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Hellow. I send you this note to your old adress since I don't know the new one. All the best

Gil

A Simple Way to Tell a Simple Polytope from its Graph

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Let P be a simple d-dimensional polytope and let $G(P)$ be the graph of P. Thus, $G(P)$ is an abstract graph defined on the set of vertices $V(P)$ of P. Two vertices v and u in $V(P)$ are adjacent in $G(P)$ if $[v,u]$ is a 1-dimensional face of P. Perles [P] conjectured and Blind and Mani [BM] recently proved that $G(P)$ determines the entire combinatorial structure of P. Here is a simple proof of this result. Let f denote the number of non-empty faces of P.

We consider the class of acyclic orientations (i.e., edge orientations with no oriented cycles) of $G(P)$. We will not distinguish between an acyclic orientation O of $G(P)$ and the partial order induced by O on $V(P)$. ($x \leq_O y$ iff there is an O-directed path from x to y.) Note that if O is an acyclic orientation of $G(P)$ then the restriction of $G(P)$ to any non-empty subset A of $V(P)$ has a sink (= element with out-degree zero) with respect to O.

An acyclic orientation O of $G(P)$ is *good* if for every non-empty face F of P, $G(F)$ has exactly one sink. Otherwise, O is *bad*. The existence of good acyclic orientations of $G(P)$ is well-known. Good acyclic orientations are obtained, e.g., by orienting the edges according to the value of a linear functional on R^d that is 1-1 on $V(P)$; see [B, Sec. 15]. Our first goal is to distinguish intrinsically between good and bad orientations of $G(P)$.

Let O be an acyclic orientation of $G(P)$. Let h_k^O be the number of vertices of $G(P)$ with in-degree k in O. Define

$$f^O = h_0^O + 2h_1^O + 4h_2^O + \cdots + 2^k h_k^O + \cdots + 2^d h_d^O .$$

If x is a vertex of $G(P)$ of in-degree k w.r.t. O then x is a sink in 2^k faces of P. (Every i edges incident to x determine an i-face F of P which includes them.) Since each face has at least one sink we obtain that

(I) $f^O \geq f$, and

(II) O is good if and only if $f^O = f$.

To distinguish between good and bad orientations from the knowledge of $G(P)$ only, compute f^O for every acyclic orientation O. The good acyclic orientations of $G(P)$ are those having the minimal value of f^O.

Now we will show how to identify the faces of P. The criterion is very simple: An induced connected k-regular subgraph H of G is the graph of some k-face of P if and only if its vertices are initial w.r.t. some good acyclic orientation O of $G(P)$. Indeed, if F is a face of P, it is well known that $V(F)$ is an initial set with respect to some good acyclic orientation: just consider a linear functional with respect to which the vertices of F lie below all other vertices. (See [B, Sec. 18].) On the other hand, let H be a connected k-regular subgraph of $G(P)$ and let O be a good acyclic orientation with respect to which $V(H)$ is an initial set. Let x be a sink of H with respect to O. There are k edges containing x in H, all oriented towards x. Therefore x is a sink in a k-face F that contains these k edges. Since the orientation O is good, x is the unique sink of F, and therefore all vertices of F are $\leq x$, with respect to O. But $V(H)$ includes the set of all vertices that are $\leq x$ with respect to O. (Remember: $V(H)$ is an initial set with respect to O.) Thus, $V(F) \subset V(H)$. Since both H and $G(F)$ are k-regular and connected, $V(F) = V(H)$ and $G(F) = H$. This completes the proof.

Remarks:

1. We do not have a practical way to distinguish between good and bad orientations. The algorithm suggested by the proof above is exponential in $|V(P)|$. We do not know of an efficient way even for computing the face numbers of P from $G(P)$.

2. It was observed already by Perles that the 2-skeleton of P determines P up to combinatorial isomorphism. His observation is based on the following fact: Let x and y be adjacent vertices in $G(P)$ and let F be the facet of P containing x but not y. Let z be a vertex adjacent to x, $z \neq y$. It is easy to identify the unique vertex w which is adjacent to z and does not belong to F. Let M be the (unique)

2-face of P containing x, y and z. Then w is the vertex adjacent to z in M, different from x. This gives a quick way to identify the facets of P, hence the entire combinatorial structure of P, from the 2-skeleton of P. Perles also observed that all induced 3-gons, 4-gons and 5-gons in $G(P)$ correspond to 2-faces of P.

3. Perles [P] proved that simplicial d-polytopes are determined by their $[d/2]$-skeleton. (Dancis [D] extended this result to a large class of simplicial manifolds.) Perles also proved that simple polytopes are determined by the incidence relations between their 1-faces and 2-faces. The proof described above can be extended to show that the combinatorial structure of a simple d-polytope is determined by the incidence relations between its i-faces and $(i+1)$-faces, whenever $i < [d/2]$. It is also possible to show that $(d-k)$-simple polytopes are determined by their k-skeleton. (P is $(d-k)$-simple if every $(k-1)$-face is included in exactly $d-k+1$ facets.) Details will appear elsewhere. (Note that general d-polytopes are determined by their $(d-2)$-skeleton, and this is best possible even for quasi-simplicial polytopes, [G, Ch. 12].)

4. Perles asked whether every connected $(d-1)$-regular subgraph of $G(P)$ which does not separate $G(P)$ is the graph of a facet of P. This is still unknown.

5. I am thankful to Micha A. Perles and Zeev Smilanski for helpful comments.

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