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4 Nowhere-zero Flow Problems

FRANÇOIS JAEGER

1. Introduction

The concept of a *flow in a graph* is a useful model in Operations Research, and is also essentially identical to the concept of a current in an electrical network. It is thus not surprising that the study of flows is a classical and important topic in graph theory, which leads to rich developments and generalizations in combinatorial optimization, polyhedral combinatorics and matroid theory.

At first sight it seems that the dual concept of *tension* (or potential difference) has less importance in the literature, and appears mainly in scheduling and shortest-path problems. However, as observed by Tutte, the whole theory of vertex-colorings of graphs can be formulated in terms of tension, and this is indeed an essential part of graph theory which is intimately related to the history of its development.

In the case of planar graphs, the duality between flow and tension corresponds to the geometric duality of graphs represented in the plane. This allows a reformulation of face-coloring properties of plane graphs in terms of flow properties. For instance, one can show that the four-color theorem is equivalent to the following statement:

Every bridgeless planar directed graph has an integer flow with all edge-values in the set $\{\pm 1, \pm 2, \pm 3\}$.

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This led Tutte to consider similar properties for arbitrary graphs. For instance, he proposed the following *5-flow conjecture*:

Every bridgeless directed graph has an integer flow with all edge-values in the set $\{\pm 1, \pm 2, \pm 3, \pm 4\}$.

The study of this kind of problem involves an extension of the usual concept of flow, where the set of flow values is an arbitrary Abelian group. Using this extended concept, it is possible to unify such problems as Tutte's 5-flow conjecture, the cycle double cover conjecture (*every bridgeless graph has a family of cycles which together cover each edge twice*) and Fulkerson's conjecture (*every bridgeless trivalent graph has six 1-factors which together cover each edge twice*) into a single framework—the class of **nowhere-zero flow problems**.

The interest of such an approach is that it brings together different methods developed independently for various conjectures, and can also help us to formulate new and pertinent problems. The present survey lays more emphasis on the unity of the different nowhere-zero flow problems than on their specific aspects.

In Section 2 we introduce the necessary definitions and notation. We then present nowhere-zero k-flows in Section 3, and discuss the main results and conjectures on the existence of such flows in Section 4. Section 5 is devoted to the cycle double cover conjecture, Section 6 deals with Fulkerson's conjecture, and in Section 7 we present a conjecture which implies the previous ones. Some results on special classes of graphs are reviewed in Section 8, and the main contributions of the reduction methods are outlined in Section 9. We conclude, in Section 10, by mentioning some relationships with other research topics.

2. Definitions and Notation

Our definition of a graph allows loops and multiple edges. For convenience, we shall not distinguish a graph G from the various digraphs which can be obtained from G by assigning an orientation to each edge of G. This motivates the following terminology: if, for each edge of a graph G, we distinguish one initial end and one terminal end, we obtain a directed graph which will be called an **orientation** of G.

If G is a directed graph, and if $S \subseteq V(G)$, we denote by $\omega^+(S)$ the set of edges with initial end in S and terminal end not in S. We write $\omega^-(S) = \omega^+(V(G) - S)$ and $\omega(S) = \omega^+(S) \cup \omega^-(S)$. A k-subset of $E(G)$ of the form $\omega(S)$, where S is a proper non-empty subset of $V(G)$, is called a ***k*-cut** of G. Thus G is k-edge-connected if and only if it has no l-cuts for $l < k$, and a bridge is an edge which forms a 1-cut.

If A is an Abelian group (with additive notation), and if G is a directed graph, then an ***A*-flow** of G is a mapping φ from $E(G)$ to A such that:

$$\text{for all } S \subseteq V(G), \quad \sum_{e \in \omega^+(S)} \varphi(e) - \sum_{e \in \omega^-(S)} \varphi(e) = 0. \tag{1}$$

The usual concept of flow corresponds to the case where A is $\mathbf{Z}$ or $\mathbf{R}$.

We make the following remarks:

(*i*) It is easy to see that the mapping φ is an A-flow if and only if equation (1) is satisfied for all sets S consisting of a single vertex of G.

(*ii*) It follows from (1) that an A-flow takes the value zero on each bridge.

(*iii*) Suppose we change the orientation of the edge e in G, and simultaneously replace $\varphi(e)$ by $-\varphi(e)$. Then (1) is still valid for all subsets S of $V(G)$, and hence the new mapping φ' is an A-flow in the new directed graph.

(*iv*) If each element of A is its own opposite—for instance, if $A = \mathbf{Z}_2{}^k$ for some $k \geq 1$—the situation is simpler. Condition (1) can then be rewritten as:

$$\text{for all } S \subseteq V(G), \quad \sum_{e \in \omega(S)} \varphi(e) = 0,$$

and this is clearly independent of the orientation of G.

The **support** $\sigma(\varphi)$ of the A-flow φ of G is the set of edges e of G such that $\varphi(e) \neq 0$. φ is said to be a **nowhere-zero flow** if $\sigma(\varphi) = E(G)$. If φ takes all its values in $B \subseteq A$, then it is called a ***B*-flow**. We shall be interested in the existence of B-flows for subsets B for which $0 \notin B$ and $B = -B$.

Note that the problem of the existence of a B-flow in a graph is trivial if $0 \in B$ (consider the flow which takes the value 0 on every edge). On the other hand, the condition $0 \notin B$ implies that we are restricting our attention to bridgeless graphs (see remark (*ii*)). Also, by remark (*iii*), the condition $B = -B$ implies that the following properties are equivalent for a graph G:

(*a*) some orientation of G has a B-flow;

(*b*) every orientation of G has a B-flow.

When (*a*) and (*b*) hold, we simply say that G *has a B-flow*. Thus we are studying a property of undirected graphs. The orientations will be used only as a reference for defining flows. By remark (*iv*), this is unnecessary if every element of A is its own opposite.

3. Nowhere-zero *k*-flows

Face Colorings and Flows

Consider a directed graph G which is 2-cell-embedded in an orientable surface S (see **[56]** and **ST1**, Chapter 2 for definitions). Assume that the embedding is *face-k-colorable*—that is, the faces of the embedding can be colored with k colors in such a way that each edge belongs to the boundary of two faces with different colors. Consider the colors as the elements of an additive group of order k—say, $\mathbf{Z}_k$. For each edge e, let $r(e)$ be the color of the face bounded by e on its right, and let $l(e)$ be the color of the face bounded by e on its left. Using remarks (*i*) and (*iii*) of Section 2, we can easily check that the mapping $r - l$ from $E(G)$ to $\mathbf{Z}_k$ is a $\mathbf{Z}_k$-flow of G. The face-coloring property is equivalent to the fact that this flow is nowhere-zero. We can formulate this result as follows:

Theorem 3.1. *If a graph has a face-k-colorable 2-cell embedding in some orientable surface, then it has a nowhere-zero $\mathbf{Z}_k$-flow.* ||

More can be said for plane embeddings (see **[49]**):

Theorem 3.2. *A plane graph is face-k-colorable if and only if it has a nowhere-zero $\mathbf{Z}_k$-flow.*

Sketch of proof. Each flow corresponds to a potential difference in the dual graph (this is the duality of flows and tensions for plane graphs mentioned in Section 1—see [**36**, Chapter 7]). This means that in the graph, each flow can be obtained by assigning a value to each face, and then assigning to each edge the difference between the value of the face on the right and the value of the face on the left. In particular, each nowhere-zero $\mathbf{Z}_k$-flow can be obtained from a face-k-coloring by the process used to prove Theorem 3.1. ||

Some Equivalence Results

Theorem 3.2 led Tutte to study nowhere-zero $\mathbf{Z}_k$-flows for arbitrary graphs. In particular he obtained two equivalence results that we now present briefly.

It is clear that in Theorems 3.1 and 3.2, the group $\mathbf{Z}_k$ can be replaced by any other additive group of the same order. This is a general phenomenon which is a consequence of Theorem 3.4 below.

For a graph G, and $F \subseteq E(G)$, we denote by $r(F)$ the maximum number of edges in a forest of G contained in F. We shall use the following lemma, which is an immediate extension of a classical result:

Lemma 3.3. *Let G be a connected directed graph, and let T be a spanning tree of G. Let A be an additive group, and let c be any mapping from $E(G) - E(T)$ to A. Then there exists exactly one A-flow φ of G such that, for each edge e of G not in T, $\varphi(e) = c(e)$.* ||

The following result is due to Tutte **[49]**:

Theorem 3.4. *Let A be a finite additive group of order λ, and let G be a directed graph. Then the number of nowhere-zero A-flows of G is*

$$F(G, \lambda) = \sum_{F \subseteq E(G)} (-1)^{|E(G) - F|} \lambda^{|F| - r(F)}.$$

Proof. It follows from Lemma 3.3 that, for every $F \subseteq E(G)$, $\lambda^{|F| - r(F)}$ is the number of A-flows of the subgraph $(V(G), F)$ of G. Equivalently, $\lambda^{|F| - r(F)}$ is the number of A-flows of G whose support is contained in F. The result now follows by the inclusion–exclusion principle. ||

The polynomial $F(G, \lambda)$ is called the **flow polynomial** of G. It is, in a sense, dual to the classical chromatic polynomial, and can be evaluated similarly by a deletion–contraction process (see **[53]**).

Another important equivalence result was obtained by Tutte **[49]**: let k be an integer, $k \geqslant 2$, and let G be a directed graph. A **nowhere-zero *k*-flow of *G*** is a $\mathbf{Z}$-flow φ of G such that $0 < |\varphi(e)| < k$ for each e in $E(G)$.

Theorem 3.5. *A directed graph G has a nowhere-zero k-flow if and only if it has a nowhere-zero $\mathbf{Z}_k$-flow.*

Sketch of proof. If we replace each edge-value of a nowhere-zero k-flow of G by the corresponding value of $\mathbf{Z}_k$, we obtain a nowhere-zero $\mathbf{Z}_k$-flow. Conversely, if we replace each edge-value of a nowhere-zero $\mathbf{Z}_k$-flow by the corresponding integer in $[1, k - 1]$, we obtain a mapping f from $E(G)$ to $\{1, 2, \ldots, k - 1\}$ which satisfies the following property:

for each vertex v of G,

$$\sum_{e \in \omega^+(\{v\})} f(e) - \sum_{e \in \omega^-(\{v\})} f(e) \equiv 0 \pmod{k}.$$

We consider a new graph G' obtained from G by adding a new vertex joined to each other vertex by a new edge. The mapping f yields a $\mathbf{Z}$-flow f' of G' which takes its values in $\{1, \ldots, k - 1\}$ for the edges of G, and in $k\mathbf{Z}$ for the new edges. A result of Tutte on regular chain-groups [**50**, Proposition 6.3] then allows us to derive from f' a nowhere-zero k-flow of G. ||

Direct proofs of Theorem 3.5 can be found in **[8]** and **[58]**.

In view of Theorems 3.4 and 3.5, we define (as in **[22]**) a graph G to be an $\boldsymbol{F_k}$ **graph** (for $k \geq 2$) if it satisfies the following equivalent properties:

(*i*) for some additive group A of order k, G has a nowhere-zero A-flow;

(*ii*) for every additive group A of order k, G has a nowhere-zero A-flow;

(*iii*) G has a nowhere-zero k-flow.

Note that, by (*iii*), if G is an F_k graph, then it is an F_l graph for each $l \geq k$.

Clearly a graph is an F_2 graph if and only if all of its vertices have even degree (by property (*i*), with $A = \mathbf{Z}_2$). The following simple result gives interesting examples for $k = 3$ and $k = 4$ (see **[33]**, **[48]**, **[49]**):

Theorem 3.6. *Let G be a trivalent graph. Then*

(*i*) *G is an F_3 graph if and only if it is bipartite*;

(*ii*) *G is an F_4 graph if and only if it is edge-3-colorable.*

Sketch of proof. Part (*i*) is proved by considering nowhere-zero $\mathbf{Z}_3$-flows as orientations for which every vertex is a source or a sink.

Part (*ii*) is proved by considering nowhere-zero $\mathbf{Z}_2{}^2$-flows as edge-colorings with three colors. ||

4. *k*-flow Conjectures and Theorems

The 4-flow Conjecture

By Theorem 3.2, the four-color theorem (see **[1]**) is equivalent to the result that every bridgeless planar graph is an F_4 graph. In **[52]** Tutte conjectured the following stronger property:

The 4-flow conjecture. *Every bridgeless graph with no subgraph contractible to the Petersen graph is an F_4 graph.*

This conjecture is discussed in **[42]**. In this direction, using the four-color theorem and matroid theory, Walton and Welsh **[54]** proved that every bridgeless graph with no subgraph contractible to the Kuratowski graph $K_{3,3}$ is an F_4 graph.

It is apparently not known whether the 4-flow conjecture is equivalent to its restriction to trivalent graphs. This restriction can be formulated as follows (see Theorem 3.6 (*ii*)):

The trivalent 4-flow conjecture. *Every bridgeless trivalent graph with no subgraph homeomorphic to the Petersen graph is edge-3-colorable.*

A first (small) step towards a proof of this conjecture is the result that bridgeless trivalent graphs with crossing-number 1 are edge-3-colorable (see **[23]**, **[13]**). Of course, the proofs rely on the four-color theorem.

The 5-flow Conjecture

Tutte also looked for an analogue of the four-color theorem for arbitrary graphs. In **[49]**, he proposed the following conjecture:

The 5-flow conjecture. *Every bridgeless graph is an F_5 graph.*

Since the Petersen graph is not an F_4 graph (Theorem 3.6 (*ii*)), this conjecture, if true, would be the best possible.

Tutte also proposed the weaker conjecture that there exists an integer $k \geq 5$ such that every bridgeless graph is an F_k graph. This was proved for $k = 8$ in 1975 independently by Kilpatrick **[30]** and Jaeger **[20]**, **[22]**, using essentially the same method. This result is now superseded by the 6-flow theorem of Seymour (see below). However, we shall present a full proof here, because it is fairly simple and uses auxiliary results which are interesting for their own sake. We shall need two lemmas; the first one is due to Kundu **[31]**:

Lemma 4.1. *Every 2k-edge-connected graph ($k \geq 1$) contains k pairwise edge-disjoint spanning trees.*

Proof. Tutte **[51]** and Nash-Williams **[35]** have proved that a graph G contains k pairwise edge-disjoint spanning trees if and only if, for each partition P of $V(G)$ into p blocks, the number $m(P)$ of edges of G joining different blocks is at least $k(p-1)$. This is clearly true if $p = 1$. If $p \geq 2$, $P = \{B_1, \ldots, B_p\}$, and G is $2k$-edge-connected, then

$$m(P) = \tfrac{1}{2} \sum_{i=1}^{p} |\omega(B_i)| \geq \tfrac{1}{2}p(2k) > k(p-1). \quad ||$$

The proofs of the next lemma given in **[22]** and **[30]** rely on a formula of Edmonds **[10]** on the minimum number of independent sets of a matroid needed to cover the elements. We give here a simpler proof:

Lemma 4.2. *Every 3-edge-connected graph has three spanning trees with empty intersection.*

Proof. Consider a 3-edge-connected graph G. Replacing every edge of G by two parallel edges, we obtain a 6-edge-connected graph G'. By Lemma 4.1, G' has three pairwise edge-disjoint spanning trees. By identifying each of these trees with a tree of G, we obtain three spanning trees of G with empty intersection. ||

We now prove the following 8-flow theorem:

Theorem 4.3. *Every bridgeless graph is an F_8 graph.*

Proof. It is easily seen that it is sufficient to prove the result for a 3-edge-connected graph G (see Section 9 below). By Lemma 4.2, we can find spanning trees T_1, T_2, T_3 of G which have empty intersection. Using Lemma 3.3, we can obtain $\mathbf{Z}_2$-flows $\varphi_1, \varphi_2, \varphi_3$ of G such that $\varphi_i(e) = 1$ for all e in $E(G) - E(T_i)(i = 1, 2, 3)$. Then $(\varphi_1, \varphi_2, \varphi_3)$ defines in the obvious way a nowhere-zero $\mathbf{Z}_2{}^3$-flow of G.||

The 6-flow Theorem

We present here an outline of Seymour's proof of this result, but from a slightly different perspective.

Consider the following constructions for a graph G:

C_0: add an isolated vertex to G;

C_1: add an edge within one connected component of G;

C_2: add two edges joining two distinct connected components of G.

Let $\mathscr{C}$ be the class of graphs which can be obtained from the graph K_1 by a finite number of constructions of the form C_0, C_1 and/or C_2. The following result implies that every graph in $\mathscr{C}$ is an F_3 graph:

Theorem 4.4. *Let G be a graph in $\mathscr{C}$, considered with an arbitrary orientation. For each mapping c from $E(G)$ to $\mathbf{Z}_3$, there exists a $\mathbf{Z}_3$-flow φ of G such that $\varphi(e) \neq c(e)$ for each e in $E(G)$.*

Proof. We proceed by induction on $|E(G)|$. If $|E(G)| = 0$, there is nothing to prove. Suppose that G' is constructed from G using construction C_1 or C_2, and let c' be a mapping from $E(G')$ to $\mathbf{Z}_3$. Let μ be a $\mathbf{Z}_3$-flow of G' whose support is a cycle containing $E(G') - E(G)$. Since $|E(G') - E(G)| \leq 2$, we may use μ to obtain a $\mathbf{Z}_3$-flow μ' of G' such that $\mu'(e) \neq c'(e)$ for each e in $E(G') - E(G)$. (μ' is equal to μ, $-\mu$ or the zero flow.) By the induction hypothesis, there exists a $\mathbf{Z}_3$-flow φ of G such that $\varphi(e) \neq c'(e) - \mu'(e)$, for each e in $E(G)$. Then $\varphi' = \varphi + \mu'$ is a $\mathbf{Z}_3$-flow of G' such that $\varphi'(e) \neq c'(e)$, for each e in $E(G')$. ||

We now present Seymour's 6-flow theorem **[41]**:

Theorem 4.5. *Every bridgeless graph is an F_6 graph.*

Sketch of proof. It is sufficient to prove the result for a simple 3-connected graph G (see Section 9 below). Seymour showed that there exist vertex-disjoint cycles $C_1, \ldots, C_r$ of G such that the graph H obtained by contracting the edges of these cycles belongs to $\mathscr{C}$. By Theorem 4.4, H has a nowhere-zero $\mathbf{Z}_3$-flow, which can be extended to a $\mathbf{Z}_3$-flow φ_3 of G with $E(G) - \bigcup_{i=1}^{r} C_i \subseteq \sigma(\varphi_3)$. Consider now a $\mathbf{Z}_2$-flow φ_2 of G with $\sigma(\varphi_2) = \bigcup_{i=1}^{r} C_i$. Then (φ_2, φ_3) defines a nowhere-zero $(\mathbf{Z}_2 \times \mathbf{Z}_3)$-flow of G. ||

Seymour **[41]** has also given a sketch of a proof of the following result, which yields an alternative proof of the 6-flow theorem:

Theorem 4.6. *Let G be a 3-connected trivalent simple graph. Then there exists a spanning tree T of G such that the contraction of the edges of $E(G) - T$ yields a graph which belongs to $\mathscr{C}$.* ||

In **[58]** Younger used Seymour's proof to obtain a polynomial-time algorithm for constructing a nowhere-zero 6-flow in any bridgeless graph. For planar graphs this algorithm can be specialized to yield a nowhere-zero 5-flow.

The 3-flow Conjecture

A theorem of Grötzsch **[17]** asserts that every loopless planar graph without triangles is vertex-3-colorable. By duality and Theorem 3.2, this can be reformulated as follows: every bridgeless planar graph without 3-cuts is an F_3 graph. This led Tutte to propose the following conjecture (see **[6**, unsolved problem 48**]**); it is easy to see that it would be sufficient to prove this conjecture for 4-edge-connected graphs.

The 3-flow conjecture. *Every bridgeless graph without 3-cuts is an F_3 graph.*

The following result appears in **[20]**, **[22]**:

Theorem 4.7. *Every bridgeless graph without 3-cuts is an F_4 graph.*

Proof. It is easy to show that it is sufficient to prove the result for 4-edge-connected graphs. By Lemma 4.1, any such graph G contains two edge-disjoint spanning trees T_1, T_2. By Lemma 3.3, there exists a $\mathbf{Z}_2$-flow $\varphi_i (i = 1, 2)$ of G such that $E(G) - E(T_i) \subseteq \sigma(\varphi_i)$. Then (φ_1, φ_2) is a nowhere-zero $\mathbf{Z}_2{}^2$-flow of G. ||

We propose the following conjecture:

The weak 3-flow conjecture. *There exists an integer k such that every k-edge-connected graph is an F_3 graph.*

A possible approach to this conjecture would be to enlarge the class $\mathscr{C}$ introduced above to a class $\mathscr{C}'$, by allowing new constructions which preserve the F_3 property (for instance, insertion of a new vertex into an edge, or identification of vertices). We might then ask whether there exists an integer k such that every k-edge-connected graph is in $\mathscr{C}'$.

A More General Conjecture

Let us call a graph **mod (2p + 1)-orientable** ($p \geq 1$) if it has an orientation such that the out-degree of each vertex is congruent (modulo $2p + 1$) to the in-degree. We denote by $U(\mathbf{Z}_{2p+1})$ the subset $\{\bar{1}, -\bar{1}\}$ of $\mathbf{Z}_{2p+1}$. We then obtain the following result:

Theorem 4.8. *For any graph G, and for any $p \geq 1$, the following properties are equivalent:*

(*i*) *G is mod* $(2p + 1)$*-orientable;*

(*ii*) *G has a* $U(\mathbf{Z}_{2p+1})$*-flow;*

(*iii*) *G has a* $(\mathbf{Z} \cap ([-p-1, -p] \cup [p, p+1]))$*-flow;*

(*iv*) *G has a* $(\mathbf{Q} \cap ([-1 - 1/p, -1] \cup [1, 1 + 1/p]))$*-flow.*

Sketch of proof. The equivalence of (*i*) and (*ii*) is immediate. The equivalence of (*iii*) and (*iv*) is a special case of a well-known property of flows. The equivalence of (*ii*) and (*iii*) is proved in two steps. The first step is that (*ii*) holds if and only if G has a $(\mathbf{Z}_{2p+1} \cap \{\bar{p}, \overline{p+1}\})$-flow; this is easily proved by using an appropriate automorphism of $\mathbf{Z}_{2p+1}$. The equivalence between (*iii*) and the existence of a $(\mathbf{Z}_{2p+1} \cap \{\bar{p}, \overline{p+1}\})$-flow of G is then proved by the same method as Theorem 3.5. ||

Note that G is mod 3-orientable if and only if it is an F_3 graph.

For every $p \geq 1$, let M_{2p+1} be the class of mod $(2p + 1)$-orientable graphs. It follows from the equivalence of (*i*) and (*iv*) of Theorem 4.8 that $M_{2p'+1} \subseteq M_{2p+1}$ for all $p' \geq p \geq 1$. It is not difficult to check that, for every $p \geq 1$, the complete graph K_{4p+2} belongs to $M_{2p+1} - M_{2p+3}$. Hence $M_{2p'+1} \subset M_{2p+1}$, for all $p' > p \geq 1$. It is then tempting to conjecture that the higher the edge-connectivity of a graph, the higher it is ranked in the hierarchy $M_3 \supset M_5 \supset \ldots \supset M_{2p+1} \supset \ldots$. To make this more precise, the following conjecture (whose name is justified in Section 5) is proposed in [**25**]:

The circular flow conjecture. *For all $p \geq 1$, every 4p-edge-connected graph is mod* $(2p + 1)$*-orientable.*

Note that, for $p = 1$, this conjecture is equivalent to the 3-flow conjecture. Moreover, it is easy to see that for $p = 2$, the conjecture implies the 5-flow conjecture. Indeed, assume that every 8-edge-connected graph is mod 5-orientable. Consider a 3-edge-connected graph G and replace each edge of G by three parallel edges. The resulting graph G' is 9-edge-connected, and hence mod 5-orientable. It is then easy to convert a $U(\mathbf{Z}_5)$-flow of G' into a nowhere-zero $\mathbf{Z}_5$-flow of G. The extension to all bridgeless graphs is immediate (see Section 9).

Balanced Valuations

Like the four-color problem, the nowhere-zero flow problems presented in this section lend themselves to various interesting reformulations. We shall consider only one of these here.

A **balanced valuation** of a graph G is a mapping b from $V(G)$ to Q such that,

$$\text{for all } S \subseteq V(G),\ \left|\sum_{v \in S} b(v)\right| \leq |\omega_G(S)|.$$

The following result is proved in [**19**]:

Theorem 4.9. *Let p, q be integers with $1 \leq p < q$. Then a graph G has a* $\mathbf{Z} \cap ([-q, -p] \cup [p, q])$*-flow if and only if it has a balanced valuation of the form* $\frac{q+p}{q-p}w$, *where w is a mapping from V(G) to* $\mathbf{Z}$ *such that, for all* $v \in V(G)$, *w(v) and the degree of v have the same parity.* ||

For instance, if we apply Theorem 4.9 with $p = 1$ and $q = 3$ to trivalent graphs, and if we then use Theorem 3.6(*ii*), we obtain the following result of Bondy [**4**]: *a trivalent graph is edge-3-colorable if and only if it has a balanced valuation with values in* $\{-2, +2\}$. Similarly, it follows from Theorem 4.9 that *a 5-regular graph is an F_3 graph if and only if it has a balanced valuation with values in* $\{-3, +3\}$ (see [**5**]). More generally, all the results and problems discussed above can be reformulated in terms of balanced valuations. For instance, the 6-flow theorem asserts that every bridgeless trivalent graph has a balanced valuation with values in $\{-\frac{3}{2}, +\frac{3}{2}\}$, and the 5-flow conjecture asks whether $\frac{3}{2}$ can be replaced by $\frac{5}{3}$ in this statement.

5. The Double Cover Conjecture

A 2-cell embedding of a graph G on a surface is a **strong embedding** if each face-boundary is a cycle. For instance, every embedding of a planar 2-connected graph in the plane is a strong embedding.

A **cycle double cover** of a graph G is a family of cycles of G such that every edge appears in exactly two cycles of this family. For instance, the family of face-boundaries of a strong embedding of G is a cycle double cover.

The following two conjectures appear in the literature (see **[18]**, **[32]**, **[39]**, **[45]** and **[57]**):

The strong embedding conjecture. *Every 2-connected graph has a strong embedding on some surface.*

The double cover conjecture. *Every bridgeless graph has a cycle double cover.*

The double cover conjecture is easily seen to be equivalent to its restriction to 2-connected graphs. Hence, *the strong embedding conjecture implies the double cover conjecture.* The first conjecture appears as a topological motivation for the second one. Both problems are reviewed in some detail in **[26]**. Our concern here is only with the double cover conjecture which, as we shall see, can be viewed as a nowhere-zero flow problem.

A cycle double cover is said to be ***k*-colorable** ($k \geq 2$) if we can color its cycles with k colors in such a way that each edge appears on two cycles of different colors. For instance, a cycle double cover consisting of k cycles is k-colorable. The family of face-boundaries of a face-k-colorable strong embedding of G is a k-colorable cycle double cover of G.

Let D_k be the subset of $\mathbf{Z}_2^k$ consisting of those elements containing exactly two 1s, and let $\varphi = (\varphi_1, \ldots, \varphi_k)$ be a D_k-flow of G. For each $i \in \{1, \ldots, k\}$, choose a partition of $\sigma(\varphi_i)$ into cycles of G. The union of these k partitions is a k-colorable cycle double cover of G. Conversely, it is easily seen that every k-colorable cycle double cover of G can be obtained in this way from some D_k-flow of G. Thus we obtain the following reformulation of the double cover conjecture:

The double cover conjecture (second version). *For every bridgeless graph G, there exists an integer $k \geq 2$ such that G has a D_k-flow.*

The following stronger conjecture appears in **[8]** and **[37]**:

The 5-colorable double cover conjecture. *Every bridgeless graph has a D_5-flow.*

A cycle double cover is said to be **orientable** if one can orient each of its cycles into a directed cycle in such a way that each edge appears once in each direction in the resulting family of directed cycles. For instance, the family of face-boundaries of a strong embedding of G in some orientable surface is an orientable cycle double cover.

Let $\vec{D}_k$ be the subset of $\mathbf{Z}^k$ ($k \geq 2$) consisting of those elements $(z_1, \ldots, z_k)$ such that exactly one z_i is 1, exactly one z_i is -1, and all the other z_i are 0. It is easy to see that *G has an orientable k-colorable cycle double cover if and only if it has a $\vec{D}_k$-flow.*

Oriented versions of the previous conjectures can be stated as follows:

The orientable double cover conjecture. *For every bridgeless graph G, There exists an integer $k \geq 2$ such that G has a $\vec{D}_k$-flow.*

The 5-colorable orientable double cover conjecture. *Every bridgeless graph has a $\vec{D}_5$-flow.*

These oriented versions are related to nowhere-zero k-flow problems. Indeed, it is clear that the proof of Theorem 3.1 can be immediately adapted to yield the following:

Theorem 5.1. *If a graph has a $\vec{D}_k$-flow ($k \geq 2$), then it is an F_k graph.* ||

An interesting consequence of this result is that the 5-colorable orientable double cover conjecture implies the 5-flow conjecture. If we drop the orientability of cycle double covers, we can prove only the following result:

Theorem 5.2. *If a graph has a D_k-flow ($k \geq 2$), then it is an F_p graph for $p = 2^{\lceil \log_2 k \rceil}$.*

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_k)$ be a D_k-flow of G. Let $r = \lceil \log_2 k \rceil$, and let f be a one-to-one mapping from $\{1, \ldots, k\}$ into $\mathbf{Z}_2^r$. For each i in $\{1, \ldots, k\}$, let φ_i' be the $\mathbf{Z}_2^r$-flow of G which takes the value $f(i)$ on $\sigma(\varphi_i)$ and the value 0 on $E(G) - \sigma(\varphi_i)$. It is easy to check that $\sum_{i=1}^{k} \varphi_i'$ is a nowhere-zero $\mathbf{Z}_2^r$-flow of G. ||

Theorems 5.1 and 5.2 have the following converses for small values of k; Theorem 5.4 was proved for trivalent graphs by Tutte **[48]**:

Theorem 5.3. *The following statements are equivalent for a graph G:*

(*i*) *G is an F_2 graph;*

(*ii*) *G has a $\vec{D}_2$-flow;*

(*iii*) *G has a D_2-flow.*

Proof. This is immediate. ||

Theorem 5.4. *The following statements are equivalent for a graph G:*

(*i*) *G is an F_3 graph;*

(*ii*) *G has a* $\vec{D}_3$*-flow.*

Proof. This is the case $p = 1$ of [**25**, Proposition 2]. ||

Theorem 5.5. *The following statements are equivalent for a graph G:*

(*i*) *G is an* F_4 *graph;*

(*ii*) *G has a* D_3*-flow;*

(*iii*) *G has a* $\vec{D}_4$*-flow;*

(*iv*) *G has a* D_4*-flow.*

Proof. (*i*) ⇒ (*ii*). If (φ_1, φ_2) is a nowhere-zero $\mathbf{Z}_2^2$-flow of G, then $(\varphi_1, \varphi_2, \varphi_1 + \varphi_2)$ is a D_3-flow of G.

(*ii*) ⇒ (*iii*) (see [**48**]). Let $(\varphi_1, \varphi_2, \varphi_3)$ be a D_3-flow of G. Fir $i = 1, 2, 3$, let φ_i' be a **Z**-flow of G which takes the value 1 or -1 on $\sigma(\varphi_i)$ and the value 0 on $E(G) - \sigma(\varphi_i)$. Let

$$\psi_1 = \tfrac{1}{2}(\varphi_1' - \varphi_2' - \varphi_3'),\ \psi_2 = \tfrac{1}{2}(-\varphi_1' + \varphi_2' - \varphi_3'),$$
$$\psi_3 = \tfrac{1}{2}(-\varphi_1' - \varphi_2' + \varphi_3') \text{ and } \psi_4 = \tfrac{1}{2}(\varphi_1' + \varphi_2' + \varphi_3').$$

Then $(\psi_1, \psi_2, \psi_3, \psi_4)$ is a $\vec{D}_4$-flow of G.

(*iii*) ⇒ (*iv*). This is immediate.

(*iv*) ⇒ (*i*). This is Theorem 5.2 for $k = 4$. ||

The property of being mod $(2p + 1)$-orientable (see Section 4) can also be studied in terms of cycle double covers of a special kind. Consider a vector $\mathbf{x} = (x_1, \ldots, x_k)$, where the x_i belong to $\mathbf{Z}_2$ or **Z**. Two of the coordinates x_i and x_j are said to be **cyclically consecutive** if $|j - i| = 1$ or $k - 1$. Let C_k be the subset of D_k consisting of those elements whose two non-zero components are cyclically consecutive. Similarly, let $\vec{C}_k$ be the subset of $\vec{D}_k$ consisting of those elements whose two non-zero components are cyclically consecutive. The following result was proved in [**25**]; Theorem 5.4 is the special case $p = 1$ of this result:

Theorem 5.6. *A graph is mod* $(2p + 1)$*-orientable* $(p \geq 1)$ *if and only if it has a* $\vec{C}_{2p+1}$*-flow.* ||

It follows from Theorem 5.6 that the circular flow conjecture of Section 4 can be reformulated as follows:

The circular flow conjecture (second version). *For all* $p \geq 1$, *every* $4p$*-edge-connected graph has a* $\vec{C}_{2p+1}$*-flow.*

In support of this conjecture, the following result is proved in [**25**]:

Theorem 5.7. *For all* $p \geq 1$, *every* $4p$*-edge-connected graph has a* C_{2p+1}*-flow.* ||

For $p = 1$, this is essentially the nowhere-zero-4-flow result of Theorem 4.7. The proof of the general case also relies on Lemma 4.1.

6. Fulkerson's Conjecture

We present here a conjecture of Fulkerson [**15**], which deals with 1-factors of trivalent graphs and can be viewed as an edge-coloring problem. As we shall see, it can be reformulated as a nowhere-zero flow problem.

Fulkerson's conjecture. *In every bridgeless trivalent graph, there exists a family of six 1-factors such that each edge appears in exactly two of them.*

A trivalent graph which satisfies this condition is said to have the **Fulkerson property**. The following result is a characterization of this property in terms of flows. We denote by Y the subset of $\mathbf{Z}_2^6$ consisting of those elements containing exactly four 1s.

Theorem 6.1. *Let G be a trivalent graph. Then G has the Fulkerson property if and only if G has a Y-flow.*

Proof. Let $(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)$ be a Y-flow of G, and let v be a vertex of G. Then

$$\sum_{i=1}^{6} |\sigma(\varphi_i) \cap \omega(\{v\})| = \sum_{e \in \omega(\{v\})} \sum_{i=1}^{6} |\sigma(\varphi_i) \cap \{e\}| = 4|\omega(\{v\})| = 12.$$

Since $|\sigma(\varphi_i) \cap \omega(\{v\})| \in \{0, 2\}$ for $i = 1, \ldots, 6$, we thus have $|\sigma(\varphi_i) \cap \omega(\{v\})| = 2$. Hence, $M_i = E(G) - \sigma(\varphi)$ is a 1-factor of G, for $i = 1, \ldots, 6$, and each edge of G belongs to exactly two of the sets M_i.

Conversely, if G has the Fulkerson property, then it is easy to construct a Y-flow of G, since to every 1-factor M there corresponds a $\mathbf{Z}_2$-flow φ with $\sigma(\varphi) = E(G) - M$. ||

It is not difficult to show that, if every trivalent bridgeless graph has a Y-flow, then the same property holds for all bridgeless graphs (see Section 9). Hence we can reformulate Fulkerson's conjecture as follows:

Fulkerson's conjecture (second version). *Every bridgeless graph has a Y-flow.*

This conjecture can also be viewed as part of a wider problem. For $p \geq 1$, let Y_p be the subset of $\mathbf{Z}_2^{3p}$ consisting of those elements with exactly $2p$ 1s. Thus Y_1 is the set D_3 introduced in Section 5, and Y_2 is the set Y defined above. It follows from work of Seymour [**40**], and from an easy extension of the above discussion, that for each bridgeless graph G, the set $P(G)$ of positive integers p such that G has a Y_p-flow contains all sufficiently large integers, or all sufficiently large even integers. Moreover, the first situation occurs if G is trivalent and has no subgraph homeomorphic to the Petersen graph—this should be compared with the trivalent 4-flow conjecture. Any upper bound for min $P(G)$, valid for all bridgeless graphs G, would constitute important progress.

7. A Unifying Conjecture

So far we have encountered a number of conjectures in which B is a subset of some additive group A, such that $0 \notin B$ and $B = -B$, and every bridgeless graph is conjectured to have a B-flow. Let us call this the ***B*-flow conjecture**.

We present here a conjecture which implies all reasonable B-flow conjectures when B is a subset of $\mathbf{Z}_2{}^k$, for some integer k. This conjecture appears in **[24]** in the context of binary spaces, but we reformulate it in purely graph-theoretical terms.

In this section we consider $\mathbf{Z}_2$ as a field—that is, we write $\mathbf{Z}_2$ for $GF(2)$. It is well known that the set of $\mathbf{Z}_2$-flows of a graph G forms a vector space over $\mathbf{Z}_2$. We denote this vector space by $\mathcal{F}(G)$ and its dimension by $\mu(G)$. Recall that a *subdivision* of a graph G is any graph which can be obtained from G by inserting new vertices of degree 2 into the edges.

Let G_1 and G_2 be two graphs. We write $G_1 \leq G_2$ if there exists a subdivision G_1' of G_1 with the following property: there exists a bijective mapping β from $E(G_2)$ to $E(G_1')$ such that, for each $\mathbf{Z}_2$-flow φ of G_1', $\varphi \circ \beta$ is a $\mathbf{Z}_2$-flow of G_2. We write $G_1 \simeq G_2$ if $G_1 \leq G_2$ and $G_2 \leq G_1$.

For instance, denote by K_2^3 the graph consisting of two vertices and three parallel edges e_1, e_2, e_3 joining these two vertices, and let G be an edge-3-colorable trivalent graph.

We show that $K_2^3 \leq G$. For this, consider an edge-3-coloring of G, and let $M_i (i = 1, 2, 3)$ be the set of edges of G of the ith color. For $i = 1, 2, 3$, replace e_i by a path P_i of length $|M_i|$, and let H be the resulting subdivision of K_2^3. There exists a bijective mapping β from $E(G)$ to $E(H)$ such that $E(P_i) = \beta(M_i) (i = 1, 2, 3)$. Then, for each $\mathbf{Z}_2$-flow φ of H which is not identically zero, $\varphi \circ \beta$ is a $\mathbf{Z}_2$-flow of G whose support is a set of bicolored cycles. Hence $K_2^3 \leq G$.

The following result is easy to prove:

Theorem 7.1. *The relation $\leq$ is a quasi-order—that is, it is reflexive and transitive. If $G_1 \leq G_2$, then $|E(G_1)| \leq |E(G_2)|$, and $\mu(G_1) \leq \mu(G_2)$; moreover, equality holds in both inequalities if and only if $G_1 \simeq G_2$.* ||

It follows that $G_1 \simeq G_2$ if and only if $\mathcal{F}(G_1)$ and $\mathcal{F}(G_2)$ are isomorphic

in a strong sense—that is, if and only if the cycle matroids of G_1 and G_2 are isomorphic (for definitions, see Chapter 3 or **[55]**).

Let $\mathscr{C}$ be a class of graphs. A graph G is said to be **$\mathscr{C}$-minimal** if G belongs to $\mathscr{C}$, and $G' \simeq G$ whenever G' belongs to $\mathscr{C}$ and $G' \leq G$.

Theorem 7.2. *For each graph G in $\mathscr{C}$, there exists a $\mathscr{C}$-minimal graph G_0 such that $G_0 \leq G$.*

Proof. This is an immediate consequence of Theorem 7.1. ||

Now let k be a positive integer, and let B be a subset of $\mathbf{Z}_2{}^k - \{0\}$. The following result motivates our study of the relation $\leq$:

Theorem 7.3. *If G_1 and G_2 are two graphs with $G_1 \leq G_2$, and if G_1 has a B-flow, then G_2 also has a B-flow.*

Proof. Let G_1' be a subdivision of G_1, and let β be a bijective mapping from $E(G_2)$ to $E(G_1')$ such that, for each $\mathbf{Z}_2$-flow φ of G_1', $\varphi \circ \beta$ is a $\mathbf{Z}_2$-flow of G_2. If G_1 has a B-flow, G_1' also has a B-flow $(\varphi_1, \ldots, \varphi_k)$. Then $(\varphi_1 \circ \beta, \ldots, \varphi_k \circ \beta)$ is a B-flow of G_2. ||

From now on, $\mathscr{C}$ denotes the class of bridgeless graphs. It follows from Theorems 7.2 and 7.3 that, for any $B \subseteq \mathbf{Z}_2{}^k - \{0\}$, the B-flow conjecture is equivalent to its restriction to $\mathscr{C}$-minimal graphs.

The only $\mathscr{C}$-minimal graphs known so far (up to equivalence under $\simeq$) are the graph L consisting of one vertex and one loop at this vertex, the graph K_2^3 defined above and the Petersen graph P. Moreover, it is easy to show that G is an F_2 graph if and only if $L \leq G$, and that G is an F_4 graph if and only if $L \leq G$ or $K_2^3 \leq G$.

In **[24]** we conjectured that L, K_2^3 and P are the only $\mathscr{C}$-minimal graphs, up to equivalence under $\simeq$. This conjecture essentially says that, for $B \subseteq \mathbf{Z}_2{}^k - \{0\}$ $(k \geq 1)$, the B-flow conjecture is true if and only if it holds for L, K_2^3 and P. Thus, for instance, our conjecture implies the 5-colorable double cover conjecture and Fulkerson's conjecture. We now reformulate our conjecture as a B-flow conjecture.

Let $\{\varphi_1, \varphi_2, \ldots, \varphi_6\}$ be a basis of the vector space $\mathcal{F}(P)$ of $\mathbf{Z}_2$-flows of the Petersen graph P, and let X be the subset of $\mathbf{Z}_2{}^6$ consisting of the fifteen elements of the form $(\varphi_1(e), \varphi_2(e), \ldots, \varphi_6(e))$, for $e \in E(P)$. We then have the following result:

Theorem 7.4. *The following properties are equivalent for a graph G:*

(*i*) *G has an X-flow;*

(*ii*) *$L \leq G$ or $K_2^3 \leq G$ or $P \leq G$.* ||

It follows that the property for a graph G to have an X-flow is independent of the choice of the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_6\}$. We therefore say that G has a **Petersen flow** if G has an X-flow.

It is now clear that our conjecture about $\mathscr{C}$-minimal graphs can be reformulated as follows:

The Petersen flow conjecture. *Every bridgeless graph has a Petersen flow.*

The Petersen flow conjecture can be restricted to trivalent graphs (see Section 9). Moreover, it is easy to see that a trivalent graph G satisfies the Petersen flow conjecture if and only if we can color its edges, using the edges of the Petersen graph P as colors, in such a way that every triple of mutually incident edges of G is colored as a similar triple of P. Another simple formulation in terms of edge-5-colorings was recently obtained in **[27]**.

A similar approach via a quasi-order relation for arbitrary B-flow problems is also undertaken in **[27]**. The quasi-order description and its properties are more complicated in this general case, and are not presented here.

8. Special Results

Most of the conjectures reviewed above are true for planar graphs and for F_4 graphs. It is therefore natural to try to prove them for classes of graphs which are either 'nearly planar' or 'nearly F_4'. This direction of research offers a variety of open problems, which should be more tractable than the general conjectures.

For graphs which are 'nearly planar', we could try, for instance, to prove some of the conjectures for graphs of small orientable or non-orientable genus. The first results we know in this direction were obtained by Steinberg, who proved the 3-flow conjecture **[43]** and the 5-flow conjecture **[44]** for graphs embeddable in the projective plane. More recently, the 5-flow conjecture has been proved independently by Möller *et al.* **[34]** and Fouquet **[14]** for graphs of orientable genus at most 2 and graphs of non-orientable genus at most 4.

We now present some results concerning graphs which are 'nearly F_4'. We define a graph G to be a **nearly-F_4 graph** if it is possible to add a new edge to G in order to obtain an F_4 graph. A graph G is a **deletion-F_4 graph** if it is possible to delete an edge of G in order to obtain and F_4 graph.

Examples of nearly-F_4 graphs are the following (see **[21]** for the first three classes):

(*i*) *F_4 graphs*: adding a loop, or an edge parallel to an existing edge, to an F_4 graph yields an F_4 graph.

(*ii*) *graphs with a Hamiltonian path*: by adding an edge, it is possible to obtain a graph with a Hamiltonian cycle—such a graph is easily seen to be an F_4 graph.

(*iii*) *trivalent graphs with a 2-factor having* 0 *or* 2 *odd components*: if the 2-factor has no odd cycles, then the graph is an F_4 graph (Theorem 3.6(*ii*)), whereas if the 2-factor has two odd cycles, then its edges can be colored with two colors (0, 1) and (1, 0) in such a way that each vertex, with the exception of two vertices v and v', is bicolored; if we now join v and v' by a new edge, and color the edges yet uncolored with (1, 1), then the resulting coloring of the edges defines a nowhere-zero $\mathbf{Z}_2^2$-flow.

(*iv*) *deletion-F_4 graphs*: if the deletion of an edge e yields an F_4 graph, adding a new edge parallel to e clearly gives an F_4 graph.

We now discuss some results on nearly-F_4 graphs, starting with the 5-flow conjecture (see **[21]**):

Theorem 8.1. *Every bridgeless nearly-F_4 graph is an F_5 graph.*

Sketch of proof. We must prove that if G is an F_4 graph and if $G - e$ is bridgeless for some $e \in E(G)$, then $G - e$ is an F_5 graph. It is possible to choose an orientation of G which has a $(\mathbf{Z} \cap \{1, 2, 3\})$-flow φ. It is easy to see that we may also assume that $\varphi(e) = 1$.

If $\omega^+(S) = \{e\}$ for some $S \subseteq V(G)$, then

$$\sum_{e' \in \omega^-(S)} \varphi(e') = \varphi(e) = 1,$$

and hence $|\omega^-(S)| = 1$. But this contradicts the hypothesis that $G - e$ is bridgeless. It follows that there exists in $G - e$ a directed path from the initial end of e to its terminal end. Hence there exists a **Z**-flow μ of G such that $\mu(e) = -1$ and $\mu(e') \in \{0, 1\}$, for all e' in $E(G) - \{e\}$. It follows that $\varphi + \mu$ is a **Z**-flow of G which takes the value 0 on e and values in $\{1, 4\}$ elsewhere, and we may consider $\varphi + \mu$ as a nowhere-zero 5-flow of $G - e$. ||

We next discuss some special cases of the double cover conjecture. The following result is due to Celmins **[8]**:

Theorem 8.2. *Every trivalent 3-edge-connected deletion-F_4 graph has a D_5-flow.*

Proof. Let G be a trivalent 3-edge-connected graph, and let $e \in E(G)$ be such that $G - e$ is an F_4 graph. It follows from the 3-edge-connectivity of G that there exists a cycle C of $G - e$ which contains both ends of e.

Let φ_0 be the $\mathbf{Z}_2$-flow of $G - e$ with $\sigma(\varphi_0) = C$, and let $(\varphi_1, \varphi_2, \varphi_3)$ be a D_3-flow of $G - e$. Then it is easy to check that $(\varphi_0, \varphi_0 + \varphi_1, \varphi_0 + \varphi_2, \varphi_0 + \varphi_3)$ is a D_4-flow of $G - e$.

The two ends of e define a partition of C into two paths P' and P''. Let φ_0' be the $\mathbf{Z}_2$-flow of G with support $P' \cup \{e\}$, and let φ_0'' be the $\mathbf{Z}_2$-flow of G with support $P'' \cup \{e\}$. Then $(\varphi_0', \varphi_0'', \varphi_0 + \varphi_1, \varphi_0 + \varphi_2, \varphi_0 + \varphi_3)$ is a D_5-flow of G. ||

The following result is due to Tarsi **[47]**; its proof relies on an ingenious construction:

Theorem 8.3. *Every bridgeless graph with a Hamiltonian path has a D_6-flow.* ||

9. Reductions

We consider here only 'B-flow conjectures', in the sense of Section 7—that is, conjectures concerning the class of all bridgeless graphs. The approach presented in this section can, of course, be used for any kind of nowhere-zero flow problem. For instance, the proof of Steinberg's 3-flow theorem for graphs embedded in the projective plane is a good example of the use of reduction techniques.

We start by presenting the reduction of the B-flow conjecture to trivalent 3-edge-connected graphs. In this section, A denotes an additive group and B is any non-empty subset of A with $0 \notin B$, $B = -B$. Recall that the B-flow conjecture states that every bridgeless graph has a B-flow. We shall assume that the B-flow conjecture is true for the graph K_2^3 defined in Section 7; equivalently, there exist b_1, b_2, b_3 in B such that $b_1 + b_2 + b_3 = 0$.

The following result is a generalized version of **[41**, Proposition 2.1], and is proved in a similar way:

Theorem 9.1. *Every minimal counter-example to the B-flow conjecture is a simple trivalent 3-edge-connected graph.*

Proof. A minimal counter-example G is clearly a loopless 2-edge-connected graph, which is not an F_2 graph, and has at least 3 edges. If $\{e_1, e_2\}$ is a 2-cut of G, we can contract e_1 and obtain a 2-edge-connected graph which, by the minimality of G, has a B-flow. This easily yields a B-flow of G, and we have a contradiction. Hence G is 3-edge-connected, and in particular has no vertices of degree smaller than 3.

We now assume that G has a vertex v of degree greater than 3. Then, by a result of Fleischner **[11]**, we can find two edges e_1 and e_2 incident to v such that we obtain a bridgeless graph by deleting e_1 and e_2 and adding a new edge joining the ends of e_1, e_2 distinct from v. By the minimality of G, this graph has a B-flow, which easily gives a B-flow of G; again, we get a contradiction.

We conclude that G is trivalent and 3-edge-connected; G must be simple because otherwise G is isomorphic to K_2^3, which is not a counter-example. ||

Note that, by a well-known result, '3-edge-connected' can be replaced by '3-connected' in Theorem 9.1.

We now assume that the B-flow conjecture is true for all F_4 graphs. Hence, by Theorem 3.6(*ii*), every minimal counter-example to the B-flow conjecture is a simple trivalent 3-edge-connected graph which is not edge-3-colorable.

We call a 3-cut of a trivalent graph G *trivial* if it is of the form $\omega(\{v\})$ for some vertex v of G. A **snark** (see **[9]** or **[13]**) is a trivalent, 3-edge-connected, non-edge-3-colorable graph in which every 3-cut is trivial. We now present a condition under which the B-flow conjecture can be reduced to snarks.

Let G be a trivalent graph, and let v be a vertex of G. We choose an orientation of G such that the three edges e_1, e_2, e_3 incident to v have initial end v. We say that v is ***B*-specifiable** if, for all b_1, b_2, b_3 in B such that $b_1 + b_2 + b_3 = 0$, there exists a B-flow φ of G such that $\varphi(e_i) = b_i$, for $i = 1, 2, 3$. We say that B has the **vertex-specification property** if, for every trivalent graph G which has a B-flow, every vertex of G is B-specifiable.

Theorem 9.2. *If B has the vertex-specification property, then every minimal counter-example to the B-flow conjecture is a snark.*

Sketch of proof. Let G be a minimal counter-example, and assume that G has a non-trivial 3-cut $\omega(S)$, so that $|S| \geq 2$ and $|V(G) - S| \geq 2$. By identifying the vertices of S to a single vertex, we obtain a bridgeless trivalent graph G'. Similarly, by identifying the vertices of $V(G) - S$ to a single vertex, we obtain a bridgeless trivalent graph G''. By the definition of G, the graphs G' and G'' have B-flows. The vertex-specification property gives B-flows of G' and G'' which can be 'pieced together' into a B-flow of G. This gives the required contradiction. ||

In most of the interesting cases, the vertex-specification property can easily be proved using symmetry considerations. This is true, for instance, when B is D_k, $\vec{D}_k$ for $k \geq 3$ (see Section 5), or Y_p for $p \geq 1$ (see Section 6). For $B = \mathbf{Z}_5 - \{0\}$, the result is non-trivial. It is proved in **[8]**, using the contraction–deletion process associated with the computation of the flow polynomial to obtain a stronger enumerative result.

An even more difficult result is proved in **[8]**:

Theorem 9.3. *Every minimal counter-example to the 5-flow conjecture is a cyclically 5-edge-connected snark of girth at least* 7.

Outline of proof. The proof that every minimal counter-example is cyclically 5-edge-connected uses sophisticated enumerative methods analogous to the Birkhoff–Lewis reduction of the 4-ring for the four-color problem **[3]**. To see how the proof of the girth property works, consider, for instance, a cycle C of length 6 in a minimal counter-example G. Delete three edges which form a perfect matching of C. Since G is cyclically 5-edge-connected, the resulting graph G' is bridgeless. It is then easy to combine a nowhere-zero $\mathbf{Z}_5$-flow of G' with a $\mathbf{Z}_5$-flow of G whose support is C to obtain a nowhere-zero $\mathbf{Z}_5$-flow of G. This gives the required contradiction. ‖

An analogous result holds for counter-examples to the 5-flow conjecture which are minimal in the class of graphs embedded in a given surface S. Such a graph G is simple, 3-edge-connected, trivalent, of girth at least 6, and has no face-boundary of length less than 7 (see **[14]**, **[34]**). It follows easily from Euler's formula that $|V(G)| \leqslant -14k(S)$, where $k(S)$ is the Euler characteristic of S. Hence the 5-flow conjecture for graphs embedded in a given surface has been reduced to a finite number of cases.

Finally, a similar result was recently obtained by Goddyn **[16]**, for the double cover conjecture:

Theorem 9.4. *Every minimal counter-example to the double cover conjecture has girth at least* 7. ‖

Theorems 9.3 and 9.4 are interesting in view of the fact that no snark with girth at least 7 is known. It is conjectured in **[28]** that such snarks do not exist.

10. Conclusion

In this chapter we have presented a brief survey of a rich class of problems which are rather strongly interrelated—the class of nowhere-zero flow problems. The basic problems appear to be difficult, but a number of reduction results and special results have been obtained, and more can be done in this direction.

To conclude, we mention some relationships between nowhere-zero flow problems and other areas of research.

The existence of nowhere-zero flows provides tools for obtaining cycle covers of graphs (that is, families of cycles covering all edges) of short total length. For instance, it is proved in **[2]** and **[46]** that every bridgeless

graph G has a cycle cover of total length at most $\frac{5}{3}|E(G)|$. It is shown in **[38]** how some improvements could be derived from the validity of various B-flow conjectures.

Recently, Bouchet has initiated the study of nowhere-zero k-flows in bidirected graphs, proposing a '6-flow conjecture' and proving a '216-flow theorem' **[7]**. This work has been pursued by different authors (**[12]**, **[29]**, **[59]**), and the universal bound of 216 has been reduced to 30.

Finally, nowhere-zero flow problems have natural extensions to matroids and chain groups. For instance, the double cover conjecture can be viewed as a problem on binary matroids, and the 5-flow conjecture is a special case of the study of the critical exponent of matroids representable over $GF(5)$ (see Chapter 3 and **[55]**). It is likely that a significant advance on nowhere-zero flow problems would have interesting repercussions in the study of more general matroid problems.

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