Configuration Spaces of Planar Polygons

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Abstract

We present some results about the topological structure of configuration spaces of arbitrary polygons in the plane, i.e. point sets $(P_1, P_2, \ldots, P_n)$ in $\mathbb{R}^2$ with given distances $|P_i - P_{i+1}|$. The configuration space of such a polygon is the set of all realizations, modulo the group of proper isometries of $\mathbb{R}^2$. We first describe the local structure of these spaces. We prove that the configuration spaces are orientable, we compute the number of components and we present a simple combinatorial criterion for connectedness. Furthermore we show how to compute the Euler-characteristic and we give a complete list of the possible types of configuration spaces for planar pentagons. Our results constitute an extension of a result found by T.F. Havel about the configuration space of the planar equilateral pentagon linkage (see the theorem in the Introduction below).

1 Introduction

Point sets with fixed distances between certain pairs of points appear in many branches, so in chemistry (molecules), mechanics (bar mechanisms, linkages) and architecture (frameworks). In the abstract mathematical world, we can assign to these structures a weighted graph $G = (V, E, \omega)$, $\omega : E \longrightarrow \mathbb{R}_+$. V is the set of atoms in a molecule or the set of flexible joints of a bar mechanism. An edge of the graph suggests that between the corresponding atoms there is a force retaining the relative position of the atoms, or that there is a rigid bar between the corresponding joints in the mechanism.

Given a weighted graph as above, at least two important questions arise:

1. Is there a realization $\varphi : V \longrightarrow \mathbb{R}^d$ such that $|\varphi(v_i) - \varphi(v_j)| = \omega(\{v_i v_j\})$ for all $\{v_i v_j\} \in E$? (Realization problem)

2. What is the set of all possible realizations (modulo the group of proper isometries of $\mathbb{R}^d$)? **(Description of the configuration space)**

Above all, we investigate the second question for the simplest nontrivial case: cycle graphs in $\mathbb{R}^2$.

The starting point is a rather surprising result (see [8] or [4] p. 199ff):

Theorem (Havel, 1987) *The topological structure of the configuration space of the planar equilateral pentagon linkage is that of a compact, connected and orientable two-dimensional manifold of genus 4.*

In the proof, the oriented area of the pentagon is used as a Morse function. With the help of this function it is possible to reconstruct the configuration space. Unfortunately, for arbitrary polygons, the oriented area is very hard to handle.

Gibson and Newstead studied planar 4-bar mechanisms with methods of algebraic geometry (see [7]). They found that for planar 4-gons, the configuration spaces consist either of one or of two circles (if they are manifolds). These results furnish the key for our new method: These two compact, one-dimensional manifolds appear also as level surfaces of the height–function on a two-dimensional torus (see [10]).

So we consider a higher-dimensional torus and define a simple Morse function on it in such a way that the configuration space of a given polygon appears as level surface of this function. Then we use a surgery technique for level surfaces to reconstruct the configuration space.

In Section 2 we give exact definitions. In Sections 3 we mention the main results and we prove them in Section 4. In Section 5, we explain how to compute the Euler–characteristic of a configuration space and we present another proof of Havel's theorem

2 Definitions

Although we will deal only with cycle graphs, we define the configuration space of an arbitrary weighted graph.

We identify a n–tuple of points $(P_1, P_2, \ldots, P_n)$ in $\mathbb{R}^d$ with an element of $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$ and define an equivalence relation $\sim$ as follows:

$$(P_1, P_2, \ldots, P_n) \sim (Q_1, Q_2, \ldots, Q_n) \quad :\Longleftrightarrow \quad \begin{array}{l}\text{There is a (proper) isometry} \\ r \text{ of } \mathbb{R}^d \text{ with } (P_1, P_2, \ldots, P_n) \\ = (rQ_1, rQ_2, \ldots, rQ_n)\end{array}$$

Let G^d be the group of (proper) isometries of $\mathbb{R}^d$ and $N = \{1, 2, \ldots, n\}$. We set $X^d := \mathbb{R}^{dn}/\sim$ and consider the map $\tilde{d} \; : \; X^d \times X^d \longrightarrow \mathbb{R}_+$ defined by

$$\tilde{d}([P],[Q]) := \min_{g \in G^d} \max_{i \in N} \; |P_i - gQ_i| \; .$$

It is not hard to prove that $\left(X^d, \tilde{d}\right)$ is a metric space (see [9]).

Definition 1 *Let $G = (V, E, \omega)$, $\omega : E \longrightarrow \mathbb{R}_+$ be a finite weighted graph. A (not necessarily injective) map $\varphi : V \longrightarrow X^d$ with $|\varphi(v_i) - \varphi(v_j)| = \omega(v_i, v_j)$ for all $v_iv_j \in E$ is called a* **realization.** *The set of all realizations is called* **configuration space of G** *and is denoted by X^d_G.*

Note that we take X^d instead of $\mathbb{R}^d$. So X^d_G is a set of equivalence classes, but we will often identify a representative with its class and write $(P_1, P_2, \ldots, P_n)$ when we mean $[(P_1, P_2, \ldots, P_n)]$.

Notation: A weighted cycle graph C_n can be characterized by a "length-vector" $l = (l_1, l_2, \ldots, l_n)$ defined by $l_i := \omega(\{v_iv_{i+1}\})$. In this case, we denote the associated configuration space by X^d_l instead of $X^d_{C_n}$. Here and in the following, indices are computed mod n.

3 Results

We consider realizations in the plane. Let $l = (l_1, l_2, \ldots, l_n) \in (\mathbb{R}_+)^n$, $n \geq 3$ be a length–vector and let X^2_l be the configuration space of the associated cycle graph.

For a general weighted graph it is a difficult problem to decide whether a realization in $\mathbb{R}^2$ or $\mathbb{R}^d$ exists. For cycle graphs, the answer is very simple and we have a good cvharacterization of the set of all length–vectors for which the configuration space is not empty.

Assumption 1 *1. Necessary and sufficient conditions for the existence of a realization in* $\mathbb{R}^2$ *are*

$$l_i \leq \sum_{\substack{k=1 \\ k \neq i}}^{n} l_k, \quad i = 1, 2, \ldots, n$$

2. The set of all length-vectors $l = (l_1, l_2, \ldots, l_n)$ *for which a realization exists is a cone* Λ^n *generated by* $\binom{n}{2}$ *vectors* $\xi^{ij} \in \mathbb{R}^n$ *defined by*

$$\xi_k^{ij} := \begin{cases} 1 & , \quad \text{if } k = i \text{ or if } k = j \\ 0 & , \quad \text{elsewhere} \end{cases} .$$

In the following, we assume that l is a point of the cone Λ^n described above, so the configuration space is not empty.

X_l^2 is a subset of the metric space X^2 (see Section 2). In the next two theorems we describe ε-neighborhoods of a point $[P] = [(P_1, P_2, \ldots, P_n)]$ in X_l^2.

Theorem 1 *Let* $(P_1, P_2, \ldots, P_n)$ *be in* X_l^2 *with* $\dim \operatorname{aff}\{P_1, P_2, \ldots, P_n\} = 2$. *Then there is a neighborhood of* $(P_1, P_2, \ldots, P_n)$ *in* X_l^2 *homeomorphic to* $\mathbb{R}^{n-3}$.

Theorem 2 *Let* $(P_1, P_2, \ldots, P_n)$ *be in* X_l^2 *with* $\dim \operatorname{aff}\{P_1, P_2, \ldots, P_n\} = 1$. *There exists a neighborhood of* $(P_1, P_2, \ldots, P_n)$ *homeomorphic to a cone defined by*

$$\sum_{i=2}^{n-1} \alpha_i x_i^2 = 0.$$

The number of coefficients with $\alpha_i < 0$ *can be interpreted and easily calculated as an index of a Morse function (see Lemma 3 in Section 4).*

The next theorems contain some general results about the global structure of X_l^2.

Theorem 3 *Let $l = (l_1, l_2, \ldots, l_n)$ be fix. If*

$$\pm l_1 + \pm l_2 + \cdots + \pm l_n \neq 0$$

for any choice of the signs, then X_l^2 is a compact and orientable $(n-3)$-dimensional manifold.

Theorem 4 *Let $\nu : N \longrightarrow N$ be a permutation of $N = \{1, 2, \ldots, n\}$. We set $l_\nu := (l_{\nu(1)}, l_{\nu(2)}, \ldots, l_{\nu(n)})$. Then $X_{l_\nu}^2$ is homeomorphic to X_l^2.*

Theorem 5 *For fixed l, the space X_l^2 consists of at most two topological components.*

If X_l^2 has two topological components, then these components are manifolds and they are diffeomorphic.

From theorem 4 we know that the order of the l_i's doesn't matter. So we can assume $l_1 \leq l_2 \leq \ldots \leq l_n$. The next result is very surprising:

Theorem 6 *For $l = (l_1, l_2, \ldots, l_n)$ with $l_1 \leq l_2 \leq \ldots \leq l_n$ the space X_l^2 is connected if and only if $l_{n-1} + l_{n-2} \leq l_1 + l_2 + \cdots l_{n-3} + l_n$.*

It is easy now to see that for example the vector $l = (\varepsilon, \varepsilon, \ldots, \varepsilon, 1, 1, 1) \in (\mathbb{R}_+)^n$, ($\varepsilon$ small), gives us a disconnected configuration space for any $n \geq 3$.

Using Theorem 4, Theorem 5 and Theorem 6, we can give a complete list of the possible types of configuration spaces for planar 5-gons (if they are manifolds).

Theorem 7 *We assume $l_1 \le l_2 \le \ldots \le l_5$ and set $c := l_1 + l_2 + l_3 + l_4 + l_5$. There are six possible types of configuration spaces for planar 5-gons:*

$$\begin{array}{rcl}
X_l^2 \text{ is a 2-sphere} & \Longleftrightarrow & \frac{c}{2} < l_1 + l_5, \\
X_l^2 \text{ is a 2-torus} & \Longleftrightarrow & l_1 + l_5 < \frac{c}{2} < l_2 + l_5, \\
X_l^2 \text{ is a surface of genus 2} & \Longleftrightarrow & l_2 + l_5 < \frac{c}{2} < \min\{l_1 + l_2 + l_5, l_3 + l_5\}, \\
X_l^2 \text{ is a surface of genus 3} & \Longleftrightarrow & l_3 + l_5 < \frac{c}{2} < l_4 + l_5, \\
X_l^2 \text{ is a surface os genus 4} & \Longleftrightarrow & l_4 + l_5 < \frac{c}{2}, \\
X_l^2 \text{ is union of two 2-tori} & \Longleftrightarrow & l_1 + l_2 + l_5 < \frac{c}{2}.
\end{array}$$

4 Proofs

Proof of Assumption 1

1. The conditions are of course necessarily.

Let $[P] \in X_l^2$. Since $l_1 \le l_2 + l_3 + \cdots + l_n$ and $l_1 + l_2 + \cdots + l_{n-1} \ge l_n$, we find $k \in \{2, 3, \ldots, n-1\}$ with

$l_1 + \cdots + l_{k-1} \le l_k + l_{k+1} + \cdots + l_n$ and with $l_1 + \cdots + l_k \ge l_{k+1} + l_{k+2} + \cdots + l_n$.

So there exists a triangle with sides $a := l_1 + \cdots + l_{k-1}$, $b := l_k$ and $c := l_{k+1} + \cdots + l_n$. With the appropriate intersections of the sides of the triangle, we get a realization in X_l^2.

2. It is easy to see, that for any vector in Λ^n the conditions of 1 . hold. In order to show that the conditions are also sufficient, we use induction on n. For $n = 3$, we have the identity

$$(l_1, l_2, l_3) = \frac{1}{2}(l_1 + l_2 - l_3)(1,1,0) + \frac{1}{2}(l_1 - l_2 + l_3)(1,0,1) + \frac{1}{2}(-l_1 + l_2 + l_3)(0,1,1)$$

with positive coordinates on the right-hand side, if $l_i \le l_j + l_k$ with $i \ne j \ne k \ne i$.

Suppose the statement is true for $n - 1$. Let $l = (l_1, l_2, \ldots, l_n)$ be a vector which fulfills the conditions above. W.l.o.g we assume $l_1 \le l_2 \le \cdots \le l_n$. We define $v_\mu := (l_1 - \mu, l_2, l_3, \cdots, l_{n-1}, l_n - \mu)$ and choose $\mu_0 \ge 0$ in such a way that the conditions above hold for v_{μ_0}, but not for v_μ, $\mu > \mu_0$.

Two cases are possible: Either $l_1 - \mu_0 = 0$, then v_{μ_0} is in Λ^n by induction, or we have

$$l_{n-1} = (l_1 - \mu_0) + l_2 + \cdots + l_{n-2} + (l_n - \mu_0).$$

(Remember that $l_1 \leq \ldots \leq l_n$.) In this case we set

$$\begin{aligned} v_{\mu_0} &= (l_1 - \mu_0)(1,0,\ldots,0,1,0) + l_2(0,1,0,\ldots,0,1,0) + \\ &\quad + l_{n-2}(0,\ldots,0,1,1,0) + (l_n - \mu_0)(0,0,\ldots,0,1,1). \end{aligned}$$

With $l = v_{\mu_0} + \mu_0(1,0,\ldots,0,1)$ we see that in both cases l is in the cone Λ^n. $\square$

Proof of Theorem 1

Let $l = (l_1, l_2, \ldots, l_n)$ be a point in the cone Λ^n (see Assumption 1).

To $[(P_1, P_2, \ldots, P_n)] \in X_l^2$ we associate the polygon $P = (P_1, P_2, \ldots, P_n)$ in $\mathbb{R}^2$. The straight lines $\overline{P_i P_{i+1}}$ suggest the fixed distances between P_i and P_{i+1}. We have to consider triangulations of the polygon P. For all these triangulations, the vertices of the triangle belong to the set $\{P_1, P_2, \ldots, P_n\}$.

The following Lemma 1 proves the existence of a special *regular* triangulation.

Lemma 1 *For $P = (P_1, P_2, \ldots, P_n) \in X_l^2$ with dim aff $\{P_1, P_2, \ldots, P_n\} = 2$, there exists a triangulation $(t_1, \ldots, t_{n-2})$ of P such that no triangle t_i is degenerate (i.e. for every triangle the affine hull of its vertices is two-dimensional).*

Proof: $\exists\ k \in \{1, 2, \ldots, n\}$ with dim aff $\{P_k, P_{k+1}, P_{k+2}\} = 2$. We assume $k = 1$. If dim aff $\{P_1, P_3, P_4, \ldots, P_n\} = 1$, then $P_4, P_5, \ldots, P_n \in$ aff $\{P_1, P_3\}$. Since $P_3 \neq P_4$ we have dim aff $\{P_2, P_3, P_4\} = 2$ and since $P_n \neq P_1$ we have dim aff $\{P_2, P_4, \ldots, P_n, P_1\} =$ dim aff$\{P_1, P_2, P_n\} = 2$.

So we can always find $j \in \{1, 2, \ldots, n\}$ with

dim aff $\{P_{j-1}, P_j, P_{j+1}\} = 2$ and with dim aff $\{P_1, \ldots, P_{j-1}, P_{j+1}, \ldots, P_n\} = 2$

By induction there is a regular triangulation. $\square$

Proof of Theorem 1: Lemma 1 shows the existence of a regular triangulation of the polygon $(P_1, \ldots, P_n)$. We can take the $(n-3)$ diagonals of the triangulation as local coordinates for a neighborhood of $(P_1, P_2, \ldots, P_n)$. Theorem 1 follows. $\square$

In order to prove the following theorems, we embed our configuration space X_l^2 in a higher dimensional torus and define a simple Morse function on this torus. X_l^2 will appear as level surface of this Morse function.

With a reconstruction technique (surgery) we can, in principle, build up X_l^2 starting with a sphere.

First we describe the embedding of X_l^2 in a torus and define the Morse function. For basic notions of Morse theory see [10].

We parametrize $(P_1, P_2, \ldots, P_n) \in X_l^2$ with polar coordinates

$$P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad P_i = \begin{pmatrix} l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + \cdots + l_{i-1} \cos\alpha_{i-1} \\ l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + \cdots + l_{i-1} \sin\alpha_{i-1} \end{pmatrix}$$

$i = 2, 3, \ldots, n$

(Later on we set α_1 equal to 0 (or π), we write α_1 for technical reasons.)

Now we forget the condition $|P_1 - P_n| = l_n$. The angles $\alpha_2, \ldots, \alpha_{n-1}$ take all the values in $[0, 2\pi)$. So we can interpret these angles as parameters for the $(n-2)$–torus T^{n-2}.

We define the function $D_n : T^{n-2} \longrightarrow \mathbb{R}$ by

$$D_n(\alpha_2, \ldots, \alpha_{n-1}) := -|P_1 - P_n|^2,$$

Obviously, we have $X_l^2 = D_n^{-1}(-l_n^2)$, and thus X_l^2 is indeed a level surface of D_n.

The following Lemma 2 is crucial.

Lemma 2 *We set* $\overline{T^{n-2}} = \{(\alpha_2, \ldots, \alpha_{n-1}) \in T^{n-2} \ : \ D_n(\alpha_2, \ldots, \alpha_{n-1}) < 0\ \}$.

Then D_n is, restricted to $\overline{T^{n-2}}$, a Morse function.

Remark: $\overline{T^{n-2}}$ is a connected subset of T^{n-2}.

Proof: D_n is C^∞. We have to show that the critical points of D_n are non-degenerate (i.e. the Hessian is not singular).

$$\begin{aligned} D_n \ &= \ -(l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + \cdots + l_{n-1} \cos\alpha_{n-1})^2 \\ &\quad\ -(l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + \cdots + l_{n-1} \sin\alpha_{n-1})^2. \end{aligned}$$

From the conditions $\frac{\partial D_n}{\partial \alpha_i} = 0$, $(i = 2, 3, \ldots, n-1)$ we obtain $n-2$ equations

$$\begin{aligned} & (l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + \ldots + l_{n-1} \cos\alpha_{n-1}) l_i \sin\alpha_i \\ = \; & (l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + \ldots + l_{n-1} \sin\alpha_{n-1}) l_i \cos\alpha_i \end{aligned}$$

We add $l_1^2 \sin\alpha_1 \cos\alpha_1$ to the sum of these $(n-2)$ equations and find

$$\begin{aligned} & (l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + \ldots + l_{n-1} \cos\alpha_{n-1}) l_1 \sin\alpha_1 \\ = \; & (l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + \ldots + l_{n-1} \sin\alpha_{n-1}) l_1 \cos\alpha_1 \end{aligned}$$

There exists an isometry of $\mathbb{R}^2$ such that α_1 becomes 0 or π. So we get

$$l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + \ldots + l_{n-1} \sin\alpha_{n-1} = 0$$

Since $P_n \neq P_1$ we have $(l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + \ldots + l_{n-1} \cos\alpha_{n-1}) \neq 0$ and finally $\sin\alpha_i = 0$, $i = 1, 2, \ldots, n-1$.

So the critical points of D_n have an easy geometrical interpretation. All the angles α_i, $i = 1, 2, \ldots, n-1$ must be 0 or π, i.e. the critical points are characterized by $\dim \operatorname{aff}(P_1, P_2, \ldots, P_n) = 1$.

Now we show that the determinant of the matrix

$$H := \left(\frac{\partial^2 D_n}{\partial\alpha_i \partial\alpha_j} \right)_{i,j=2,\ldots,n-1}$$

is not zero at critical points.

The second derivatives of D_n are

$$\begin{aligned} \frac{\partial^2 D_n}{\partial\alpha_i \partial\alpha_j} &= -2l_i l_j \sin\alpha_i \sin\alpha_j - 2l_i l_n \cos\alpha_n \frac{\partial}{\partial\alpha_j}(\sin\alpha_i) \\ & \quad -2l_i l_j \cos\alpha_i \cos\alpha_j + 2l_i l_n \sin\alpha_n \frac{\partial}{\partial\alpha_j}(\cos\alpha_i) \end{aligned}$$

$$\begin{aligned} \text{with} \quad l_n \cos\alpha_n &:= -(l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + \cdots + l_{n-1} \cos\alpha_{n-1}); \\ l_n \sin\alpha_n &:= -(l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + \cdots + l_{n-1} \sin\alpha_{n-1}) \end{aligned}$$

For critical points we have (by setting $\varepsilon_i := \cos\alpha_i = \pm 1$)

$$\frac{\partial^2 D_n}{\partial\alpha_i \partial\alpha_j} = -2l_i l_j \varepsilon_i \varepsilon_j - 2l_i l_n \varepsilon_n \frac{\partial}{\partial\alpha_j}(\sin\alpha_i), \quad i, j \in \{2, 3, \ldots, n-1\}$$

Now we have to compute the determinant of H. H is given by

$$H = \begin{pmatrix} -A_2^2 - A_2A_n & \dots & -A_2A_i & \dots & -A_2A_{n-1} \\ \vdots & \ddots & \vdots & & \vdots \\ -A_2A_i & \dots & -A_i^2 - A_iA_n & \dots & -A_iA_{n-1} \\ \vdots & & \vdots & \ddots & \vdots \\ -A_2A_{n-1} & \dots & -A_iA_{n-1} & \dots & -A_{n-1}^2 - A_{n-1}A_n \end{pmatrix}$$

where $A_i := \sqrt{2}l_i\varepsilon_i$.

Using elementary matrix operations (see [9]) we obtain

$$\det H = 2^{n-2}l_1l_2\cdots l_{n-1}\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}(l_1\varepsilon_1 + l_2\varepsilon_2 + \cdots + l_{n-1}\varepsilon_{n-1})^{n-3}.$$

Since $D_n < 0$ we have $(l_1\varepsilon_1 + l_2\varepsilon_2 + \cdots + l_{n-1}\varepsilon_{n-1}) \neq 0$. All terms in the product are different from zero and so $\det H \neq 0$. □

Now, we want to calculate the indices of the Morse function D_n for critical points. For a definition see [10]).

In our situation, the indices of D_n have a very simple geometrical interpretation (see Lemma 3 below).

With regard to the following advance, we focus on critical points with $(l_1\varepsilon_1 + l_2\varepsilon_2 + \cdots + l_{n-1}\varepsilon_{n-1}) > 0$.

Lemma 3 *Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})$ be a critical point of the Morse function D_n.*

Then the index of D_n for ε equals the number of -1 in the set $\{\varepsilon_1, \dots, \varepsilon_{n-1}\}$.

Proof: Let $l = (l_1, l_2, \dots, l_n)$ be given. From a theorem of Jacobi we know that the index of D_n at the critical point ε is equal to the number of variations of sign in the sequence $1, U^1, U^2, \dots, U^{n-2}$, where U^j denotes the sign of the j-th principal minor of the matrix H (see [6], p. 303–304).

With the same elementary operations as above, we obtain for the j-th principal minor $\det H^j$ of H

$$\begin{aligned} \det H^j &= 2^jl_2l_3\cdots l_{j+1}\varepsilon_2\varepsilon_3\cdots\varepsilon_{j+1}(l_1\varepsilon_1 + \cdots + l_{n-1}\varepsilon_{n-1})^{j-1} \\ &\quad \cdot(l_1\varepsilon_1 + l_{j+2}\varepsilon_{j+2} + l_{j+3}\varepsilon_{j+3} + \cdots + l_{n-1}\varepsilon_{n-1}) \end{aligned}$$

and therefore

$$\begin{aligned}
U^j &= \operatorname{sign}\det H^j \\
&= \varepsilon_2\varepsilon_3\cdots\varepsilon_{j+1}\operatorname{sign}(l_1\varepsilon_1 + l_{j+2}\varepsilon_{j+2} + l_{j+3}\varepsilon_{j+3} + \cdots + l_{n-1}\varepsilon_{n-1}), \\
& \quad (j = 1, \ldots, n-3) \\
U^{n-2} &= \varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}
\end{aligned}$$

Some of these minors of H may be 0, but we can easily see that *never* two consecutive minors vanish. In this case we can apply the so-called Gundenfinger–rule which allows us to drop zeros in the sequence $1, U^1, U^2, \ldots, U^{n-2}$.

We consider two cases:

1. There exists $i \in N$ with $l_i\varepsilon_i + l_{i+1}\varepsilon_{i+1} \neq 0$
2. $l_i\varepsilon_i + l_{i+1}\varepsilon_{i+1} = 0, \quad i \in N$.

Case 1. Induction on n. For $n = 4$, a simple examination of the few possible cases shows the correctness of the statement.

We assume that for $n-1$ Lemma 3 is true.
Since $l_1\varepsilon_1 + l_2\varepsilon_2 + \cdots + l_{n-1}\varepsilon_{n-1} > 0$, there must be $i \in N$ with $l_i\varepsilon_i + l_{i+1}\varepsilon_{i+1} > 0$. Without loss of generality we assume $i = n-2$. Setting $s := l_{n-2}\varepsilon_{n-2} + l_{n-1}\varepsilon_{n-1} > 0$, we see that for the sequences $1, U^1, U^2, \ldots, U^{n-2}$ resp. $1, V^1, V^2, \ldots, V^{n-3}$ of the principal minors for $(l_1, l_2, \ldots, l_{n-2}, l_{n-1})$ resp. for $(l_1, l_2, \ldots, l_{n-3}, s)$ we have $U^i = V^i, \ i = 1, 2, \ldots, n-5$.

For $(l_1, l_2, \ldots, l_{n-2}, l_{n-1})$ we find

$$\begin{aligned}
U^{n-4} &= \varepsilon_{n-3}\operatorname{sign}(l_1\varepsilon_1 + l_{n-2}\varepsilon_{n-2} + l_{n-1}\varepsilon_{n-1}) \\
U^{n-3} &= \operatorname{sign}(l_1\varepsilon_1 + l_{n-1}\varepsilon_{n-1}) \\
U^{n-2} &= \varepsilon_1\varepsilon_{n-1}
\end{aligned}$$

and for $(l_1, l_2, \ldots, l_{n-3}, s)$ we have

$$\begin{aligned}
V^{n-4} &= \varepsilon_{n-3}\operatorname{sign}(l_1\varepsilon_1 + s) = U^{n-4} \\
V^{n-3} &= \varepsilon_1\varepsilon_{n-2}
\end{aligned}$$

(To simplify matters, we have multiplied both sequences with $\varepsilon_2\varepsilon_3\cdots\varepsilon_{n-2}$. The number of variations of sign doesn't change.)

We set $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-3}, \varepsilon_{n-2}, \varepsilon_{n-1})$ and $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-3}, 1)$.

The numbers N_ε resp. $N_{\tilde{\varepsilon}}$ of -1 for the two ε-vectors are connected by the formula

$$N_\varepsilon = N_{\tilde{\varepsilon}} + \frac{1}{2}(1 - \varepsilon_{n-2}) + \frac{1}{2}(1 - \varepsilon_{n-1})$$

and again an examination of a few possible cases shows the correctness of the Lemma for n.

Case 2. Here, the index can be calculated directly. We have

$$\varepsilon_{2k} = -1, \quad \varepsilon_{2k+1} = 1$$

and can assume that $l_1 = l_2 = \dots = l_n = 1$. Such configurations exist only for even n because of $(l_1\varepsilon_1 + l_2\varepsilon_2 + \dots + l_{n-1}\varepsilon_{n-1}) > 0$. So $N_\varepsilon = \frac{n}{2} - 1$.

On the other hand $U^j = \varepsilon_2\varepsilon_3 \cdots \varepsilon_{j+1} \operatorname{sign}(\varepsilon_1 + \varepsilon_{j+2} + \varepsilon_{j+3} + \dots + \varepsilon_{n-1})$ and $U^{n-2} = \varepsilon_1\varepsilon_2 \cdots \varepsilon_{n-1} = \varepsilon_2\varepsilon_3 \cdots \varepsilon_{n-1}$. A straightforward calculation gives us

$$U^j = \begin{cases} (-1)^{\frac{j}{2}} & \text{if } j \text{ odd} \\ (-1)^{\frac{j+1}{2}} & \text{if } j \text{ even} \end{cases}$$

and the sequence $1, U^1, \dots, U^{n-2}$ becomes $1, -1, -1, 1, 1, -1, -1, \dots, (-1)^{\frac{n-2}{2}}$. So the index equals

$$\frac{n}{2} - 1 = N_\varepsilon. \qquad \square$$

Example 1: At the point $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}) = (1, 1, \dots, 1)$ we have

$$U^j = 1 \ , \ j = 1, 2, \dots, n-2 \ .$$

There are no variations of sign, all eigenvalues of H are positive, the index is 0.

Proof of Theorem 2

Let $\varepsilon = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n-1})$ be a critical point of D_n. The Lemma of Morse (see [10]) guarantees the existence of local coordinates $y_2, y_3, \dots, y_{n-1}$ in a neighborhood U of ε with

$$D_n(\xi) = D_n(\varepsilon) - y_2^2 - y_3^2 - \dots - y_{\lambda+1}^2 + y_{\lambda+2}^2 + \dots + y_{n-1}^2$$

for $\xi \in U$. Here λ is the index of the critical point.

We intersect U with the level surface through the critical value $D_n(\varepsilon)$ and Theorem 2 follows. $\square$

Proof of Theorem 3

The fact that X_l^2 is a manifold follows easily from Theorem 1. The orientability of X_l^2 is a consequence of the fact that the torus T^{n-2}, in which our $(n-3)$-dimensional configuration space is embedded, is orientable. $\square$

Proof of Theorem 4

It is enough to prove the statement for a transposition $\nu = (i, i+1)$. We assume $i = 1$ and define the map $\varphi \; : \; X_l^2 \longrightarrow X_{l_\nu}^2$ in the following way:

To $(P_1, P_2, \ldots, P_n) \in X_l^2$ correspond via φ the configuration $(P_1', P_2', \ldots, P_n') \in X_{l_\nu}^2$ with

$$P_i' \equiv P_i, \quad i = 1, 3, 4, \ldots, n \quad \text{and with} \quad P_2' = \begin{pmatrix} l_2 \cos \alpha_2 \\ l_2 \sin \alpha_2 \end{pmatrix}$$

Points with cone–neighborhoods are mapped on points with cone–neighborhoods of the same index (see Lemma 3).

Let P in X_l be a point with $\dim \operatorname{aff}\{P_1, P_2, \ldots, P_n\} = 2$ and set $P' := \varphi(P)$. We show the existence of a triangulation of the abstract n–gon such that the corresponding triangulations of P *and* P' are regular in the sense of Lemma 1.

If we can choose the diagonal $P_1P_3 = P_1'P_3'$ for the triangulation, we are done. If not, then we have either

$$\dim \operatorname{aff}\{P_1, P_3, P_4, \ldots, P_n\} = 1$$

or

$$\dim \operatorname{aff}\{P_1, P_2, P_3\} = \dim \operatorname{aff}\{P_1, P_2', P_3\} = 1.$$

In the first case the only possible triangulation of P has the diagonals $P_2P_4, P_2P_5, \ldots, P_2P_n$, the only possible triangulation of P' has the diagonals $P_2'P_4', P_2'P_5', \ldots, P_2'P_n'$.

For the second case we use induction: If $l_1 = l_2$ the Theorem is proven. So P_1 and P_3 do never coincide. For $n = 4$ the diagonal P_2P_4 resp. $P_2'P_4'$ furnish regular triangulations.

For $n \geq 5$ we choose a point $P_i = P_i' \notin \text{aff}\{P_1, P_2, P_3\}$. We have $\dim \text{aff}\{P_1, P_2, P_i, P_{i+1}, \dots, P_n\} = 2$ and $\dim \text{aff}\{P_2, P_2, P_4, \dots, P_i\} = 2$. The same for P'. By induction we find a triangulation with the required property.

It is clear that φ is a homeomorphism. □

Proof of Theorem 5

From the proof of Assumption 1 we know that there exists a triangle with sides $a := l_1 + \cdots l_{k-1}$, $b := l_k$ and $c := l_{k+1} + \cdots + l_n$.

For the partial configuration $(P_1, P_2, \dots, P_k)$ and $(P_{k+1}, P_{k+2}, \dots, P_n, P_1)$ we have

$$\max_{i \in \{1,\dots,k-1\}} \{0, 2l_i - \sum_{j=1}^{k-1} l_j\} \leq |P_1 - P_k| \leq \sum_{j=1}^{k-1} l_j$$

resp.

$$\max_{i \in \{k+1,\dots,n\}} \{0, 2l_i - \sum_{j=k+1}^{n} l_j\} \leq |P_{k+1} - P_1| \leq \sum_{j=k+1}^{n} l_j$$

If we fix P_k and P_{k+1} in the plane, then the point P_1 lies in a domain of the plane which is bounded by two pairs of concentric circles with centres P_k resp. P_{k+1}. In X_l^2 there are two configurations $Q = (Q_1, \dots, Q_n)$ and $Q' = (Q_1', \dots, Q_n')$ with $|Q_1 - Q_k| = |Q_1' - Q_k'| = l_1 + l_2 + \cdots + l_{k-1}$, $|Q_k - Q_{k+1}| = |Q_k' - Q_{k+1}'| = l_k$ and with $|Q_1 - Q_{k+1}| = |Q_1' - Q_{k+1}'| = l_{k+1} + l_{k+2} + \cdots + l_n$. ($Q$ and Q' can coincide.)

With an isometry we get $P_k = Q_k = Q_k'$ and $P_{k+1} = Q_{k+1} = Q_{k+1}'$. Then Q_1 and Q_1' also lies in the intersection of the two pairs of circles. If we stretch the partial configurations $(P_1, P_2, \dots, P_k)$ and $(P_{k+1}, P_{k+2}, \dots, P_n, P_1)$, we can find a path from P to Q or to Q' in X_l^2. This proves the first part of the Theorem.

Set $l = (l_1, l_2, \dots, l_n)$. There exists $k \in \{2, 3, \dots, n-1\}$ such that P_1 lies in the intersection of two pairs of concentric circles with centres P_k resp. P_{k+1}.

X_l^2 consists of two components, say X_1 and X_2, so the intersection of the two pairs of circles has also two components. For every configuration in X_1 the point P_1 is an inner point of one of the two halfplanes bounded by $\mathrm{aff}\{P_k, P_{k+1}\}$. So X_1 is a manifold (see Theorem 1). To a configuration in X_1 we associate the mirror image with respect to $\mathrm{aff}\{P_k, P_{k+1}\}$. This is of course a diffeomorphism. □

Proof of Theorem 6

We use the Morse function of Lemma 2.

We forget the condition $|P_1 - P_n| = l_n$ and consider the function $D_n = -|P_1 - P_n|^2$ on T^{n-2}. The configuration space $D_n^{-1}(-(l_1 + l_2 + \cdots + l_{n-1})^2)$ consists of a single point. The index of D_n at this point is 0 (see Example 1).

We decrease the distance between P_1 and P_n up to $|P_1 - P_n| = l_n$.

If no critical points appear, then the type of the configuration space is constant (see [5], p. 85).

If we pass through a critical point, the type of the configuration space changes depending on the index of the critical point and we have to apply a reconstruction technique (surgery) (see [5]).

The dimension of X_l^2 is $n-3$. For making the configuration space disconnected, we have to do surgery with maximal possible index $(n-3)$. The first critical value with this index is $l_{n-1} + l_{n-2} - l_{n-3} - \cdots - l_2 - l_1$. So we must have $l_{n-1} + l_{n-2} - l_{n-3} - \cdots - l_2 - l_1 > l_n$ or $l_{n-1} + l_{n-2} > l_1 + l_2 + \cdots + l_{n-3} + l_n$ for a disconnected configuration space. On the other hand, if $l_{n-1} + l_{n-2} > l_1 + l_2 + \cdots + l_{n-3} + l_n$, then we consider the triangle $\Delta P_{n-2} P_{n-1} P_n$. We have $l_1 + l_2 + \cdots + l_{n-3} + l_{n-1} \leq l_1 + l_2 + \cdots + l_{n-3} + l_n < l_{n-1} + l_{n-2} \leq l_{n-2} + l_n$ and therefore $l_{n-1} - l_{n-2} < l_n - l_{n-3} - \cdots - l_1$. There is no possibility to connect the triangle $\Delta P_{n-2} P_{n-1} P_n$ with its mirror image. □

Remark: R. Connelly found a proof by induction for the criterion of Theorem 6 ([3]).

Proof of Theorem 7

W.l.o.g we assume $l_1 \leq l_2 \leq l_3 \leq l_4 \leq l_5$ (see Theorem 4). We look for critical points $(\varepsilon_2, \varepsilon_3, \varepsilon_4)$ of D_5 on the torus T^3 with $l_1\varepsilon_1 + l_2\varepsilon_2 + l_3\varepsilon_3 + l_4\varepsilon_4 \geq$

l_5. There are only six such points

$$
\begin{array}{rclll}
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (+1,+1,+1,+1) & \text{with} & |P_1-P_5| = +l_1+l_2+l_3+l_4 =: a \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,+1,+1,+1) & \text{with} & |P_1-P_5| = -l_1+l_2+l_3+l_4 =: b \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (+1,-1,+1,+1) & \text{with} & |P_1-P_5| = +l_1-l_2+l_3+l_4 =: c \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (+1,+1,-1,+1) & \text{with} & |P_1-P_5| = +l_1+l_2-l_3+l_4 =: d \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (+1,+1,+1,-1) & \text{with} & |P_1-P_5| = +l_1+l_2+l_3-l_4 =: e \\
(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) & = & (-1,-1,+1,+1) & \text{with} & |P_1-P_5| = -l_1-l_2+l_3+l_4 =: f
\end{array}
$$

For the critical point in the level surface $D^{-1}(-a^2)$ the index is 0, for b,c,d,e the index is 1 and for f the index is 2 (see Lemma 3).

The conditions $l_1 \leq \ldots \leq l_5$ implies a partial order of the set $\{a,b,c,d,e,f\}$ of critical values. In our case, this order is given by $e \leq d \leq c \leq b \leq a$ and $f \leq c$.

Now we decrease the value of D_5 to $-l_5^2$. If we pass solely through critical points of index 1, at every passage a handle is added and the genus of the surface increases by 1 (see [5]). Obviously, we can have at most four such handles.

If we pass through the critical point of index 2, we must have $l_5 < f = -l_1-l_2+l_3+l_4$. This is exactly the condition for a disconnected configuration space.

A straightforward calculation gives us the condition $d < f < c$. So we have to pass the two critical values b and c, each with a critical point of index 1, before passing f. Since the two components have to be diffeomorphic, they have to be two tori.

A further critical point of index 1 is impossible.

Finally we give an example of a length–vector for any type of configuration space: The vectors $(4,5,6,7,15)$, $(4,5,6,7,13)$, $(4,5,6,7,11)$, $(4,5,6,7,9)$, and $(4,5,6,7,7.5)$ furnish surfaces of genus $0,1,2,3,4$, the vector $(1,1,3,3,3)$ has a disconnected configuration space. $\square$

5 Further results

1. The Euler–characteristic of X_l^2

Let $l = (l_1, l_2, \ldots, l_n)$ be as above. With the presented method it is possible to compute the Euler–characteristic of X_l^2.

n even: X_l^2 is an odd–dimensional compact manifold and so χ is always 0 (see for example [2]).

n odd: X_l^2 is level surface of the function $D_n : T^{n-2} \longrightarrow \mathbf{R}$. Consider the set $M^{-l_n^2} := \{x \in T^{n-2} \ : \ D_n \leq -l_n^2\}$. Obviously X_l^2 is the boundary of $M^{-l_n^2}$.

The Euler–characteristic of $M^{-l_n^2}$ is given by

$$\chi(M^{-l_n^2}) = \sum_k (-1)^k c_k$$

where c_k is the number of critical points of index k in $M^{-l_n^2}$ (see [5]).

It is a well–known fact that $\chi(X_l^2) = 2\chi(M^{-l_n^2})$ (see for example [2]).

2. Havel's example

Using the Morse function D_n we can give another proof of Havel's Theorem concerning the configuration space of the equilateral pentagon linkage (see Introduction).

$l = (1, 1, 1, 1, 1)$. The function $D_5 = -|P_1 - P_5|^2$ has in

$$D_5^{-1}([-(l_1 + l_2 + l_3 + l_4)^2, -(l_5)^2]) = D_5^{-1}([-4^2, -1^2])$$

the following critical points: One critical point of index 0 is in the one-point level surface $D_5^{-1}(-16)$ and four critical points of index 1 are in $D_5^{-1}(-4)$. If we increase the value of D_5 up to -1^2 (i.e. if we decrease $|P_1 - P_5|$), then the single point $D_5^{-1}(-16)$ becomes first a sphere and after the crossing of the four critical points of index 1 the compact surface of genus 4. To every critical point of index 1 corresponds one handle. If several critical points of D_n lie within the same level surface, then the reconstructions can be done independently (see [1]).

Final Remarks: 1. Unfortunately, in the general case, the surgery technique doesn't give a precise description of X_l^2 because we don't know *where* the reconstruction has to be done. Even for dim $X_l^2 = 2$, different reconstructions of the same index can produce different manifolds. Nevertheless, the presented Morse function is very useful. In some cases, it's possible to decompose the configuration space in a product of lower–dimensional spheres and/or tori.

2. For polygons in $\mathbf{R}^3$, the configuration space of the chain $(l_1, l_2, \ldots, l_n)$ is no longer a manifold, so we cannot define a Morse Function in a similar way. But there is some hope that our technique will work for other types of configurations in $\mathbf{R}^3$.

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