NORTH-HOLLAND

Symmetric Magic Squares and Multivariate Splines

Rong-Qing Jia*
Department of Mathematics
University of Alberta
Edmonton, Canada T6G 2G1

Submitted by Richard A. Brualdi

ABSTRACT

We use multivariate splines to investigate linear diophantine equations and related problems in graph theory. In particular, we solve a conjecture of Stanley about symmetric magic squares. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

In this paper we give a nice application of analysis to combinatorics. Specifically, we use multivariate splines to solve the conjecture of Stanley about symmetric magic squares (see [21], [23, p. 40], [24, p. 262]).

An $m \times m$ matrix with nonnegative integer entries is called a *magic r-square of order m* if every row and column sums to $r \in \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. Let $H_m(r)$ denote the number of all magic r-squares of order m. For instance, $H_1(r) = 1$ and $H_2(r) = r + 1$. It seems that MacMahon [17, §407] first computed $H_3(r)$:

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

* Supported in part by NSERC Canada under Grant OCP 121336. E-mail: jia@xihu.math.ualberta.ca.

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opened a new way of attacking the more difficult problem of Stanley's conjecture about symmetric magic squares, which had remained unsolved for a long time by using commutative algebra.

Here is an outline of the paper. In the next section we shall describe the relationship between linear diophantine equations and discrete truncated powers. Since our intended audience might be unfamiliar with multivariate splines, we devote Sections 3, 4, and 5 to the basic theory of truncated powers and discrete truncated powers. While most of the results in these sections were known before, we often give new and straightforward proofs for them. This makes the paper almost self-contained, and I do hope that mathematicians working in the area of combinatorics will enjoy studying multivariate splines. In Section 6 and 7 we apply the theory of multivariate splines to magic labelings of graphs. The reader will find in these sections that the discrete truncated power associated with a given graph has many nice properties. These properties enable us to gain sufficient information about the number of magic labelings of a graph, so that we can solve Stanley's conjecture on symmetric magic squares in Section 8.

We shall adopt the common terminology of multiset theory (e.g., see [24, p. 10]). Intuitively, a *multiset* is a set with possible repeated elements; for instance $\{1, 1, 2, 5, 5, 5\}$. More precisely, a finite multiset M on a set S is a function $\mu : S \to \mathbb{N}$ such that $\sum_{x \in S} \mu(x)$ is finite. One regards $\mu(x)$ as the number of repetitions of x. The integer $\sum_{x \in S} \mu(x)$ is called the *cardinality* or *number of elements* of M and is denoted by $\#M$. If M' is another multiset of S corresponding to $\mu' : S \to \mathbb{N}$, then we say that M' is a submultiset of M if $\mu'(x) \leqslant \mu(x)$ for all $x \in S$. The *complement* of M' in M, denoted by $M \setminus M'$, is the multiset on S corresponding to $\mu'' : S \to \mathbb{N}$, where $\mu''(x) = \mu(x) - \mu'(x)$ for all $x \in S$. If $M' = \{y\}$, where y is an element of M, we often write $M \setminus y$ instead of $M \setminus \{y\}$. The meaning of the union and intersection of two multisets is also clear.

As usual, we denote by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the set of integers, real numbers, and complex numbers, respectively. For $i, j \in \mathbb{Z}$, we denote by δ_{ij} the Kronecker symbol; that is, $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. We use the notation $(a..b)$ to denote the interval $\{x \in \mathbb{R} : a < x < b\}$, where a is a real nubmer or $-\infty$ and b is a real number or ∞. The meaning of $[a..b)$, $(a..b]$ or $[a..b]$ is also clear.

For a positive integer m, we denote by $\mathbb{R}^m$ and $\mathbb{C}^m$ the linear space of all real and complex m-tuples, respectively. Elements of $\mathbb{R}^m$ are regarded as row or column m-vectors depending on circumstances. The linear space $\mathbb{R}^m$ is equipped with the norm $|\cdot|$ given by

$$|x| := \sum_{1 \leqslant j \leqslant m} |x_j| \qquad \text{for} \quad x = (x_1, \dots, x_m) \in \mathbb{R}^m.$$

where

$$p_j(x) = \sum_{|\alpha|=j} a_\alpha x^\alpha, \qquad x \in \mathbb{R}^m.$$

The largest j for which $p_j \neq 0$ is called the *degree* of p, and denoted by deg p. When $p = 0$, deg p is interpreted as -1. If $k = \deg p$, then p_k is called the *leading part* of p. When $m = 1$, the leading part consists of only one term; hence we may call it the *leading term* of p. Its coefficient is called the *leading coefficient* of p. Given $k \in \mathbb{Z}$, we denote by $\Pi_k = \Pi_k(\mathbb{R}^m)$ the linear space of polynomials of degree $\leqslant k$. If k is a negative integer, then we interpret Π_k as the trivial linear space $\{0\}$.

2. LINEAR DIOPHANTINE EQUATIONS

Magic squares and symmetric magic squares both are special cases of magic labelings of graphs. Further, as indicated by Stanley [20], the theory of magic labelings can be put into the more general context of linear diophantine equations. A study of linear diophantine equations naturally leads to truncated powers and discrete truncated powers.

We shall adopt the graph-theoretic terminology used in [26]. Thus a *graph* is defined to be a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set of elements called *vertices*, and $E(G)$ is a multiset of unordered pairs of (not necessarily distinct) elements of $V(G)$ called *edges*. Note that this definition of graph permits the existence of loops and multiple edges. We shall call $V = V(G)$ the *vertex set* and $E = E(G)$ the *edge multiset* of G. Two vertices v and w are said to be *adjacent* if there is an edge joining them, i.e., there is an edge of the form vw. The vertices v and w are then said to be *incident* to such an edge.

Let $r \in \mathbb{N}$. According to Stanley [20], a magic labeling of G of index r is an assignment $L: E \to \mathbb{N}$ of a nonnegative integer label to each edge of G such that for each vertex v of G the sum of the labels of all edges incident to v is r (counting each loop at v once only). We denote by $H_G(r)$ the number of magic labelings of G of index r. If G has no edge, then $H_G(r) = \delta_{0r}$. In what follows, we assume that G has at least one edge.

If G is the complete bipartite graph $K_{m,m}$, then there is a one-to-one correspondence between a magic labeling of G of index of r and a magic r-square of order m. Furthermore, if G is the graph obtained by adding one

In some simple cases, $t(\alpha|M)$ can be calculated directly. For instance, if M is the 1×1 matrix [1], then for $\alpha \in \mathbb{Z}$ we have

$$t(\alpha|M) = \begin{cases} 1 & \text{if} \quad \alpha \geqslant 0 \\ 0, & \text{if} \quad \alpha < 0. \end{cases}$$

This is the discrete counterpart of the well-known Heaviside function

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geqslant 0, \\ 0, & \text{if} \quad x < 0. \end{cases}$$

If M is the $1 \times n$ matrix $[1, 1, \dots, 1]$ $(n > 1)$, then for $\alpha \in \mathbb{Z}$ we have

$$t(\alpha|M) = \begin{cases} \begin{pmatrix} \alpha + n - 1 \\ n - 1 \end{pmatrix} & \text{if} \quad \alpha \geqslant 0, \\ 0 & \text{if} \quad \alpha < 0. \end{cases}$$

This should be compared with the truncated power $x_+^{n-1}/(n-1)!$, where

$$x_+ := \begin{cases} x, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

In general, following Dahmen and Micchelli [7], we shall call the function $t(\cdot|M)$ defined on $\mathbb{Z}^m$ by $\alpha \mapsto t(\alpha|M)$ the *discrete truncated power* associated with M. In order to understand discrete truncated powers we shall first investigate their continuous counterparts—truncated powers.

3. TRUNCATED POWERS

Multivariate truncated powers were first introduced by Dahmen [6]. Also see [7]. In this section we review some basic properties of truncated powers. Their piecewise polynomial structure is highlighted.

Let M be an $m \times n$ real matrix. Recall that M is also viewed as the multiset of its column vectors. Throughout this section we assume that the convex hull of M does not contain the origin. The truncated power $T(\cdot|M)$ associated with M is defined to be the distribution given by

$$\phi \mapsto \langle T(\cdot|M), \phi \rangle := \int_{[0..\infty)^n} \phi(Mu)\, du, \qquad \phi \in C_c^\infty(\mathbb{R}^m), \qquad (3.1)$$

THEOREM 3.1. *Let M be an $m \times n$ real matrix with* rank $M = m \leqslant n$. *Suppose the convex hull of M does not contain* 0. *Then $T(\cdot|M)$ is locally integrable and is a homogeneous function of degree $n - m$. Moreover, $T(\cdot|M)$ is continuous and positive on* cone°(M).

Proof. The theorem certainly is true if M is an $m \times m$ invertible matrix. The general case can be proved by induction on $\#M$, using the recurrence relation (3.2). ■

Truncated powers have some nice differential properties. Let D_j denote the partial derivative with respect to the jth coordinate, $j = 1, \ldots, m$. Given $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, let

$$D_y := \sum_{j=1}^{m} y_j D_j.$$

Then D_y is the directional derivative in the direction y. The following differential formula was given in [6]: For $y \in M$,

$$D_y T(\cdot|M) = T(\cdot|M \setminus y). \tag{3.3}$$

This can be easily derived from the recurrence relation (3.2). More generally,

$$D_Y T(\cdot|M) = T(\cdot|M \setminus Y), \tag{3.4}$$

where Y is a submultiset of M and

$$D_Y := \prod_{y \in Y} D_y.$$

When Y is the empty set, we interpret D_Y as the identity operator. The differentiation formula (3.4) motivates us to define two sets. The first is the set $\mathscr{Y}(M)$ consisting of those submultisets Y of M for which $M \setminus Y$ does *not* span $\mathbb{R}^m$. The second set $c(M)$ is the union of span$(M \setminus Y)$ where Y runs over $\mathscr{Y}(M)$. A connected component of cone°$(M) \setminus c(M)$, according to [11], is called a *fundamental M-cone*.

Let $D(M)$ denote the linear space of those infinitely differentiable complex-valued functions f on $\mathbb{R}^m$ which satisfy the following system of linear partial differential equations:

$$D_Y f = 0, \qquad Y \in \mathscr{Y}(M). \tag{3.5}$$

4. DISCRETE TRUNCATED POWERS

Discrete truncated powers were first introduced by Dahmen and Micchelli in [7]. In this section we review their basic properties and study their piecewise structure.

Let M be an $m \times n$ integer matrix such that $\mathrm{conv}(M)$ does not contain the origin. The discrete truncated power $t(\cdot|M)$ was defined in Section 2 as the function given by $\alpha \mapsto t(\alpha|M)$, where $\alpha \in \mathbb{Z}^m$ and $t(\alpha|M)$ is the number of solutions to the system (2.2) of linear diophantine equations. Evidently, $t(\cdot|M)$ depends only on the multiset of the column vectors of M. We also note that $t(\alpha|M) = 0$ for $\alpha \notin \mathrm{cone}(M)$. Thus a discrete truncated power is a *sequence* on $\mathbb{Z}^m$, i.e., a mapping from $\mathbb{Z}^m$ to $\mathbb{C}$. We denote by S the linear space of all sequences on $\mathbb{Z}^m$ over the field $\mathbb{C}$. Given two sequences a and b on $\mathbb{Z}^m$, their convolution $a * b$ is the sequence defined by

$$a * b(\alpha) := \sum_{\beta \in \mathbb{Z}^m} a(\alpha - \beta) b(\beta), \qquad \alpha \in \mathbb{Z}^m.$$

Let δ be the sequence on $\mathbb{Z}^m$ given by

$$\delta(\alpha) = \begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then for any $f \in S$, $f * \delta = f$. When M is the empty set, we interpret $t(\cdot|M)$ as the sequence δ. If M is the union of two multisets M_1 and M_2 of integer vectors in $\mathbb{R}^m$, then

$$t(\cdot|M) = t(\cdot|M_1) * t(\cdot|M_2).$$

This has a simple combinatorial proof as follows. Suppose $n_j = \#M_j$, $j = 1, 2$. From the very definition of $t(\cdot|M)$ we see that for $\alpha \in \mathbb{Z}^m$,

$$\begin{aligned} t(\alpha|M) &= \sum_{\mu \in \mathbb{Z}^m} \#\{\beta \in \mathbb{N}^{n_1} : M_1 \beta = \mu\} \#\{\gamma \in \mathbb{N}^{n_2} : M_2 \gamma = \alpha - \mu\} \\ &= \sum_{\mu \in \mathbb{Z}^m} t(\mu|M_1) t(\alpha - \mu|M_2). \end{aligned}$$

Given $y \in \mathbb{Z}^m$, the backward difference operator ∇_y is defined by the rule

$$\nabla_y f := f - f(\cdot - y), \qquad f \in S.$$

Note that $[\![M]\!]$ is the support of the box spline associated with M (see [4]). In what follows, for a subset Ω of $\mathbb{R}^m$, we set

$$\nu(\Omega|M) := \mathbb{Z}^m \cap \left(\Omega - [\![M]\!]\right).$$

When $\Omega = \{y\}$ we denote this set by $\nu(y|M)$. Moreover, we denote by $\mathscr{B}(M)$ the collection of all subsets of M which are bases for $\mathbb{R}^m$. The following theorem was proved in [9].

THEOREM 4.1. *For any $y \notin c(M) + \mathbb{Z}^m$ one has*

$$\dim \nabla(M) = \sum_{B \in \mathscr{B}(M)} |\det B| = \#\nu(y|M). \tag{4.5}$$

It is easily seen that $\#\nu(y|M)$ equals the volume of the zonotone $[\![M]\!]$. The volume of $[\![M]\!]$ was first computed by Shephard [19], who essentially proved the second equality in (4.5). Also, see [22]. Based on Theorem 4.1, Dahmen and Micchelli proved the following important result in [11].

THEOREM 4.2. *Let M be a multiset of integer vectors in $\mathbb{R}^m$ such that M spans $\mathbb{R}^m$ and the convex hull of M does not contain the origin. Then for any fundamental M-cone Ω, there exists a unique element $f_\Omega \in \nabla(M)$ such that f_Ω agrees with $t(\cdot|M)$ on $\nu(\Omega|M)$.*

This result has been extended by the author in [15] to the following theorem, in which Ω is only required to be a connected set. Moreover, the proof given in [15] does not rely on Theorem 4.1.

THEOREM 4.3. *Let Ω be a nonempty connected subset of $\mathbb{R}^m$, and M a multiset of integer vectors in $\mathbb{R}^m$ such that M spans $\mathbb{R}^m$ and the convex hull of M does not contain the origin. Let g be a sequence on $\nu(\Omega|M)$ satisfying the condition that for every $Y \in \mathscr{Y}(M)$,*

$$\nabla_Y g(\alpha) = 0 \qquad \text{for all} \quad \alpha \in \nu(\Omega|M \setminus Y). \tag{4.6}$$

Then there exists a unique element $f \in \nabla(M)$ such that f agrees with g on $\nu(\Omega|M)$.

Let z be an arbitrary point in Ω. Since Ω is open and connected, we can find a finite sequence of points $y_0, \dots, y_k$ satisfying the following conditions: (1) $y_0 = y$ and $y_k = z$; (2) for every $j \in \{1, \dots, k\}$, the line segment $[y_{j-1} .. y_j]$ is contained in Ω; (3) for each $j \in \{1, \dots, k\}$, $y_j - y_{j-1} = ax$ for some $a \in (-\frac{1}{2} .. \frac{1}{2})$ and $x \in M$. Then by what has been proved we conclude that g vanishes on $\nu(y_j|M)$, $j = 1, \dots, k$. This shows that g vanishes on $\nu(\Omega|M)$ and completes the proof for the case $\#X = m$.

Now let $\#X > m$. Suppose the theorem is true for any multiset M' of integer vectors with $\operatorname{span}(M') = \mathbb{R}^m$ and $\#M' < \#M$. Let $B \subset M$ be a basis for $\mathbb{R}^m$. Denote by Ω' the set $\Omega - [\![M \setminus B]\!]$. Then Ω' is also a connected set and $\nu(\Omega'|B) = \nu(\Omega|M)$.

Let $w \in B$. If $M \setminus w$ does not span $\mathbb{R}^m$, then by (4.6) we have

$$\nabla_w g(\alpha) = 0 \qquad \text{for all} \quad \alpha \in \nu(\Omega|M \setminus w) = \nu(\Omega'|B \setminus w). \quad (4.9)$$

If $M \setminus w$ spans $\mathbb{R}^m$, then (4.6) implies that for any $Y \in \mathscr{Y}(M \setminus w)$, $\nabla_w g$ satisfies the equation

$$\nabla_Y(\nabla_w g)(\alpha) = 0 \qquad \text{for all} \quad \alpha \in \nu(\Omega|M \setminus w \setminus Y).$$

Moreover, $\nabla_w g$ vanishes on $\nu(y|M \setminus w)$, since g vanishes on $\nu(y|M)$. Thus, by the induction hypothesis, $\nabla_w g$ vanishes on $\nu(\Omega|M \setminus w)$, i.e., (4.9) is also valid in this case. Furthermore, g vanishes on $\nu(y|B)$. Applying the previous argument to g and the set B, we conclude that g vanishes on $\nu(\Omega'|B) = \nu(\Omega|M)$. This completes the induction procedure.

We have thus proved that the *restriction mapping* R from $\nabla(M)$ to $S(\nu(y|M))$ given by $f \mapsto f|_{\nu(y|M)}$ is one-to-one, where $S(\nu(y|M))$ denotes the linear space of all sequences on $\nu(y|M)$. But $\dim \nabla(M) = \#\nu(y|M) = \dim S(\nu(y|M))$ by Theorem 4.1; hence the mapping R must be onto. Let g be a sequence on $\nu(\Omega|M)$ satisfying (4.6) for every $Y \in \mathscr{Y}(M)$. Then there exists a unique $f \in \nabla(M)$ such that f agrees with g on $\nu(y|M)$. The sequence $\tilde{g} := g - f$ vanishes on $\nu(y|M)$ and satisfies (4.6) for every $Y \in \mathscr{Y}(M)$. By what has been proved before, $\tilde{g}$ vanishes on $\nu(\Omega|M)$, i.e., f agrees with g on $\nu(\Omega|M)$. This is just the desired result. ■

In the applications of Theorem 4.3, the following fact is often useful.

LEMMA 4.4. *Let Ω be a nonempty open cone contained in* cone(M). *Then $\overline{\Omega}$, the closure of Ω, is contained in $\Omega - [\![M]\!]$.*

where $q := p - \theta^{-y}p(\cdot - y)$; in particular,

$$\theta^y = 1 \quad \Rightarrow \quad \nabla_y(\theta^{()}p) = \theta^{()}(\nabla_y p). \tag{5.2}$$

This motivates us to consider the set

$$M_\theta := \{y \in M : \theta^y = 1\}. \tag{5.3}$$

Let

$$A(M) := \{\theta \in (\mathbb{C} \setminus \{0\})^m : \mathrm{span}(M_\theta) = \mathbb{R}^m\}. \tag{5.4}$$

Evidently, $e \in A(M)$ and $M_e = M$. We claim that $\theta^{()}p \in \nabla(M)$ for $p \in D(M_\theta)$, where $D(M)$ was defined in Section 3 as the linear space of all solutions to the system (3.5) of partial differential equations. To see this, let $Y \in \mathcal{Y}(M)$ and $Z = Y \cap M_\theta$. Then $M_\theta \setminus Z$ does not span $\mathbb{R}^m$, because $M_\theta \setminus Z \subseteq M \setminus Y$ and $M \setminus Y$ does not span $\mathbb{R}^m$. Hence $D_Z p = 0$ for $p \in D(M_\theta)$. But p is a polynomial, $D_Z p = 0$ implies $\nabla_Z p = 0$. Thus, by (5.2) we have

$$\nabla_Y(\theta^{()}p) = \nabla_{Y \setminus Z}\nabla_Z(\theta^{()}p) = \nabla_{Y \setminus Z}(\theta^{()}\nabla_Z p) = 0.$$

This proves the "if" part of the following theorem of Dahmen and Micchelli [10].

THEOREM 5.1. *A sequence $f \in \nabla(M)$ if and only if it has the form*

$$f(\alpha) = \sum_{\theta \in A(M)} \theta^\alpha p_\theta(\alpha), \qquad \alpha \in \mathbb{Z}^m,$$

where p_θ is some polynomial in $D(M_\theta)$ for each $\theta \in A(M)$.

By computing the dimension of $\nabla(M)$ and those of $D(M_\theta)$, $\theta \in A(M)$, Dahmen and Micchelli [10] found that

$$\dim \nabla(M) = \sum_{\theta \in A(M)} \dim D(M_\theta).$$

This proves the "only if" part of Theorem 5.1. See [12, Proposition 2.2] and [16, Theorem 4.1] for some more general results concerning the kernels of linear partial difference operators with constant coefficients.

from which we conclude that $Jf_\Omega = 1/|\det M|$, which agrees with $T(\cdot|M)$ on Ω, as was shown in Section 3. This completes the proof for the case $\#M = m$.

Let $\#M > m$, and suppose the theorem is true for any M' with $\#M' < \#M$ and $\operatorname{span}(M') = \mathbb{R}^m$. Let F_Ω be the polynomial in $D(M)$ such that F_Ω agrees with $T(\cdot|M)$ on Ω. By Theorem 3.2, F_Ω is a homogeneous polynomial of degree $n - m$. Pick $w \in M$. If $M \setminus w$ does not span $\mathbb{R}^m$, then both $D_w F_\Omega$ and $\nabla_w f_\Omega$ vanish. If $M \setminus w$ spans $\mathbb{R}^m$, then $D_w F_\Omega$ agrees with $T(\cdot|M \setminus w)$ on Ω and $\nabla_w f_\Omega$ agrees $t(\cdot|M \setminus w)$ on $\nu(\Omega|M \setminus w)$. By the induction hypothesis,

$$J(\nabla_w f_\Omega) - D_w F_\Omega \in \Pi_{n-m-2}.$$

But from (5.1) we find that J and ∇_w commute with each other; hence $J(\nabla_w f_\Omega) = \nabla_w(Jf_\Omega)$. Moreover, since $Jf_\Omega \in \Pi_{n-m}$, it is easily seen that

$$D_w(Jf_\Omega) - \nabla_w(Jf_\Omega) \in \Pi_{n-m-2}.$$

We have thus shown that for every $w \in M$,

$$\begin{aligned} D_w(Jf_\Omega - F_\Omega) &= [D_w(Jf_\Omega) - \nabla_w(Jf_\Omega)] + [\nabla_w(Jf_\Omega) - D_w F_\Omega] \\ &\in \Pi_{n-m-2}. \end{aligned}$$

Since M contains a basis for $\mathbb{R}^m$, the above inclusion relation implies that

$$Jf_\Omega - F_\Omega \in \Pi_{n-m-1},$$

and therefore the leading part of Jf_Ω agrees with $T(\cdot|M)$ on Ω. ∎

6. MAGIC LABELINGS OF GRAPHS

Let G be a graph with m vertices and n edges. Given $r \in \mathbb{N}$, the number of magic labelings of G of index r is denoted by $H_G(r)$. Let M be the incidence matrix of G. We showed in Section 2 that

$$H_G(r) = t(re|M),$$

where $t(\cdot|M)$ is the discrete truncated power associated with M, and e is the m-vector whose components are all 1. In this section, we shall investigate the integer. Since G is connected, by using induction one can find vertices $v_{k+1}, \ldots, v_m$ such that v_j is adjacent to some v_i with $i < j$ for all $j = k + 1, \ldots, m$. We choose m edges of G as follows. Let $v_j v_{j+1}$ be the jth edge ($j = 1, \ldots, k-1$), $v_k v_1$ the kth edge, and choose the jth edge ($j = k + 1, \ldots, m$) to be some edge joining v_j with v_i, $i < j$. Let G' be the subgraph of G consisting of all vertices of G and the edges chosen above. Then the incidence matrix of G' has the form

$$\begin{bmatrix} N_k & K \\ 0 & Q \end{bmatrix}, \tag{6.2}$$

where N_k is the $k \times k$ matrix given in (6.1). We are in a position to prove the following theorem concerning the rank of the incidence matrix of G (cf. [13, Theorem 13.6]).

THEOREM 6.1. *Let G be a graph with m vertices, and let M be its incidence matrix. Then*

$$\operatorname{rank} M = m - b,$$

where b is the number of bipartite connected components of G.

Proof. First, let G be a connected graph which is not bipartite. Then G has a subgraph G' whose incidence matrix M' has the form (6.2) with k an odd integer. For any $j > k$, the jth column of M' has exactly two nonzero entries in rows i and j, $i < j$; hence Q is a unit upper-triangular matrix. Thus $\det Q = 1$. Since k is odd, we also have $\det N_k \neq 0$. This shows that $\operatorname{rank} M' = m$, so that $\operatorname{rank} M = m$.

Second, let G be an arbitrary graph, and let $G_1, \ldots, G_s$ be its connected components. Suppose the incidence matrices of $G, G_1, \ldots, G_s$ are $M, M_1, \ldots, M_s$, respectively. Then

$$\operatorname{rank} M = \operatorname{rank} M_1 + \cdots + \operatorname{rank} M_s. \tag{6.3}$$

If none of the components of G is bipartite, then $\operatorname{rank} M_j$ equals the number of vertices of G_j for $j = 1, \ldots, s$. This together with (6.3) implies that $\operatorname{rank} M = m$.

Third, consider the case when G is a connected bipartite graph. Then the vertex set V of G can be partitioned into two subsets V_1 and V_2 such that every edge of G joints V_1 with V_2. We arrange the vertices of G in such an

THEOREM 6.2. *Let G be a graph with m vertices, and let M be its incidence matrix. If G has no bipartite connected components, then*

$$\theta = (\theta_1, \ldots, \theta_m) \in A(M) \quad \Rightarrow \quad \theta_j = 1 \text{ or } -1 \quad \text{for all } j = 1, \ldots, m.$$

Proof. Recall that $\theta \in A(M)$ if and only if M_θ spans $\mathbb{R}^m$, where M_θ is as given in (5.3). Observe that

$$\theta^y = \begin{cases} \theta_i & \text{if} \quad y = e_i, \\ \theta_i \theta_j & \text{if} \quad y = e_{ij}. \end{cases}$$

Hence $e_i \in M_\theta$ implies $\theta_i = 1$, while $e_{ij} \in M_\theta$ implies $\theta_i \theta_j = 1$. Let G_θ be the subgraph of G which consists of all vertices of G and all the edges of G corresponding to the column vectors of M_θ. Then the incidence matrix of G_θ is M_θ. From the above discussion we see that if G_θ contains a loop around v_i, then $\theta_i = 1$, and if G_θ contain an edge joining v_i with v_j, then $\theta_i \theta_j = 1$, i.e., $\theta_j = \theta_i^{-1}$. Furthermore, if there is a path in G_θ of length k from v_i to v_j, then

$$\theta_j = \begin{cases} \theta_i & \text{if } k \text{ is even;} \\ \theta_i^{-1} & \text{if } k \text{ is odd.} \end{cases} \tag{6.4}$$

Let K be a connected component of G_θ. Since M_θ spans $\mathbb{R}^m$, K is not bipartite by Theorem 6.1; hence K contains a circuit of length k with k an odd integer. This circuit passes through a vertex, say v_i. Then by (6.4) we have $\theta_i = \theta_i^{-1}$, since k is odd. It follows that $\theta_i = 1$ or -1. Let v_j be an arbitrary vertex in K. Since K is connected, there is a path in K from v_i to v_j. By (6.4) we have $\theta_j = \theta_i$ or $\theta_j = \theta_i^{-1}$. This shows that $\theta_j = 1$ or -1 for any vertex v_j in K. Evidently, this conclusion is valid for any vertex in G_θ. ∎

As an application of Theorem 6.2, we re-prove the following result of Stanley [20] concerning magic labelings of graphs.

THEOREM 6.3. *Let G be a graph and $r \in \mathbb{N}$. Then either $H_G(r) = \delta_{0r}$, or else there exist polynomials P_G and Q_G such that*

$$H_G(r) = P_G(r) + (-1)^r Q_G(r) \qquad \textit{for all} \quad r \in \mathbb{N}.$$

Thus, in order to verify our claim it suffices to consider the case when G is connected. Let G be a nondegenerate connected bipartite graph with m vertices and n edges. Then the vertex set V of G can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 and V_2. Let m_j be the number of vertices in V_j, $j = 1, 2$. We arrange the vertices of G in such an order that any vertex in V_1 precedes any vertex in V_2. Let M be the incidence matrix of G. Since G is nondegenerate, there is a positive integer r such that the equation $M\beta = re$ has a solution $\beta \in \mathbb{N}^n$. But the sum of the first m_1 rows and the sum of the last m_2 rows of M both equal the n-vector $(1, 1, \ldots, 1)$; hence the sum of the first m_1 components and the sum of the last m_2 components of $M\beta = re$ are equal. This shows that $m_1 r = m_2 r$ and therefore $m_1 = m_2$. In particular, the number of vertices of G is even. Note that the incidence matrix M' of G' is obtained from M by removing one of its rows, say the first row. Let e' denote the $(m - 1)$-vector whose components are all 1. If $\beta \in \mathbb{N}^n$ satisfies $M\beta = re$, then $M'\beta = re'$. Conversely, if $M'\beta = re'$, then the last $m - 1$ components of $M\beta$ are all r. But the sum of the first m_1 components and the sum of the last m_2 components of $M\beta$ are equal and $m_1 = m_2$; hence the first component of $M\beta$ is also r; that is, $M\beta = re$. This shows that $H_G(r) = H_{G'}(r)$ for all $r \in \mathbb{N}$. ∎

7. POSITIVE GRAPHS

According to Stanley [20], a magic labeling L of G is called a *positive labeling* if every edge of G receives a positive label. A graph G is said to be *positive* if there is a positive labeling for G. It is easily seen that G is positive if and only if the vector e lies in $\text{cone}^\circ(M)$. If G is not positive, then there are some edges of G that are always labeled 0 in any magic labeling. After removing these edges, the resulting graph G' is positive and $H_G(r) = H_{G'}(r)$ for all $r \in \mathbb{N}$. Thus, as far as magic labelings are concerned, we may assume without loss of generality that G is a positive graph.

Let P_G and Q_G be the polynomials in Theorem 6.3. We wish to find the exact degree of P_G and Q_G. For a positive graph G, the exact degree of P_G has been determined by Stanley [20]. In this section we shall use our methods to give Stanley's result a new proof. Furthermore, we shall also establish some results about Q_G. These results are essential to our solution of Stanley's conjecture on symmetric magic squares.

In this section a multiinteger $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ is said to be *even* if $\sum_{j=1}^m \alpha_j$ is even; otherwise, α is said to be *odd*.

does not change any component of G with an odd number of vertices. On the other hand, all the new components of G' have loops. Therefore, G has a connected component with an odd number of vertices but no loops if and only if G' does. Thus, without loss of generality, we assume that G has no bipartite connected components.

Suppose G has m vertices and n edges and the vertices of G are labeled as $v_1, \ldots, v_m$. Let M be the incidence matrix of G, and let Ω be a fundamental M-cone such that $\overline{\Omega}$ contains e. By Theorems 4.2 and 4.4, there exists an element f_Ω of $\nabla(M)$ such that f_Ω agrees with the discrete truncated power $t(\cdot|M)$ on $\Omega - [\![M]\!] \supset \mathbb{R}_+ e$. Moreover, f_Ω has a decomposition of the form (6.6). Let $A_+(M)$ denote the set of those elements of $A(M)$ which have an even number of negative components, and let $A_-(M) := A(M) \setminus A_+(M)$. It follows from (6.7) that

$$P_G(r) = \sum_{\theta \in A_+(M)} p_\theta(re) \quad \text{and} \quad Q_G(r) = \sum_{\theta \in A_-(M)} p_\theta(re), \qquad (7.2)$$

where p_θ is a polynomial in $D(M_\theta)$ for each $\theta \in A(M)$.

Let $\tilde{p}_e$ denote the leading part of p_e. By Theorem 5.2, $\tilde{p}_e$ agrees with $T(\cdot|M)$ on Ω. Since G is a positive graph, $re \in \text{cone}^\circ(M)$ for all $r > 0$; hence by Theorem 3.1, $T(\cdot|M)$ is continuous at $re \in \overline{\Omega}$, $r > 0$. This shows that for $r > 0$

$$\tilde{p}_e(re) = T(re|M) = T(e|M) r^{n-m},$$

where we have used the fact that $T(\cdot|M)$ is a homogeneous function of degree $n - m$. Furthermore, $T(e|M) > 0$ by Theorem 3.1. Thus, the leading term of $p_e(re)$ is $T(e|M) r^{n-m}$.

Assume that G is connected for the time being. We claim that in this case

$$\deg p_\theta < n - m \qquad \text{for all} \quad \theta \in A(M) \setminus \{e, -e\}. \qquad (7.3)$$

Indeed, since $p_\theta \in D(M_\theta)$, by (3.6) we have $\deg p_\theta \leqslant \#M_\theta - m$; hence $\deg p_\theta < n - m$ unless $M_\theta = M$. Suppose $M_\theta = M$ for some $\theta = (\theta_1, \ldots, \theta_m) \in A(M)$. Since G is connected, for any $i, j \in \{1, \ldots, m\}$ there is a path from v_i to v_j, so it follows from (6.4) that θ_i and θ_j have the same sign. But Theorem 6.2 tells us that θ_j is either 1 or -1 for all $j = 1, \ldots, m$; hence θ is either e, or else $-e$. This verifies (7.3).

In order to determine the leading term of $P_G(r)$ we divide our investigation into three cases.

conclude from (7.2) and (7.3) that the leading term of $p_G(r)$ is $2T(e|M)r^{n-m}$ and $\deg Q_G < \deg P_G$.

Now let G be an arbitrary positive graph. Suppose $G_1, \ldots, G_s$ are the connected components of G. Then for every $j = 1, \ldots, s$, G_j is positive and hence

$$H_{G_j}(r) = P_j(r) + (-1)^r Q_j(r), \qquad r \in \mathbb{N}, \tag{7.6}$$

for some polynomials P_j and Q_j in r. It follows from (6.8) and (7.6) that

$$P_G(r) + (-1)^r Q_G(r) = \prod_{j=1}^{s} \left[P_j(r) + (-1)^r Q_j(r) \right], \qquad r \in \mathbb{N}. \tag{7.7}$$

If one of the connected components of G, say G_j, has an odd number of vertices but no loops, then by Lemma 7.1, H_{G_j} vanishes on positive odd integers, and therefore so does H_G by (6.8). Thus $P_G(r) - Q_G(r) = 0$ for all odd $r \in \mathbb{N}$. It follows that $P_G = Q_G$. Suppose otherwise that every G_j either has a loop or has an even number of vertices. Then $\deg Q_j < \deg P_j$ for $j = 1, \ldots, s$; hence by (7.7) we have $\deg Q_G < \deg P_G$. ∎

As a consequence of the above theorem, we prove Stanley's result concerning the exact degree of P_G.

THEOREM 7.3. *Let G be a positive graph with m vertices and n edges, and let P_G be the polynomial in Theorem* 6.3. *Then* $\deg P_G = n - m + b$, *where b is the number of connected components of G which are bipartite.*

Proof. For each bipartite component of G we remove one of its vertices and replace any edge incident to this vertex by a loop around the other vertex. The resulting graph G' has $m - b$ vertices and n edges but no bipartite connected components. We showed that $H_G(r) = H_{G'}(r)$ for all $r \in \mathbb{N}$ in the proof of Theorem 6.3. Thus we may assume that $b = 0$. Let $G_1, \ldots, G_s$ be the connected components of G. Suppose each G_j has m_j vertices and n_j edges. It was shown in the proof of Theorem 7.2 that $\deg P_j = n_j - m_j$ for $j = 1, \ldots, s$. Moreover, for each j, either $\deg Q_j < \deg P_j$ or $Q_j = P_j$; hence it follows from (7.7) that

$$\deg P_G = \sum_{j=1}^{s} \deg P_j = \sum_{j=1}^{s} (n_j - m_j) = n - m. \qquad ∎$$

LEMMA 8.2. *Let G_θ be the subgraph of G whose incidence matrix is M_θ. Then G_θ is positive for every $\theta \in A(M)$. Moreover, for each $\theta \in A_-(M)$, $p_\theta(re)$ is a polynomial in r of exact degree $\#M_\theta - m$ with a positive leading coefficient.*

We shall assume that Lemma 8.2 is valid and leave its proof to the end of this section. From (8.1) and Lemma 8.2 we see that the exact degree of Q_m is the maximum of $\#M_\theta - m$ when θ runs over $A_-(M)$. Thus the proof of Theorem 8.1 reduces to counting $\#M_\theta$. To this end, let θ be an element of $A(M)$ with k negative components and $m - k$ positive components, $0 \leqslant k \leqslant m$. If $1 \leqslant k \leqslant m - 1$, then G_θ has exactly two connected components: One is a complete k-graph with no loops, and the other is a complete $(m - k)$-graph with one loop attached to its every vertex. A complete 1-graph has no edges, while a complete 2-graph is bipartite; hence in both cases M_θ does not span $\mathbb{R}^m$. In other words, $\theta \in A(M)$ implies that k is neither 1 nor 2. Furthermore, since $\#M_\theta$ equals the number of edges of G_θ, we have

$$\#M_\theta = \binom{k}{2} + \binom{m-k}{2} + (m-k). \tag{8.2}$$

Evidently, (8.2) is also true for the case $k = 0$ or $k = m$. It remains to find the maximum of $\#M_\theta$ when θ runs over $A_-(M)$. For this purpose we rewrite (8.2) as follows:

$$\#M_\theta = \left(k - \frac{m+1}{2}\right)^2 - \left(\frac{m+1}{2}\right)^2 + \binom{m+1}{2}. \tag{8.3}$$

By the previous remark, $\theta \in A_-(M)$ implies that k is an odd integer $\geqslant 3$. If m is odd, then we deduce from (8.3) that

$$\#M_\theta \leqslant \left(m - \frac{m+1}{2}\right)^2 - \left(\frac{m+1}{2}\right)^2 + \binom{m+1}{2} = \binom{m}{2},$$

and equality holds if and only if $k = m$. If m is even, then the largest odd integer $\leqslant m$ is $m - 1$. It follows from (8.3) that

$$\#M_\theta \leqslant \left(m - 1 - \frac{m+1}{2}\right)^2 - \left(\frac{m+1}{2}\right)^2 + \binom{m+1}{2} = \binom{m-1}{2} + 1,$$

where $q_\xi \in D(M_\theta \cap M_\xi)$ for each $\xi \in A(M_\theta)$. This shows that

$$t(re|M_\theta) = g_\Omega(re) = P_\theta(r) + (-1)^r Q_\theta(r), \qquad r \in \mathbb{N},$$

where

$$P_\theta(r) = \sum_{\xi \in A_+(M_\theta)} q_\xi(re) \quad \text{and} \quad Q_\theta(r) = \sum_{\xi \in A_-(M_\theta)} q_\xi(re). \quad (8.5)$$

Note that $\xi \in A(M_\theta)$ if and only if $M_\theta \cap M_\xi$ spans $\mathbb{R}^m$. Evidently, $e \in A_+(M_\theta)$ and $\theta \in A_-(M_\theta)$. We claim that

$$\xi = (\xi_1, \dots, \xi_m) \in A(M_\theta) \setminus \{e, \theta\} \quad \Rightarrow \quad M_\xi \cap M_\theta \neq M_\theta. \quad (8.6)$$

Suppose to the contrary that $M_\xi \cap M_\theta = M_\theta$. Then M_ξ contains all e_{ij} $(1 \leqslant i, j \leqslant k)$; hence $\xi_1, \dots, \xi_k$ must have the same sign. Moreover, M_ξ contains all e_j $(j = k+1, \dots, m)$, so that $\xi_j = 1$ for $j = k+1, \dots, m$. Thus either $\xi = e$, or else $\xi = \theta$. This confirms our claim (8.6). Thus by (3.6) and (8.6) we have

$$\deg q_\xi \leqslant \#(M_\xi \cap M_\theta) - m < \#M_\theta - m \qquad \text{for all} \quad \xi \in A(M_\theta) \setminus \{e, \theta\}. \quad (8.7)$$

We showed in the proof of Theorem 7.2 that the leading term of $q_e(re)$ is $T(e|M_\theta)r^{\#M_\theta - m}$ with $T(e|M_\theta) > 0$. From (8.5) and (8.7) we see that $P_\theta(r)$ and $q_e(re)$ have the same leading term. But G_θ has a connected component which has an odd number of vertices but no loops, hence $Q_\theta = P_\theta$ by Theorem 7.2. Invoking (8.5) and (8.7) again, we see that $Q_\theta(r)$ and $q_\theta(re)$ have the same leading term. This shows that the leading term of $q_\theta(re)$ is also $T(e|M_\theta)r^{\#M_\theta - m}$.

Consider $\nabla_{M \setminus M_\theta} f_\Omega$ and g_Ω. They both are elements of $\nabla(M_\theta)$ and agree with $t(\cdot|M_\theta) = \nabla_{M \setminus M_\theta} t(\cdot|M)$ on $\Omega - [\![M_\theta]\!]$. Hence by Theorem 4.3, $\nabla_{M \setminus M_\theta} f_\Omega = g_\Omega$. Recall that J_ξ is the projection from E to E_ξ defined in Section 5. From (5.1) it is easily seen that J_ξ and $\nabla_{M \setminus M_\theta}$ commute with each other. In particular,

$$\theta^{()} q_\theta = J_\theta(g_\Omega) = J_\theta(\nabla_{M \setminus M_\theta} f_\Omega) = \nabla_{M \setminus M_\theta}(J_\theta f_\Omega) = \nabla_{M \setminus M_\theta}(\theta^{()} p_\theta). \quad (8.8)$$

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