by $n \times A$; a part $n \times 1$ is denoted by n.

Proposition 3.1 *An arbitrary primitive convex triangulation of T with the Harnack distribution of signs at the vertices produces a T-curve of degree $m = 2k$ with the real scheme*

$$< \frac{3k^2 - 3k}{2} \cup 1 < \frac{(k-1)(k-2)}{2} >>$$

Remark A curve with this real scheme has $\frac{(m-1)(m-2)}{2} + 1$ connected components of the real point set. So, it is an M-curve.

Proof Let us, first, notice that the number of interior (i.e. lying strongly inside of the triangle T) integer points is equal to $\frac{(m-1)(m-2)}{2}$, the number of even interior points is equal to $\frac{(k-1)(k-2)}{2}$, and the number of odd interior points is equal to $\frac{3k^2-3k}{2}$.

Take an arbitrary even interior vertex of a triangulation of the triangle T. It has the sign "-". All neighbouring vertices (i.e. the vertices connected with the taken vertex by edges of the triangulation) are odd, and thus they all have the sign "+". It means that the star of an even interior vertex contains an oval of the curve L (*the star* of a vertex of the triangulation is the union of all triangles of the triangulation containing this vertex). The number of such ovals is equal to $\frac{(k-1)(k-2)}{2}$.

Take now an odd interior vertex of the triangulation. It has the sign "+". There are two vertices with "-" and one vertex with "+" among three symmetric images of the taken vertex under $s = s_x \circ s_y$ and s_x, s_y (where s_x, s_y are the reflections with respect to the coordinate axes). Consider the symmetric copy of the taken vertex with the sign "+". It is easy to verify, that all its neighbouring vertices have the sign "-". It means again that the star of this copy contains an oval of the curve L. The number of such ovals is equal to $\frac{3k^2-3k}{2}$.

Remark that

$$\frac{(k-1)(k-2)}{2} + \frac{3k^2 - 3k}{2} = \frac{(m-1)(m-2)}{2}$$

and, thus, we can have only one oval more.

This oval exists, because, for example, the curve L intersects the coordinate axes.

To finish the proof it remains to notice that the union of the segments

$$\{x - y = -m, \ -m \leq x, y \leq m\} \ \cup$$

$$\{x \leq 0, \ y = 0, \ -m \leq x, y \leq m\} \ \cup \ \{x = 0, \ y \leq 0, \ -m \leq x, y \leq m\}$$

is not contractible in T_* and contains only the signs "-". It meens that $\frac{3k^2-3k}{2}$ ovals corresponding to odd interior points and containing the signs "+" inside of them are situated outside of the non-empty oval. •

3.2 Counter-examples to Ragdsdale conjecture

Let us consider a non-singular real algebraic plane projective curve of even degree $m = 2k$. The real point set $\mathbf{R}A$ of this curve divides the real projective plane $\mathbf{R}P^2$ in two parts with a common boundary $\mathbf{R}A$ (these parts are the subsets of $\mathbf{R}P^2$ where a polynomial defining the curve has the positive or, respectively, the negative values). One of these parts is non-orientable, we will denote it by $\mathbf{R}P^2_-$. The other one will be denoted by $\mathbf{R}P^2_+$.

The topology of $\mathbf{R}P^2_-$ and $\mathbf{R}P^2_+$ is closely connected with the topological type of the pair $(\mathbf{R}P^2, \mathbf{R}A)$. Let p be the number of connected components of $\mathbf{R}P^2_+$, and $n+1$ be the number of connected components of $\mathbf{R}P^2_-$ (exactly one component of $\mathbf{R}P^2_-$ is non-orientable).

The numbers p and n can be described in another way. An oval of a curve is called *even* (resp. *odd*) if it lies inside of even (resp. odd) number of other ovals of this curve.

It is easy to see that p is the number of even ovals of a curve, and n is the number of odd ovals.

In 1906 V. Ragsdale [Ra] studing the results of Harnack's and Hilbert's constructions proposed two conjectures :

$$p \leq \frac{3k^2-3k+2}{2}, \quad n+1 \leq \frac{3k^2-3k+2}{2}$$

and

$$p-n \leq \frac{3k^2-3k+2}{2}, \quad n-p+1 \leq \frac{3k^2-3k+2}{2}$$

In 1938 I. Petrovsky [Pe] proved the second Ragsdale conjecture and also proposed a conjecture similar to the first one :

$$p \leq \frac{3k^2-3k+2}{2}, \quad n \leq \frac{3k^2-3k+2}{2}$$

In 1980 O. Viro [Vi 2] constructed the curves of degree $2k$ with $n = \frac{3k^2-3k+2}{2}$ for any even $k \geq 4$. These curves are counter-examples to the original Ragsdale conjecture, but not to the conjecture of Petrovsky.

The following theorem gives the counter-examples to the "corrected" Ragsdale conjecture (or to the conjecture of Petrovsky) (see also [It]).

Theorem 3.2 *For any integer number $k \geq 1$*

a) there exists a non-singular real algebraic plane projective curve of degree $2k$ with

$$p = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right]$$

b) there exists a non-singular real algebraic plane projective curve of degree $2k$ with

$$n = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right] - 1$$

Proof We will construct T-curves with the stated proprieties. Let us show, first, how to construct a curve of degree $m = 2k$ with $p = \frac{3k^2-3k+2}{2} + 1$.

Suppose that the hexagon S shown in Figure 3 is placed inside of the triangle $T = \{x \geq 0, \quad y \geq 0, \quad x + y \leq m\}$ in such a way that his center has the both coordinates odd. Any convex primitive triangulation of a convex part of a convex polygon is extendable to a convex primitive triangulation of the polygon. Inside of the hexagon S, let us take the convex primitive triangulation shown in Figure 3 and extend it to T.

To apply Theorem 2.1 we need to choose signs at the vertices in T. Inside of S put signs according to Figure 3, outside, use the Harnack rule of distribution of signs (see subsection 3.1) : a vertex (i, j) gets sign "$-$", if i, j are even, and sign "$+$" otherwise.

It is easy to calculate that the corresponding piecewise-linear curve L has exactly one even oval more than the M-curve constructed in the subsection 3.1 (i.e. now $p = \frac{3k^2-3k+2}{2} + 1$). One can verify that the curve obtained has the real scheme

$$< \frac{3k^2 - 3k - 2}{2} \sqcup 1 < 2 > \sqcup 1 < \frac{(k-1)(k-2)}{2} - 4 >>$$

This curve is an (M - 2)-curve (it means that the number of the connected components of the real point set is equal to $\frac{(m-1)(m-2)}{2} - 1$).

Now, consider a partition of the triangle T shown in Figure 4. Let us take in each marked hexagon the triangulation and the signs of S. The triangulation of the union of the marked hexagons can be extended to the primitive convex triangulation of T. Let us fix such an extension. Outside of the union of the marked hexagons again shoose the signs at the vertices of the triangulation using the Harnack rule.

One can calculate that for the corresponding piecewise-linear curve L

$$p = \frac{3k^2 - 3k + 2}{2} + a$$

where a is the number of the marked sextagons, and

$$a = \left[\frac{(k-3)^2 + 4}{8} \right]$$

The curve constructed has the following real scheme

$$< \frac{3k^2 - 3k - 2a}{2} \sqcup a \times 1 < 2 > \sqcup 1 < \frac{(k-1)(k-2)}{2} - 4a >>$$

To prove the part b) of the statement of the theorem, let us take again the partition of the triangle T shown in Figure 4 with the triangulation and the signs of each marked sextagon coinciding with the triangulation and the signs of S. Fix, in addition, a triangulation of some part P of a neighbouhood of the axe OY and the signs at the vertices of this triangulation as it shown in

Figure 5 (more precisely, only the case $k \equiv 1 \ mod\ 4$ is presented in this figure, if $k \not\equiv 1 \ mod\ 4$ one should change a little the triangulation near the point $(0, m)$). The chosen primitive convex triangulation of the union of the marked hexagons and of the part P can be extended to a primitive convex triangulation of the triangle T. Outside of the union of the marked hexagons and of the part P, let us shoose again the signs at the vertices of the triangulation using the Harnack rule.

For the corresponding piecewise-linear curve L

$$n = \frac{3k^2 - 3k + 2}{2} + \left[\frac{(k-3)^2 + 4}{8}\right]$$

(the case $k \equiv 1 \ mod\ 4$) or

$$n = \frac{3k^2 - 3k + 2}{2} + \left[\frac{(k-3)^2 + 4}{8}\right] - 1$$

(the case $k \not\equiv 1 \ mod\ 4$). •

Recently, B. Haas [Has] constructed examples of T-curves of degree $2k$ with

$$p = \frac{3k^2 - 3k + 2}{2} + \left[\frac{k^2 - 3k - 6}{6}\right]$$

3.3 M-curves with one non-empty oval

Recall that an M-curve is a curve with the maximal possible number of connected components of the real point set for a given degree. This maximal number is equal to $\frac{(m-1)(m-2)}{2} + 1$ for the degree m.

In this subsection we discuss a classification of the real schemes of M-curves of the degree $2k = 4l + 2$ with one non-empty oval.

Each non-singular curve of even degree with one non-empty oval has the real scheme

$$< p - 1 \cup 1 < n >>$$

Restrictions

We need to use here two well-known restrictions for the topology of real plane projective curves (see, for example, the survey articles [Vi 5], [Wi]).

Gudkov - Rokhlin congruence

$p - n \equiv k^2 \ mod\ 8$ *for an M-curve of degree* $2k$

Improved Petrovsky inequalities

Let A be a curve of degree $2k$. Denote by p_- (resp. by n_-) the number of even (resp. odd) ovals of $\mathbb{R}A$ *bounding from the exterior* *the components of* $\mathbf{R}P^2 \setminus \mathbf{R}A$ *with the negative Euler characteristic. Then*

$$p - n_- \leq \frac{3k^2 - 3k + 2}{2}, \quad n - p_- + 1 \leq \frac{3k^2 - 3k + 2}{2}$$

Remark that $p_- \leq 1$, $n_- = 0$ in the case of curves with one non-empty oval, and the improved Petrovsky inequalities give the following ones :

$$p \leq \frac{3k^2 - 3k + 2}{2}, \quad n \leq \frac{3k^2 - 3k + 2}{2}$$

It is easy to see, using the Gudkov - Rokhlin congruence, that for M-curves with one non-empty oval, the second inequality can be improved by 1 :

$$n \leq \frac{3k^2 - 3k}{2}$$

Construction

The following theorem states that there is no other restristions (except the Gudkov - Rokhlin congruence and the improved Petrovsky inequalities) for the topology of M-curves of degree $m = 4l + 2$ with one non-empty oval.

Theorem 3.3 *Suppose that* $m = 2k = 4l + 2$, *where* l *is a positive integer number. Than for each positive integer numbers* p, n *such that*

$$p + n = \frac{(m-1)(m-2)}{2} + 1$$

satisfying the Gudkov - Rokhlin congruence and the improved Petrovsky inequalities there exists a real algebraic plane projective M-curve of degree m *with the real scheme*

$$< p - 1 \cup 1 < n >>$$

Proof Recall that the Harnack distribution of signs in the vertices of a triangulation is the rule :

> a vertex (i, j) gets the sign "-", if i, j are both even, and it gets the sign "+" in the opposite case.

Let us call *the inverse Harnack distribution of signs* the following one :

> a vertex (i, j) has the sign "-", if i, j are both odd, and has the sign "+" in the opposite case.

Remark that the inverse Harnack distribution of signs can be formulated as the Harnack one for the 3-rd quadrant of the plane exchanging "+" and "-" :

> an integer point (i, j) of the 3-rd quadrant gets the sign "+", if i, j are both even, and it gets the sign "-" in the opposite case.

Thus, Proposition 3.1 is also true for the inverse Harnack distribution.

Let us divide the triangle T in two polygons T_1 and T_2 (T_1 is a quadrangle, T_2 is a triangle) by a segment with the following properties :

(i) the ends of the segment lie on the boundary of T and are odd integer points,

(ii) the segment contains no integer points except the ends.

Consider an arbitrary convex primitive triangulation in each polygon T_1, T_2 (the union of these triangulations is a convex primitive tirangulation of T, because the chosen segment does not contain vertices of the triangulations except the ends). We can choose the Harnack distribution of signs in T_1 and the inverse Harnack distribution of signs in T_2 (these distributions are compatible on the common boundary of the polygons due to the assuptions on the segment).

The arguments of the proof of Proposition 3.1 show again that the triangulation and the distribution of signs described above give an M-curve with one non-empty oval.

Compute the number of even ovals of this curve. Let P_1, P_2 be the numbers of interior even points of T_1 and T_2, and N_1, N_2 be the numbers of interior odd points of these polygons.

It is easy to see that for the curve obtained

$$p = P_1 + N_2 + 1, \quad n = N_1 + P_2$$

Remark The Gudkov - Rokhlin congruence has a nice corollary:

> The numbers P_2, N_2 of even and odd interior points of the triangle T_2 are congruent modulo 4.

To prove Theorem 3.3 it is enough now to divide the triangle T by the segments with the described properties in sufficiently many parts and to choose in each part an arbitrary convex primitive triangulation with an appropriate distribution of signs (the Harnack one or the inverse Harnack one) at the vertices. In the case $m = 2k = 4l + 2$ a required partition of T is shown in Figure 6 (the vertices on the axe OX have the coordinates $(k+(4i+2),\ 0)$ the vertices on the axe OY - the coordinates $(0,\ k+(4i+2))$, and the vertices on the line $x+y=m$ - the coordinates $(k \pm 4i,\ k \mp 4i)$ with appropriate values of a non-negative integer i). It is easy to verify that all possible (in the sense of the statement of Theorem 3.3) pairs p, n can be realized using this partition. For example, to realize two extremal cases one can take the Harnack distribution of signs in T (the case $p = \frac{3k^2-3k+2}{2}$, $n = \frac{(k-1)(k-2)}{2}$ - a Harnack curve) or the Harnack distribution in the quadrangle $OABC$ and the inverse Harnack distribution in $T \setminus OABC$ (the opposite extremal case $p = \frac{(k-1)(k-2)}{2} + 1$, $n = \frac{3k^2-3k}{2}$).

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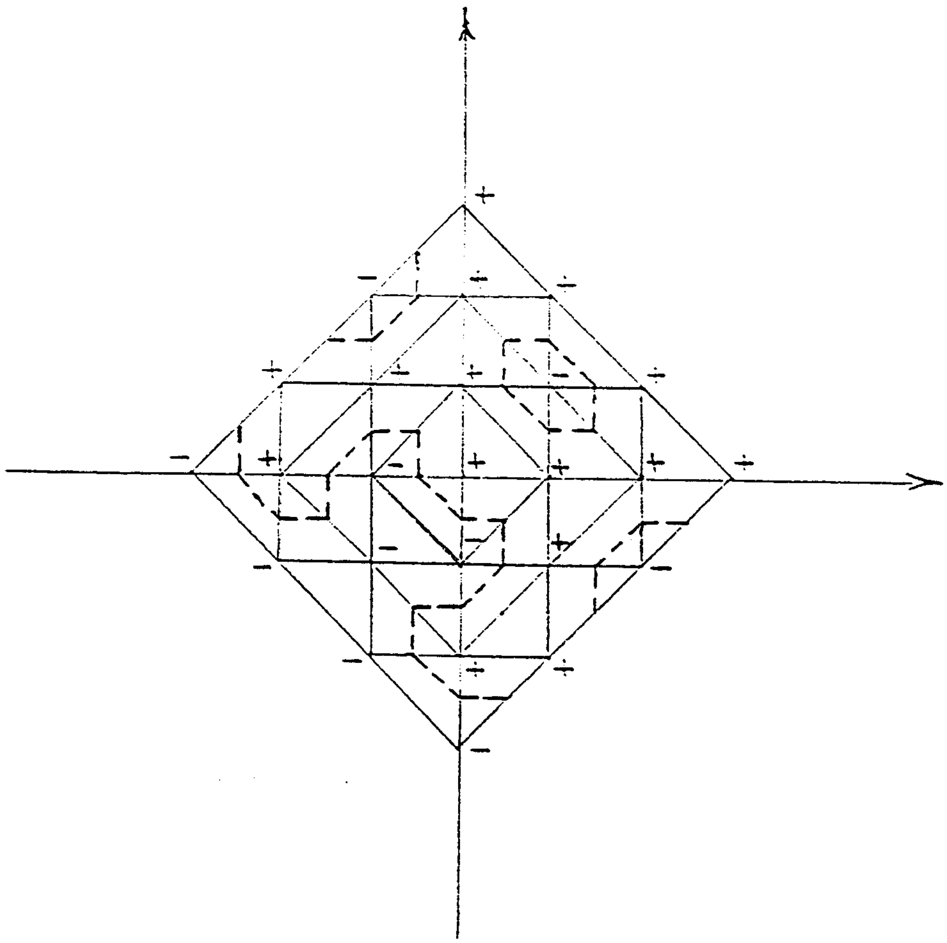

Fig 1

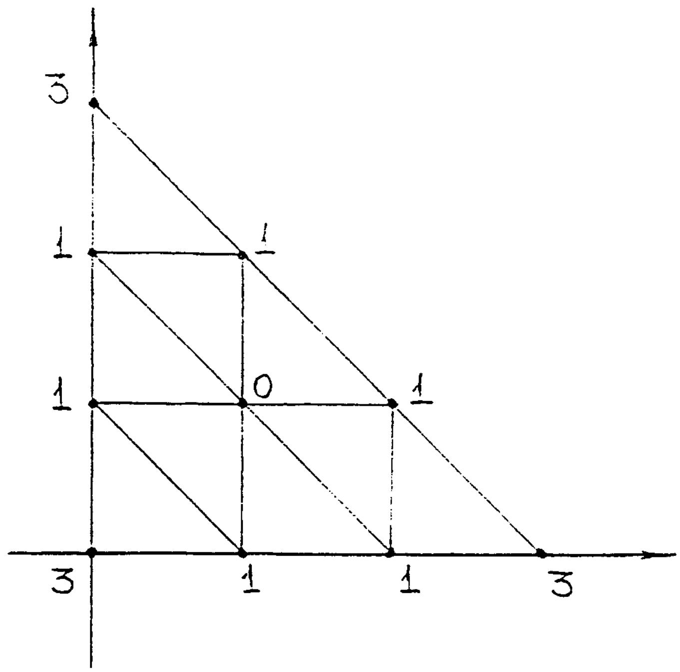

Fig. 2

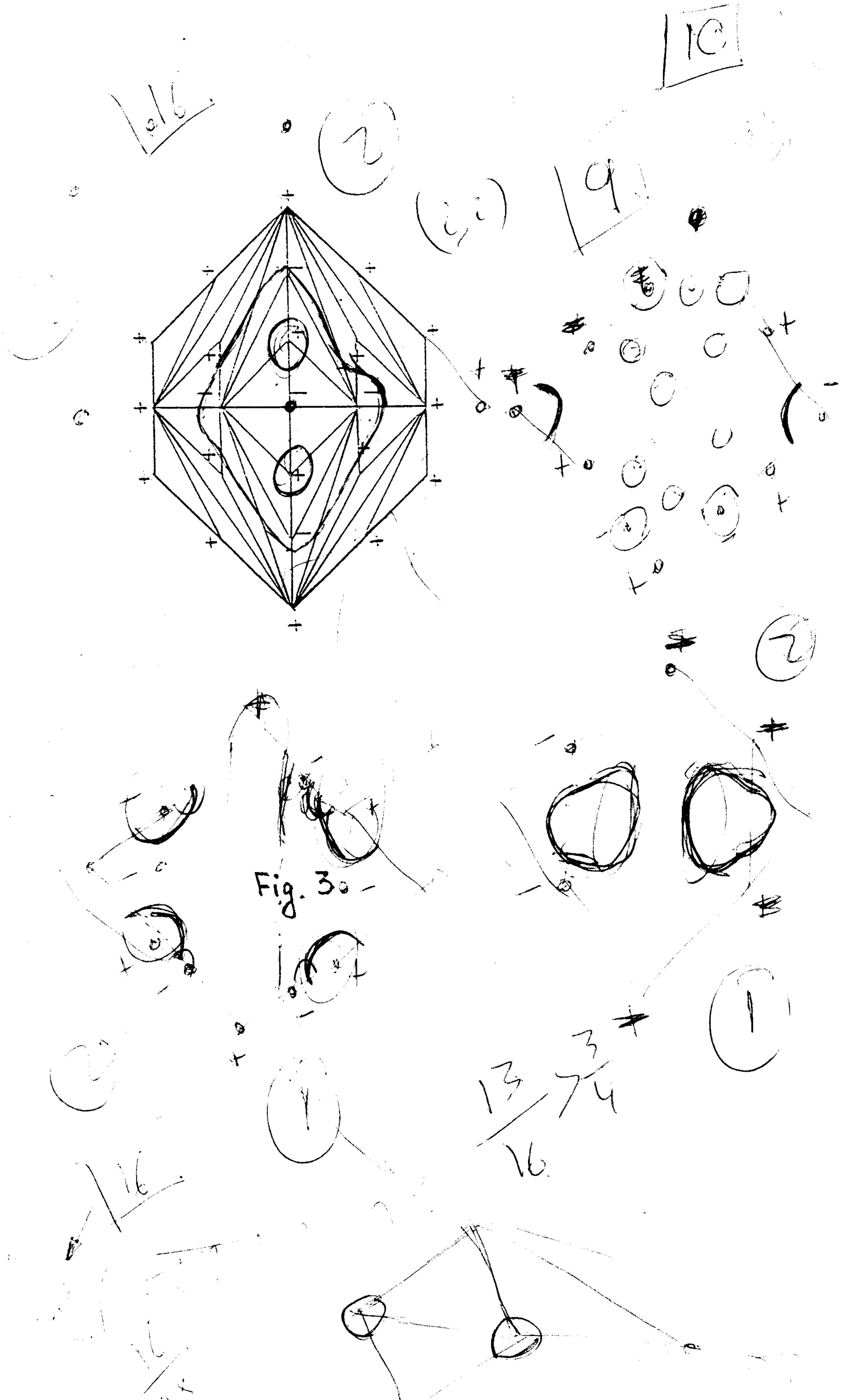

Fig. 3

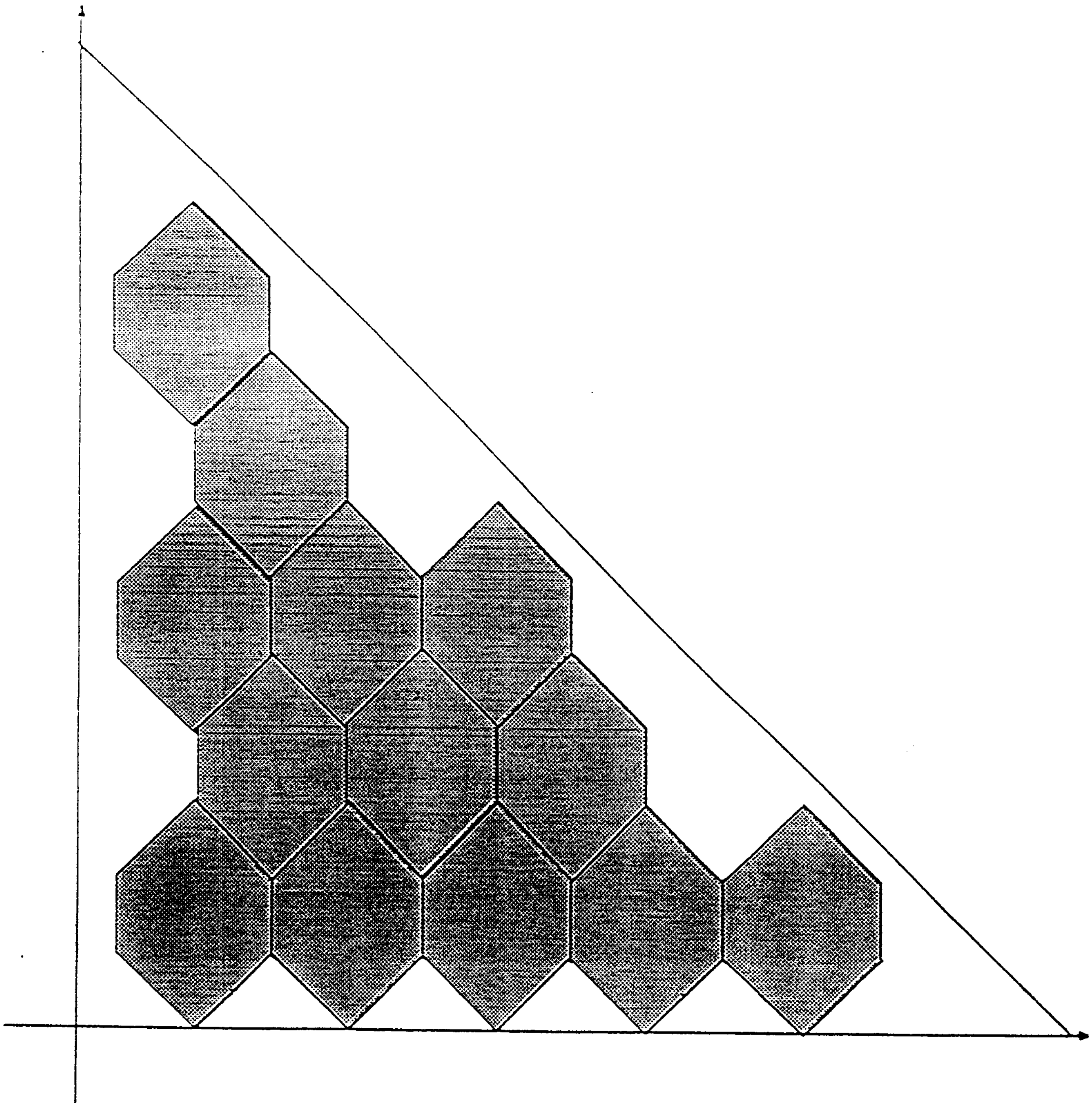

Fig. 4

área 1.8 → 16 ovalos

área 6 → [illegible] ovalos

$\frac{20}{24} = \frac{5}{6}$

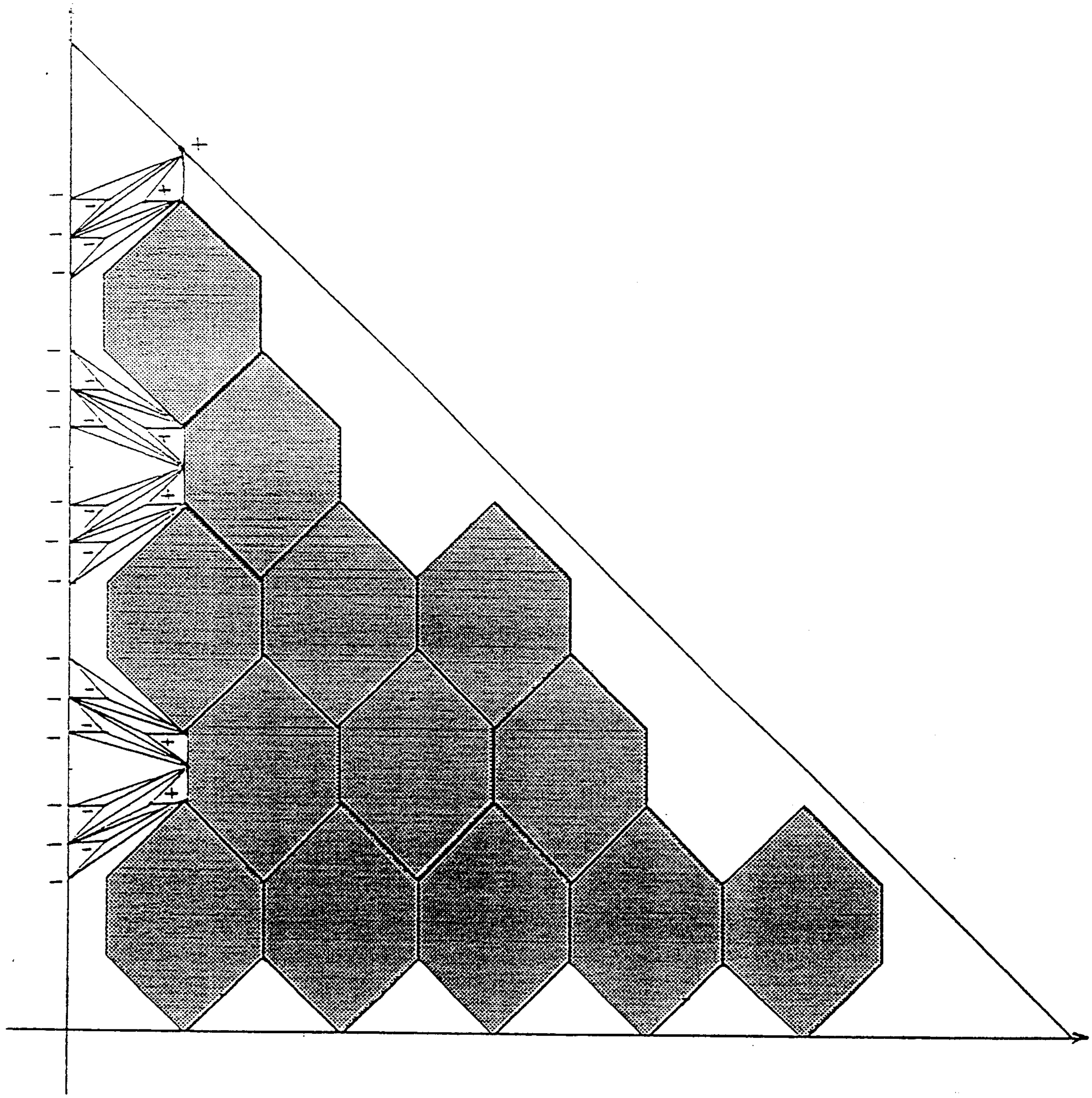

Fig. 5

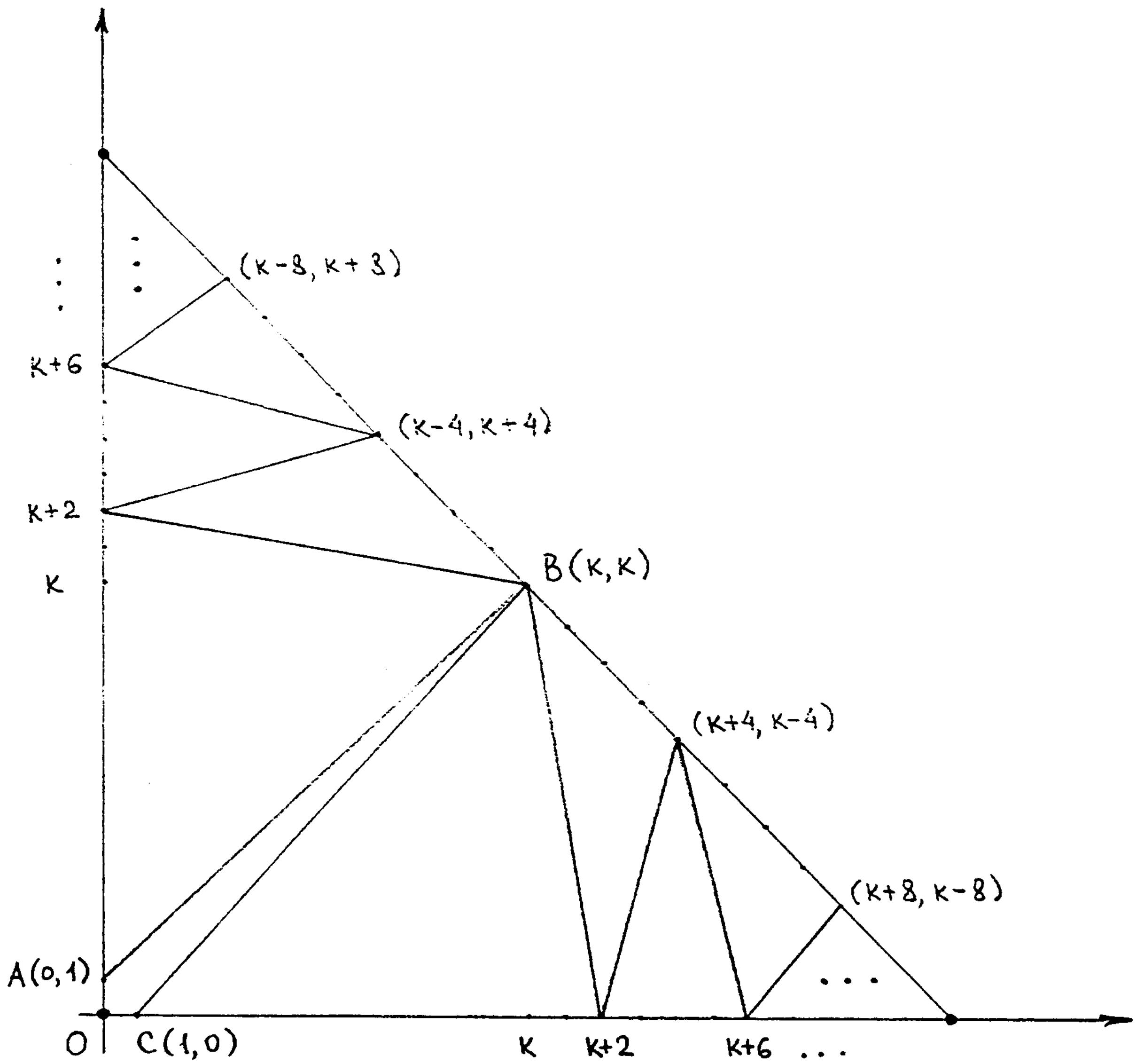

Fig. 6