

CHOW QUOTIENTS OF GRASSMANNIANS I.

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References.

Introduction.

The study of the action of the maximal torus $H \subset GL(n)$ on the Grassmann variety $G(k, n)$ is connected with numerous questions of geometry and analysis. Among these let us mention the general theory of hypergeometric functions [18], K-theory [7], combinatorial constructions of characteristic classes [17,20,40]. It was noted by I.M.Gelfand and R.W.MacPherson [20,40] that this problem is equivalent to the classical problem of study the projective equivalence classes of configurations of n points in the projective space P^{k-1} .

In the present paper we propose a geometric approach to this problem which is based on the study of the behavior of orbit closures. Namely, the closures of generic orbits are compact varieties "of the same type". Now there is a beautiful construction in algebraic geometry — that of Chow varieties [50]. It produces compact varieties whose points parametrize algebraic cycles (= positive integral combinations of irreducible subvarieties) in a given variety with given dimension and degree. In particular, any one-parameter family of subvarieties "of the same type" has a limit in the Chow variety. We define the *Chow quotient* $G(k, n)//H$ to be the space of such limits of closures of generic orbits. Any point of $G(k, n)//H$ represents an $(n-1)$ -dimensional family of $(k-1)$ -dimensional projective subspaces in P^{n-1} .

Families of subvarieties in a variety X whose parameter space has the same dimension as X are classically known as *complexes*. We call closures of generic orbits in $G(k, n)$ Lie complexes and their limit positions- generalized Lie complexes. In the simplest case of $G(2, 4)$ Lie complexes are the so-called tetrahedral complexes of lines in P^3 which have a long history (see the bibliography in [20]).

The variety $G(k, n)//H$ can be defined in two more ways:

- a) As the space of limits of closures of generic orbits of the group $GL(k)$ in the Cartesian power $(P^{k-1})^n$ (Theorem 2.2.4).
- b) As the space of limits of special Veronese varieties in the Grassmannian $G(k-1, n-1) = G(k-1, \mathbf{h})$, where \mathbf{h} is the Lie algebra of the torus H (Theorem 3.3.14).

According to interpretation a), the space $G(k, n)//H$ is obtained by adding some "ideal" elements to the space of projective equivalence classes of configurations of n points (or hyperplanes) in P^{k-1} in general position. These ideal elements are more subtle than just non-general configurations: a limit position of closures of generic orbits can be the union of several orbit closures, each of which represents some configuration.

In fact, it turns out that these elements behave in many respects as if they actually were configurations in general position. In particular, there are restriction and projection maps (Theorem 1.6.6)

$$G(k, n-1)/(C^*)^{n-1} \xleftarrow{\tilde{b}_i} G(k, n)/(C^*)^n \xrightarrow{\tilde{a}_i} G(k-1, n-1)/(C^*)^{n-1}$$

The map \tilde{a}_i corresponds to the restriction of the generic hyperplane configuration (M_1, \dots, M_n) to the hyperplane M_i . The map \tilde{b}_i corresponds to deleting M_i . The maps \tilde{a}_i, \tilde{b}_i (regarded on the generic part of Grassmannian) were at the origin of the use of Grassmannians in the problem of combinatorial calculation of Pontryagin classes [20,40]. These maps were also used in [7] to define the so-called Grassmannian complexes serving as an approximation to K-theory. Note that \tilde{a}_i cannot be, in general, extended to arbitrary configurations. It is the Chow quotient approach that permits this extension.

Veronese varieties (which make their appearance in the interpretation b) above) are defined classically as images of a projective space in the projective embedding given by all homogeneous polynomials of given degree. In particular, Veronese curves are just rational normal curves and possess a lot of remarkable geometric properties, see [46]. In our situation Veronese varieties in fact lie on a Grassmannian (in Plücker embedding): a $(k-1)$ -dimensional variety lies in $G(k-1, \mathbf{h})$ so that for $k=2$ we obtain the curve in a projective space.

The Veronese variety associated to a Lie complex Z is obtained as its *visible contour* i.e. the locus of subspaces from Z which contain a given generic point p , say $p = (1 : \dots : 1) \in P^{n-1}$. The consideration of visible contours is a classical method of analyzing complexes of subspaces [24,27] which, to our knowledge, has yet never been applied to Lie complexes.

Along with the visible contour of Z lying in $G(k-1, \mathbf{h})$ we consider the so-called *visible sweep* $Sw(Z)$. This is a subvariety in the projective space $P(\mathbf{h})$ which is the union of all subspaces from the visible contour. This variety can be found very explicitly. Thus, if our Lie complex has the form $Z = \overline{H} \cdot \overline{L}$ where $L \in G(k, n)$ is the graph of a linear operator $A = ||a_{ij}|| : \mathbb{C}^k \rightarrow \mathbb{C}^{n-k}$, $k \leq n-k$ (this assumption does not restrict the generality) then the sweep is the projectivization of the following determinantal cone:

$$\{(t_1, \dots, t_n) \in \mathbf{h} : \text{rank } ||a_{ij}(t_i - t_j)||_{i=1, \dots, k, j=1, \dots, n-k} \leq k\}.$$

The linear forms $t_i - t_j$ entering the matrix above are roots of \mathbf{h} , the Cartan subalgebra of $\mathfrak{pgl}(n)$. More precisely, we encounter those roots which enter into the weight decomposition of the parabolic subalgebra defining the Grassmannian.

Veronese varieties in Grassmannian which arise naturally in our constructions, seem to be "right" generalizations of Veronese curves in projective spaces. We show in §3.5 that these varieties admit a Steiner-type construction, which is well known for curves [24].

The homology class in the Grassmannian of such a variety is given by an extremely beautiful formula (Theorem 3.9.8). To state it, recall that the homology of Grassmannian is freely generated by Schubert cycle σ_α which correspond to Young diagrams. It turns out that the multiplicity of the cycle σ_α in the class of Veronese variety equals the dimension of the space $\Sigma^{\alpha^*}(\mathbb{C}^{n-k})$, the irreducible representation of the group $GL(n-k)$ corresponding to the Young diagram dual to α .

In the particular case $k=2$ the construction b) realizes points of our variety as limit positions of rational normal curves (Veronese curves, for short) in P^{n-2} through a fixed set of n points in general position. We deduce from that that $G(2, n)/H$ is isomorphic to the Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$ of stable n -pointed curves of genus 0 (Theorem 4.1.8). This is certainly the most natural compactification of the space $M_{0,n}$ of projective equivalence classes of n -tuples of distinct points on P^1 . It is smooth and the complement to $\overline{M}_{0,n}$ is a divisor with normal crossings.

In general, Veronese varieties in Grassmannians arising in construction b) are Grassmannian embeddings of P^{k-1} corresponding to vector bundles on of the form $\Omega^1(\log M)$, where $M = (M_1, \dots, M_n)$ is a configuration of hyperplanes in general position. They become Veronese varieties after the Plücker embedding of the Grassmannian. The configuration of hyperplanes on P^{k-1} can be read off the corresponding Veronese variety by intersecting it with natural sub-Grassmannians. This is explained in §3.

The Chow quotients of toric varieties by the action of a subtorus of the defining torus were studied in [30,31]. It was found that this provides a natural setting for the theory of secondary polytopes introduced in [22,23] and their generalizations - fiber polytopes [8]. By considering the Plücker embedding of Grassmannian we can apply the results of [30,31]. This gives a description of possible degenerations of orbit closures in $G(k, n)$ (i.e. of generalized Lie complexes) in terms of polyhedral decomposition of a certain polytope $\Delta(k, n)$ called the hypersimplex [17,19,20]. For the case of $G(2, n)$ these decompositions are in bijection with trees which describe combinatorics of stable curves.

It is now well-known [19,21,40] that various types of closures of torus orbits in $G(k, n)$ correspond to *matroids* i.e. types of combinatorial behavior of a configurations of n hyperplanes in projective space. To each matroid the corresponding *stratum* in $G(k, n)$ is associated. It is formed by orbits of the given type [18,19,21]. Matroids are in one-to-one correspondence with certain polytopes in the hypersimplex [19,21]. From our point of view, however, a natural object is not an individual matroid but a collection of matroids such that the corresponding polytopes form a polyhedral decomposition of hypersimplex. We call such collection *matroid decompositions*.

It should be said that our approach differs considerably from that of geometric invariant theory developed by D.Mumford [42]. In particular, Mumford's quotients of $G(2, n)$ by H and of $(P^1)^n$ by $GL(2)$ though isomorphic to each other, do not coincide with the space $\overline{M}_{0,n}$ which provides a finer compactification. Note that Mumford's quotient depends upon a choice of a projective embedding (and this is felt for varieties like $(P^{k-1})^n$ with large Picard group) and upon the choice of linearization i.e. the extension of the action to the graded coordinate ring of the embedding (and this is felt for groups like the torus). We prove in §0 that the Chow quotient always maps to any Mumford quotient by a regular birational map.

Instead of Chow variety one can use Hilbert schemes and obtain a different compactification. Such a construction was considered in 1985 by A.Bialynicki-Birula and A.J.Sommese [10] and later by Y.Hu [26].

The advantage of Hilbert schemes is that they represent an easily described functor so it is easy to construct morphisms into them. In our particular example of the torus action on $G(k, n)$ both constructions lead to the same answer.

In the forthcoming second part of this paper we shall study the degenerations of Veronese varieties which provide a higher-dimensional analog of stable curves of

Grothendieck and Knudsen. The main idea of stable pointed curves is that points are never allowed to coincide. When they try to do so, the topology of the curve changes in such a way that the points remain distinct. In higher dimensions instead of a collection of points we have a divisor on a variety. The analog of the condition that points are distinct is that the divisor has normal crossings. In particular, a generic configuration of hyperplanes defines a divisor with normal crossings. When the hyperplanes "try" to intersect non-normally, the corresponding Veronese variety degenerates in such a way as to preserve the normal crossing.

I am grateful to W.Fulton who suggested that the space $\overline{M}_{0,n}$ might be related to Chow quotients and informed me on his joint work with R.MacPherson [15] on a related subject. I am also grateful to Y.Hu for informing me about his work [26] and about earlier work of A.Bialynicki-Birula and A.J.Sommese [10].

I am happy to be able to dedicate this paper to Izrail Moiseevich Gelfand.

Chapter 0. CHOW QUOTIENTS.

(0.1) Chow varieties and Chow quotients.

Let H be an algebraic group acting on a complex projective variety X . We shall describe in this section an approach to constructing of the algebraic "coset space" of X by H which was introduced in [31]. (A similar approach was introduced earlier in [10], see §0.5 below).

(0.1.1) Setup of the approach. For any point $x \in X$ we consider the orbit closure $\overline{H.x}$ which is a compact subvariety in X . For some sufficiently small Zariski open subset $U \subset X$ of "generic" points all these varieties have the same dimension, say, r and represent the same homology class $\delta \in H_{2r}(X, \mathbf{Z})$. The set U may be supposed H -invariant. Moreover, since we are free to delete bad orbits from U , the construction of the quotient U/H presents no difficulty and the problem is to construct a "right" compactification of U/H . A natural approach to this is to study the limit positions of the varieties $\overline{H.x}$ when x tends to the infinity of U (i.e. ceases to be generic). One of the precise ways to speak about such "limits" is provided by Chow varieties of algebraic cycles. Before proceeding further we recall the main definitions (cf. [30,50]).

(0.1.2) By a (positive) r -dimensional algebraic cycle on X we shall understand a finite formal non-negative integral combination $Z = \sum c_i Z_i$ where $c_i \in \mathbf{Z}_+$ and Z_i are irreducible r -dimensional closed algebraic subvarieties in X . Denote by $\mathcal{C}_r(X, \delta)$ the set of all r -dimensional algebraic cycles in X which have the homology class δ . It is known that $\mathcal{C}_r(X, \delta)$ is canonically equipped with a structure of a projective (in particular, compact) algebraic variety (called the Chow variety). In this form this result is due to D.Barlet [3].

(0.1.3) A more classical approach to Chow varieties is that of Chow forms [50]. This approach first gives the projective embedding of $\mathcal{C}_r(P(V), d)$, the Chow variety of r -dimensional cycles of degree d in the projective space $P(V)$. The Chow form of any cycle Z , $\dim(Z) = r$, $\deg(Z) = d$, is a polynomial $R_Z(l_0, \dots, l_r)$ in the coefficients of $r+1$ indeterminate linear forms $l_i \in V^*$ which is defined, up to a constant factor, by the following properties, see [30,50]:

(0.1.3.1) $R_{Z+W} = R_Z \cdot R_W$.

(0.1.3.2) If Z is an irreducible subvariety then R_Z is an irreducible polynomial which vanishes for given l_0, \dots, l_r if and only if the projective subspace $\Pi(l_0, \dots, l_r) = \{l_0 = \dots = l_r = 0\}$ of codimension $r+1$ intersects Z .

(0.1.4) It is now classical [50] that the correspondence $Z \mapsto R_Z$ identifies $\mathcal{C}_r(P(V), d)$ with a Zariski closed subset of the projective space of polynomials $F(l_0, \dots, l_r)$ homogeneous of degree d in each l_i .

(0.1.5) If $X \subset P(V)$ is any projective subvariety and $\delta \in H_{2r}(X, \mathbf{Z})$, then $\mathcal{C}_r(X, \delta)$ becomes a subset of $\mathcal{C}_r(P(V), d)$ where $d \in H_{2r}(P(V), \mathbf{Z}) = \mathbf{Z}$ is the image of δ . The result of Barlet mentioned in (0.1.2) shows, in particular, that (over a field of complex numbers) this subset is Zariski closed and the resulting structure of algebraic variety on $\mathcal{C}_r(X, \delta)$ does not depend on the projective embedding. (The fact that for a given projective subvariety X the set of $Z \in \mathcal{C}_r(P(V), d)$ lying on X is Zariski closed, is classical).

In the case of base field of characteristic p , which we do not consider here, the situation is more subtle, see [43,44].

(0.1.6) Let us return to the situation (0.1.1) of the group H acting on X . We see that for $x \in U$ as in (0.1.1) the subvariety $\overline{H.x}$ is a point of the variety $\mathcal{C}_r(X, \delta)$. The correspondence $x \mapsto \overline{H.x}$ defines therefore an embedding of the quotient variety U/H into $\mathcal{C}_r(X, \delta)$.

(0.1.7) **Definition.** [31] *The Chow quotient $X//H$ is the closure of U/H in $\mathcal{C}_r(X, \delta)$.*

Thus $X//H$ is a projective algebraic variety compactifying U/H . "Infinite" points of $X//H$ are some algebraic cycles in X which are limits (or "degenerations") of generic orbit closures.

(0.1.8) **Remarks.** a) Definition (0.1.7) does not depend on the freedom in the choice of U since deletion from U of orbits which are already "generic" results in their reappearance as points in the closure.

b) The notion of "genericity" used in Definition (0.1.7) is usually much more restrictive than Mumford's notion of stability [42]. In fact (0.1.7) makes no appeal to stability and is defined entirely in terms of X and the action of H .

(0.2) Torus action on a projective space: secondary polytopes.

If H is an algebraic torus acting on a projective variety X , then X may be equivariantly embedded into a projective space with H -action. The case of torus action on a projective space recalled in this subsection will be basic for our study of more general torus actions in this paper.

(0.2.1) Let H be an algebraic torus $(\mathbf{C}^*)^k$. A character of H is the same as a Laurent monomial $t^\omega = t_1^{\omega_1} \dots t_k^{\omega_k}$ where $\omega = (\omega^{(1)}, \dots, \omega^{(k)}) \in \mathbf{Z}^k$ is an integer vector. A collection $A = \{\omega_1, \dots, \omega_N\}$ of vectors from \mathbf{Z}^k defines therefore a diagonal homomorphism from H to $GL(N)$. It is well-known that any representation of the torus can be brought into a diagonal form.

(0.2.2) The homomorphism $H \rightarrow GL(N)$ constructed from the set A above defines an H -action on the projective space P^{N-1} . The homogeneous coordinates in P^{N-1} are naturally labelled by elements of A . So we shall denote this space by $P(A)$ and the coordinates by $(x_\omega)_{\omega \in A}$ thus dropping the (unnatural) numeration of ω 's.

The Chow quotient $P(A)//H$ was described in [30,31]. We shall use this description so we recall it here.

(0.2.3) First of all, $P(A)//H$ is a projective toric variety.

To see this, we note that the "big" torus $(\mathbf{C}^*)^A$ acts on $P(A)$ (by dilation of homogeneous coordinates) commuting with H (which is just a subtorus of $(\mathbf{C}^*)^A$). Therefore $P(A)//H$ is the closure, in the Chow variety, of the $(\mathbf{C}^*)^A$ -orbit of the variety $X_A = \overline{H \cdot x}$, where $x \in P(A)$ is the point with all coordinates equal to 1.

(0.2.5) It is known that projective toric varieties are classified by lattice polytopes, see [49]. In what follows we shall describe the polytope corresponding to the toric variety $P(A)//H$.

(0.2.6) Let $Q \subset \mathbf{R}^k$ be the convex hull of the set A . A *triangulation* of the pair (Q, A) is a collection of simplices in Q whose vertices lie in A intersecting only along common faces and covering Q . To any such triangulation T we associate its *characteristic function* $\phi_T : A \rightarrow \mathbf{Z}$ as follows. By definition, the value of ϕ_T on $\omega \in A$ is the sum of volumes of all simplices of Q for which ω is a vertex. The volume form is normalized by the condition that the smallest possible volume of a lattice simplex equals 1.

(0.2.7) The *secondary polytope* $\Sigma(A)$ is, by definition, the convex hull of all characteristic functions ϕ_T in the space \mathbf{R}^A .

Secondary polytopes were introduced in [22,23] in connection with Newton polytopes of multi-dimensional discriminants. It was shown in [22] that the vertices of $\Sigma(A)$ are precisely functions ϕ_T where the triangulation T is *regular* i.e. possesses a strictly convex piecewise-linear function.

(0.2.8) Theorem. [30] *The toric variety $P(A)//H$ corresponds to the convex lattice polytope $\Sigma(A)$.*

(0.2.9) Complements. All the faces of the secondary polytope $\Sigma(A)$ possess a complete description. We shall use in this paper only the case when elements of A are exactly vertices of Q so we shall restrict ourselves to this case, see [22] for general case. Let us call a *polyhedral decomposition* of Q a collection of convex polytopes in Q whose vertices lie in A , which intersect only along common faces and cover Q . A polyhedral decomposition \mathcal{D} is called *regular*, if it possesses a strictly convex piecewise-linear function. It was shown in [22] that vertices of $\Sigma(A)$ are in bijection with regular polyhedral decompositions of Q . Vertices of the face of $\Sigma(A)$ corresponding to such a decomposition \mathcal{D} are precisely functions ϕ_T for all regular triangulations T refining \mathcal{D} .

(0.2.10) It was shown in [31] that any cycle from $P(A)//H$ is a sum of toric subvarieties (closures of H

-orbits), see [31], Proposition 1.1. In particular, a regular triangulation T represents a 0-dimensional torus orbit in $P(A)//H$ i.e. some algebraic cycle. This cycle has the form $\sum_{\sigma \in T} \text{Vol}(\sigma) \cdot \mathbf{L}(\sigma)$ where $\mathbf{L}(\sigma)$ is the coordinate k -dimensional projective subspace in $P(A)$ spanned by basis vectors corresponding to vertices of σ .

(0.3) Structure of cycles from Chow quotient.

(0.3.1) Theorem. *Let H be a reductive group acting on a smooth projective variety X . Suppose that the stationary subgroups $H_x, x \in X$ are trivial for generic x and are never unipotent. Then any component Z_i of any cycle $Z = \sum c_i Z_i \in X//H$ is a closure of a single H -orbit.*

Proof: For the case when H is a torus, this statement follows from results of [30,31]. Indeed, we can take an equivariant embedding of X into a projective space P^N with H -action in such a way that the dimension of a generic H -orbit on X is the same as the dimension of a generic H -orbit on P^N . The degeneration of torus orbits on a projective space (and, more generally, on toric varieties) was studied in [30,31] where it was found that any orbit degenerates in a union of finitely many orbits ([31], Proposition 1.1).

Consider now the general case. Let $C(t), t \neq 0$, be a 1-parameter family of closures of generic orbits, $C(0) = \lim_{t \rightarrow 0} C(t)$ is their limit in the Chow variety. Let C be any component of $C(0)$. Suppose, contrary to our statement, that C is not a closure of a single orbit. Then, for all points of C the stabilizer H_x has positive dimension. By our assumption, these stabilizers are all non-unipotent. Let x be some fixed generic point of C . Then H_x contains some torus T . Include x in a 1-parameter family of points $x(t) \in C(t)$ such that for $t \neq 0$ the point $x(t)$ lies in the orbit open in $C(t)$. Consider the closures of orbits $\overline{T \cdot x(t)} \subset C(t)$. When $t \rightarrow 0$ these closures should degenerate into some cycle Z whose support contains x . But we know that each component of Z is a closure of one T -orbit and so x should lie on the intersection of components of Z . This means that each generic point $x \in C$ lies in the closure of some orbit $H \cdot y$ not coinciding with $H \cdot x$. This is impossible.

(0.3.2) Example. The assumption that the stabilizers of points are never unipotent in the formulation of Theorem 0.3.1 can not be dropped. To construct an example, consider the group $H = SL(2, \mathbb{C})$. It has a standard action on \mathbb{C}^2 . Let X_0 be the product $\mathbb{C}^2 \times \mathbb{C}^2$ on which H acts diagonally. Let $X = P^2 \times P^2$ be the natural compactification of X_0 with the obviously extended action of $H = SL(2)$. Generic H -orbits on X are 3-dimensional: two pairs of independent vectors (e_1, e_2) and (f_1, f_2) can be brought to each other by a unique transformations from $SL(2)$ if and only if $\det(e_1, e_2) = \det(f_1, f_2)$. Thus a generic orbit depends on one parameter namely $\det(e_1, e_2)$. However, when this parameter approaches 0, the orbit degenerates into the 3-dimensional variety of proportional pairs (e_1, e_2) . This variety is a union of a 1-parametric family of 2-dimensional orbits $O_\lambda = \{(e_1, e_2) : e_1 = \lambda e_2\}$. The stabilizer of each of this orbit is a unipotent subgroup in $H = SL(2)$.

(0.4) Relation to Mumford quotients.

It is useful to have a comparison of the Chow quotient with the more standard constructions, namely Mumford's geometric invariant theory quotients [42].

(0.4.1) To define the Mumford's quotient, we should choose an H -equivariant projective embedding of X and extend the H -action to the homogeneous coordinate ring $\mathbb{C}[X]$ of X with respect to this embedding. This is equivalent to extending the action to the ample line bundle \mathcal{L} defining the embedding. Such an extension is called *linearization*. Denote the chosen linearization by α . The Mumford's quotient $(X/H)_\alpha$ or $(X/H)_{\mathcal{L}, \alpha}$ corresponding to \mathcal{L}, α is defined as $\text{Proj} \mathbb{C}[X]^H$, the projective spectrum of the invariant subring [42]. Thus there are two choices in the definition of Mumford quotient: that of an ample line bundle and that of extension of the action to the chosen line bundle.

(0.4.2) By general theory of [42], points of $(X/H)_{\mathcal{L}, \alpha}$ are equivalence classes of α -semistable orbits in X .

More precisely, two semistable orbits O, O' are equivalent if any invariant homogeneous function vanishing on O does so on O' and conversely. A Zariski open set in $(X/H)_{\mathcal{L},\alpha}$ is formed by α -stable orbits [42]. They have the property that no two α -stable orbits are equivalent. We shall say that the linearization α is non-degenerate if there are α -stable orbits.

The following result was proven in [31] for the case of a torus acting on a toric variety and, (independently and simultaneously) in [26] for torus action on an arbitrary variety.

(0.4.3) Theorem. *Let H be a reductive group acting on a projective variety X , \mathcal{L} - an ample line bundle on X and α be a linearization i.e. an extension of the H -action on X to \mathcal{L} . Suppose that α is non-degenerate. Then there is a regular birational morphism $p_\alpha : X//H \rightarrow (X/H)_{\mathcal{L},\alpha}$.*

For any algebraic cycle $Z = \sum n_i Z_i$ in X we shall call its support and denote $\text{supp}(Z)$ the union $\cup Z_i$. The proof of Theorem 0.4.3 consists of three steps:

(0.4.4) For any cycle $Z \in X//H$ as above there is at least one orbit in $\text{supp}(Z)$ which is α -semistable.

(0.4.5) All the α -semistable orbits in $\text{supp}(Z)$ are equivalent i.e. represent the same point of the Mumford quotient $(X/H)_{\mathcal{L},\alpha}$.

(0.4.6) The map $p_\alpha : X//H \rightarrow (X/H)_{\mathcal{L},\alpha}$ which takes $Z \in X//H$ to the point of $(X/H)_{\mathcal{L},\alpha}$ represented by any of the semistable orbit in $\text{supp}(Z)$, is a morphism of algebraic varieties.

(0.4.7) Proof of (0.4.4): We use the interpretation of semistability via the moment map [1,35]. Let H_c be the compact real form of H with Lie algebra \mathcal{H} and $\mu : X \rightarrow \mathcal{H}^*$ be the moment map associated to an H_c -invariant Kähler form on X . Then an orbit O is semistable if and only if $\mu(\bar{O})$ contains zero element of \mathcal{H}^* . Let $O(t), t \neq 0$, be a 1-parameter family of generic orbits and $Z(t)$ be the closure of $O(t)$. Let $Z(0)$ be the limit of $Z(t)$ is the Chow variety. Since $\mu(Z(t))$ is, for $t \neq 0$, a closed set containing 0, the set $\mu(Z(0))$ also contains 0 thus proving that at least one orbit constituting $Z(0)$, is semistable.

(0.4.8) Proof of (0.4.5): Denote by \mathcal{L} the equivariant ample invertible sheaf given by the linearization α . By definition, two semistable orbits O and O' are equivalent in $(X/H)_{\mathcal{L},\alpha}$ if any section of $\mathcal{L}^{\otimes k}$ vanishing at O , does so at O' . But the cycle Z is a limit position of closures of single orbits. So our assertion follows by continuity.

(0.4.9) Proof of (0.4.6): Let $X \subset P(V)$ be the equivariant projective embedding given by the linearization. We shall use the approach to Chow variety via Chow forms, see (0.1.3). Let d be the degree of a generic orbit closure $\bar{H}.x, x \in X$. Recall that the Chow form of any cycle $Z, \dim(Z) = r, \deg(Z) = d$, is a polynomial $R_Z(l_0, \dots, l_r)$ in the coefficients of $r+1$ indeterminate linear forms $l_i \in V^*$.

(0.4.9.1) Since the property of being a morphism is local, it suffices to prove it in a suitable open covering of $X//H$. More precisely, we are reduced to the following situation.

(0.4.9.2) Let f be an invariant rational function on V homogeneous of degree 0 (so it represents a regular function on some open set of $(X/H)_{\mathcal{L},\alpha}$). We must express the (constant) value of f on a generic orbit O as a rational function of the coefficients of the Chow form R_Z , where $Z = \bar{O}$.

(0.4.9.3) We can write (for characteristic 0 only!)

$$f|_Z = \frac{1}{d} \sum_{\mathbf{x} \in Z \cap L} f(\mathbf{x}),$$

where L is a generic projective subspace in $P(V)$ of codimension r . On the other hand, let l_1, \dots, l_r be equations of L . Then we have the equality of polynomials in $l \in V^*$:

$$(0.4.9.4) \quad R_Z(l, l_1, \dots, l_r) = c \cdot \prod_{\mathbf{x} \in Z \cap L} l(\mathbf{x})$$

where c is a non-zero number depending on l_1, \dots, l_r .

(0.4.9.4) Let $V = \mathbf{C}^{N+1}$ with coordinates x_0, \dots, x_N and let ξ_i be dual coordinates in V^* , so an indeterminate linear form on V is $(\xi, \mathbf{x}) = \sum \xi_i x_i$ for some ξ_0, \dots, ξ_N . The Chow form of any 0-cycle $W = \mathbf{x}^{(1)} + \dots + \mathbf{x}^{(d)}$ is the polynomial $\prod (\mathbf{x}^{(i)}, \xi)$. Let us restrict the considerations to the affine chart, say, $\mathbf{C}^N = \{x_0 \neq 0\}$ in $P(V) = P^N$. The coordinate x_0 can then set to be 1 and we can set $\xi_0 = 1$ as well thus obtaining the Chow form of a 0-cycle $W \subset A^N$ as before in the form

$$\Phi_W(\xi) = \prod (1 + x_1^{(i)} \xi_1 + \dots + x_N^{(i)} \xi_N).$$

The coefficients of this polynomial at various monomials in ξ 's are known as elementary symmetric functions in d vector variables $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(d)}$, see [28,39]. By formula (0.1) elementary symmetric functions of the d points of intersection $Z \cap L$, for any generic L of codimension r , can be polynomially expressed through the coefficients of R_Z . Therefore we are reduced to the following lemma.

(0.4.9.5) Lemma. *Let $f(\mathbf{x}), \mathbf{x} = (x_1, \dots, x_N)$ be a rational function in N variables and $d > 0$. Then there is a rational function $U_f = U_f(\Phi)$ in the coefficients of an indeterminate homogeneous polynomial $\Phi(\xi_1, \dots, \xi_N)$, $\deg(\Phi) = d$ satisfying the following property. If $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(d)}$ are points not lying on the polar locus of f then*

$$\sum f(\mathbf{x}^{(i)}) = U_f(\Phi_W)$$

where $\Phi_W = \prod (1 + (\mathbf{x}^{(i)}, \xi))$.

Proof: It is known since P.A. MacMahon [28,39] that any symmetric polynomial in $\mathbf{x}^{(i)}$ (in characteristic 0) can be polynomially expressed via elementary symmetric polynomials (in many different ways, if $N > 1$). If $f(\mathbf{x}) = P(\mathbf{x})/Q(\mathbf{x})$, where P, Q are relatively prime polynomials, then

$$\sum f(\mathbf{x}^{(i)}) = \frac{1}{\prod Q(\mathbf{x}^{(i)})} \sum_i P(\mathbf{x}^{(i)}) \prod_{j \neq i} Q(\mathbf{x}^{(j)})$$

is a ratio of two symmetric polynomials and the assertion follows.

The proof of Theorem 0.4.3 is finished.

(0.4.10) Remark. In [31] it was shown that for the case of torus action on a toric variety the Chow quotient $X//H$ is, in some sense, the "least common multiple" of all Mumford's quotients corresponding to different linearizations. From the point of view of general reductive groups a more typical case is when the group has 0-dimensional center and hence there is only one linearization. However, we shall see that in this case Chow quotient still differs drastically from the Mumford one. The reason for this, as we would like to suggest, is that the Chow quotient takes into account not only Mumford quotients corresponding to various linearizations, but also more general symplectic quotients [1] corresponding to coadjoint orbits of H_c . For a torus, a different choice of a coadjoint orbit amounts to a change of a linearization, see [31] so all the symplectic quotients are reduced to Mumford's ones. In general case the symplectic quotients corresponding to non-zero orbits may not have an immediate algebro-geometric interpretation [1]. Nevertheless, their presence is somehow felt in $X//H$.

(0.5) Hilbert quotients (the Bialynicki-Birula-Sommese construction).

A different way of speaking about limit positions of generic orbit closures is that of Hilbert schemes. Such a construction was considered by A. Bialynicki - Birula and A.J. Sommese [10] and later by Y. Hu [26].

(0.5.1) Recall [25,47] that for any projective variety X there is the *Hilbert scheme* \mathcal{H}_X parametrizing all subschemes in X . By definition, a morphism $S \rightarrow \mathcal{H}_X$ is a flat family of subschemes in X parametrized by S . The scheme \mathcal{H}_X is of infinite type since no bound on "degrees" of subschemes is imposed. The connected components of \mathcal{H}_X are, nevertheless, finite-dimensional projective schemes.

(0.5.2) Any connected component of the scheme \mathcal{H}_X is canonically mapped into the Chow variety "corresponding" to this component. More precisely, if K is any such connected component then dimensions of subschemes from K are the same and equal, say, r . For any scheme $Z \in K$ we define the algebraic cycle

$$(0.5.3) \quad Cyc(Z) = \sum_{C \subset supp(Z) - \text{irred.}, dim(C)=r} Mult_C(Z).C$$

where C runs over all r -dimensional irreducible components of the algebraic variety $supp(Z)$ and $Mult_C Z$ is the multiplicity given by the scheme structure, see [30,42].

(0.5.4) In the situation of (0.5.2) it follows from results of [42] that the cycles $Cyc(Z)$ for all subschemes $Z \subset K$ have the same homology class, say δ_K and the formula (0.5.3) defines a regular morphism $K \rightarrow \mathcal{C}_r(X, \delta_K)$, see [42].

(0.5.5) Hilbert schemes have the advantage over Chow varieties in that they are defined as objects representing an easily described functor (that of flat families of subschemes, see (0.5.1)). In particular, the Zariski tangent space to the scheme \mathcal{H}_X at a point given by a subscheme Z equals (see [47], Proposition 8.1):

$$(0.5.6) \quad T_Z \mathcal{H}_X = H^0(Z, \mathcal{N}_Z), \quad \text{where } \mathcal{N}_Z = \underline{Hom}(J_Z/J_Z^2, \mathcal{O}_Z).$$

Here J_Z is the sheaf of ideals of the subscheme Z . The sheaf \mathcal{N}_Z is called the normal sheaf of Z . It is locally free if Z is a locally complete intersection. If Z is a smooth variety then \mathcal{N}_Z is the sheaf of sections of the normal bundle of Z .

(0.5.6) Consider the situation of (0.1.1) i.e. an action of an algebraic group H on a projective variety X . Then for a small open H -invariant set $U \subset X$ the orbit closures $\overline{H.x}$ form a flat family. We obtain an embedding $U/H \hookrightarrow \mathcal{H}_X$.

(0.5.7) Definition. The Hilbert quotient $X///H$ is the closure of U/H in the Hilbert scheme \mathcal{H}_X .

Thus $X///H$ is a projective algebraic variety compactifying U/H . "Infinite" points of $X///H$ correspond to subschemes in X which are "degenerations" of generic orbit closures.

(0.5.8) The cycle map (0.5.3) provides a canonical regular birational morphism

$$(0.5.9) \quad \pi : X///H \longrightarrow X//H$$

from the Hilbert quotient to the Chow quotient. This morphism may be very non-trivial even in the case when the group H is finite. So $X///H$ provides a still finer compactification.

(0.5.10) In general the Hilbert quotient is rather hard to describe. For instance, in the case of torus action on the projective space considered in section (0.2) the Hilbert quotient is the toric variety corresponding to the so-called *state polytope* of the toric subvariety X_A introduced by D.Bayer, I.Morrison and M.Stillman [5,6]. However, its exact description depends not only on the geometry of the set A (as is the case for the secondary polytope) but also on the arithmetic nature of relation between elements of A .

We shall see later that for the torus action on the Grassmannian Hilbert and Chow quotients coincide thus allowing us to use the advantages of both approaches.

Chapter 1. GENERALIZED LIE COMPLEXES.

(1.1) Lie complexes and the Chow quotient of Grassmannian

(1.1.1) Let \mathbf{C}^n be the coordinate n - dimensional complex vector space with coordinates x_1, \dots, x_n . By $G(k, n)$ we shall denote the Grassmannian of k - dimensional linear subspaces in \mathbf{C}^n . The group $(\mathbf{C}^*)^n$ of diagonal matrices acts on $G(k, n)$. Since homotheties act trivially we obtain in fact an action of the $n - 1$ - dimensional algebraic torus $H = (\mathbf{C}^*)^n / \mathbf{C}^*$. Our main object of study in this paper will be the Chow quotient $G(k, n) // H$.

(1.1.2) For each subset $I \subset \{1, \dots, n\}$ denote by L_I the coordinate subspace in \mathbf{C}^n defined by equations $x_i = 0, i \in I$ and by \mathbf{C}^I the coordinate subspace spanned by basis vectors from I . Thus the codimension of L_I and the dimension of \mathbf{C}^I equal to $|I|$, the cardinality of I .

Call a k -dimensional subspace $L \in G(k, n)$ *generic* if for any $I \subset \{1, \dots, n\}, |I| = k$ we have $L \cap L_I = 0$. The space $G^0(k, n)$ of all generic subspaces is an open H - invariant subset in $G(k, n)$. It is called the *generic stratum*. It will serve as the open set U from §0.

(1.1.3) The Grassmannian $G(k, n)$ can be seen as the variety of $(k - 1)$ - dimensional projective subspaces in the projective space P^{n-1} . Using the terminology going back to Plücker, one usually calls $(n - 1)$ - dimensional families of subspaces in P^{n-1} *complexes*.

(1.1.4) **Definition.** By a *Lie complex* we shall mean an algebraic subvariety in $G(k, n)$ which is the closure of the H - orbit $\overline{H.L}$ of some generic subspace $L \in G^0(k, n)$.

(1.1.5) **Proposition.** [19] Each Lie complex is a $(n - 1)$ - dimensional variety containing all the H - fixed points on $G(k, n)$ given by coordinate subspaces $\mathbf{C}^I, |I| = k$. These $\binom{n}{k}$ points are the only singular points of a Lie complex. Near each of these points a Lie complex looks like the cone over $P^{k-1} \times P^{n-k-1}$ in the Segre embedding.

(1.1.6) **Example.** Lie complexes in $G(2, 4)$ were extensively studied in classical literature under the name of *tetrahedral complexes*. see [2], [27] and references in [20]. Let us describe them in more detail. Let x_1, \dots, x_4 be homogeneous coordinates in P^3 and L_i be the coordinate plane $\{x_i = 0\}$. The configuration of four planes L_i can be thought of as a tetrahedron. A line $l \in P^3$ lies in generic stratum $G^0(2, 4)$ if and only if it does not intersect any of the 6 lines given by the edges of our tetrahedron. For such a line the four points of intersections $l \cap L_i$ are distinct and, as any four distinct points on a projective line, possess the cross-ratio $r(l \cap L_1, \dots, l \cap L_4) \in \mathbf{C} - \{0, 1\}$. Let $\lambda \in \mathbf{C} - \{0, 1\}$ be a fixed number. The tetrahedral complex K_λ is, by definition, the closure of the set of those $l \in G^0(2, 4)$ for which the cross-ratio $r(l \cap L_1, \dots, l \cap L_4)$ equals λ . Its equation in Plücker coordinates is

$$p_{12}p_{34} + \lambda p_{13}p_{24} = 0.$$

This can be commented as follows. The classical Plücker relation gives that three quadratic polynomials $p_{12}p_{34}, p_{13}p_{24}$ and $p_{14}p_{23}$ on $G(2, 4)$ are linearly dependent i.e. generate a 1-dimensional linear system (pencil) of hypersurfaces. The tetrahedral complexes are just hypersurfaces from this pencil. They are, therefore, particular cases of quadratic line complexes. As was pointed out in [20], the definition of a tetrahedral complex as the closure of a torus orbit is due to F.Klein and S.Lie.

(1.1.7) Clearly all Lie complexes represent the same class in $(2n - 2)$ - dimensional homology of the Grassmannian. Denote this class δ . Let us recall an explicit formula for δ found by A.Klyachko [36]. For any Young diagram $\alpha = (\alpha_1 \geq \dots \geq \alpha_k)$ with no more than k rows and no more than $(n - k)$ columns we shall denote by $|\alpha| = \sum \alpha_i$ the number of cells in α and by σ_α the Schubert class in $H_{2|\alpha|}(G(k, n))$ corresponding to α (see [24] and §3.9 below for details on Schubert cycles). These classes form an integral basis in the homology and the formula of Klyachko gives a decomposition of δ with respect to this basis.

(1.1.8) **Proposition.** [36] Let α be a Young diagram with $(n - 1)$ cells. The coefficient at σ_α in the

decomposition of the fundamental class δ of a Lie complex with respect to Schubert cycles equals

$$\sum_{i=0}^k (-1)^i \binom{n}{i} \dim \Sigma^\alpha(\mathbf{C}^{k-i}),$$

where $\Sigma^\alpha(\mathbf{C}^{k-i})$ is the irreducible representation of $GL(k-i)$ with highest weight α .

(1.1.9) **Example.** For a Lie complex in $G(2, n)$ the above formula gives

$$\delta = (n-2)\sigma_{n-2,1} + (n-4)\sigma_{n-3,2} + (n-6)\sigma_{n-4,3} + \dots$$

In particular, for a Lie complex in $G(2, 4)$ the formula gives $\delta = 2\sigma_{2,1}$ and $\sigma_{2,1}$ is the class of hyperplane section of $G(2, 4)$ in the Plücker embedding. This agrees with the fact that Lie complexes in $G(2, 4)$ are quadratic complexes.

(1.1.10) The collection of all Lie complexes is naturally identified with $G^0(k, n)/H$, the quotient of the generic stratum (1.1.2). We are interested in the Chow quotient $G(k, n)//H$ which is a projective subvariety in the Chow variety $\mathcal{C}_{n-1}(G(k, n), \delta)$, namely the closure of the set of all Lie complexes.

Any algebraic cycle from $G(k, n)//H$ will be called a *generalized Lie complex*. It is our point of view that generalized Lie complexes are the "right" generalizations of generic torus orbits in the Grassmannian. We shall see later that each generalized Lie complex can be seen as a (possibly reducible) algebraic subvariety in $G(k, n)$.

(1.1.11) **Example.** The Chow quotient $G(2, 4)//H$ is isomorphic to the projective line P^1 . The isomorphism is given by the cross-ratio of four points of intersection $l \cap L_i$ in Example (1.1.6). There are exactly three generalized Lie complexes in $G(2, 4)$ which are not closures of single orbits (i.e. are not genuine Lie complexes). They are limit positions of tetrahedral complexes corresponding to values $0, 1, \infty$ not taken by the cross ratio. Denote by $Z_{ij} \subset G(2, 4)$ the space of lines intersecting the coordinate line $x_i = x_j = 0$ (the edge of the tetrahedron). This is a linear section of $G(2, 4)$ given by the equation $p_{ij} = 0$. The three limit complexes are

$$Z_{12} + Z_{34}, Z_{13} + Z_{24}, Z_{14} + Z_{23}.$$

(1.2) Chow strata and matroid decompositions of the hypersimplex.

(1.2.1) Call two k -dimensional linear subspaces $L, L' \subset \mathbf{C}^n$ *equivalent* if $\dim(L \cap L_I) = \dim(L' \cap L_I)$ for any $I \subset \{1, \dots, n\}$. Corresponding equivalence classes are called *strata*. They are H -invariant subsets in $G(k, n)$. A *base* of a subspace $L \in G(k, n)$ is a k -element subset $I \subset \{1, \dots, n\}$ such that $L \cap L_I = 0$. It is well-known that two subspaces L and L' lie in the same stratum (i.e. are equivalent) if and only if their sets of bases coincide. As a particular case we obtain the generic stratum $G^0(k, n) \subset G(k, n)$ defined as follows. A space L lies in $G^0(k, n)$ if and only if each k -element subset is a base for L . \swarrow UNIFORM MATROID ??

This stratification was introduced in [18, 19, 21]. The set of bases for any subspace $L \subset G(k, n)$ introduces on $\{1, \dots, n\}$ the structure of a *matroid* of rank k . Because of this, this stratification is often referred to as the *matroid stratification* of the Grassmannian.

(1.2.2) It was remarked in [19], §5.1 that the matroid stratification of the Grassmannian is not a stratification in the sense of Whitney. In particular, the closure of a stratum may happen not to be a union of other strata.

(1.2.3) Let e_1, \dots, e_n be standard basis vectors in the coordinate space \mathbf{R}^n . We define the convex polytope $\Delta(k, n)$ called the (k, n) -*hypersimplex* to be the convex hull of $\binom{n}{k}$ points $e_{i_1} + \dots + e_{i_k}$ where $1 \leq i_1 < \dots < i_k \leq n$.

All these points are vertices of $\Delta(k, n)$. We shall denote these vertices shortly by $e_I = \sum_{i \in I} e_i$ where $I \subset \{1, \dots, n\}$, $|I| = k$. For any subspace $L \in G(k, n)$ we define its *matroid polytope* $M(L)$ as the convex hull of e_I , where I runs over all bases for L . Thus $\Delta(k, n)$ itself is the matroid polytope for a generic subspace.

The hypersimplex was introduced in [17] and serves as a combinatorial model both for the Grassmannian with torus action and for any Lie complex.

(1.2.4) Proposition. [19] *Let $L \in G(k, n)$. Then:*

- a) *Any edge of $M(L)$ is parallel to a vector of the form $e_i - e_j, i \neq j$.*
- b) *The (complex) dimension of the orbit $H.L$ coincides with the real dimension of the polytope $M(L)$. The closure $\overline{H.L}$ is a projective, normal, toric variety and $M(L)$ is the corresponding polytope (i.e. the fan of $\overline{H.L}$ is the normal fan of $M(L)$). In particular:*
- c) *Any Lie complex is a projective toric variety and the corresponding polytope is $\Delta(k, n)$. So p -dimensional H -orbits on any Lie complex are in bijection with p -dimensional faces of $\Delta(k, n)$.*

The following description of faces of $\Delta(k, n)$ was given in [17].

(1.2.5) Proposition. a) *Each face of $\Delta(k, n)$ is itself a hypersimplex.*

b) *Edges of $\Delta(k, n)$ are segments $[e_I, e_J]$ where J differs from I by replacing one element $i \in I$ by another $j \notin I$.*

c) *For $k > 1$ there are exactly $2n$ facets (faces of codimension 1) of $\Delta(k, n)$. They are*

$$\Gamma_i^+ = \text{Conv}\{e_I, i \in I\} \quad \text{and} \quad \Gamma_i^- = \text{Conv}\{e_I, i \notin I\}$$

for $1 \leq i \leq n$. Each polytope Γ_i^+ is linearly isomorphic to the hypersimplex $\Delta(k-1, n-1)$ whereas each Γ_i^- is isomorphic to $\Delta(k, n-1)$.

(1.2.6) By a *matroid polytope* in $\Delta(k, n)$ we shall mean any subpolytope $M \subset \Delta(k, n)$ whose vertices are among vertices of $\Delta(k, n)$ and edges have the form described in part a) of Proposition 1.2.4 (i.e. are among edges of $\Delta(k, n)$). According to Proposition 1.2.5 the polytope $M(L)$ for any subspace $L \in G(k, n)$ is a matroid polytope. Matroid polytopes of such form are called *realizable*.

The notion of matroid polytope in $\Delta(k, n)$ was introduced in [19, 21]. It was shown in these papers that such polytopes are in bijection with the structures of rank k matroid on a set $\{1, \dots, n\}$.

(1.2.7) Consider the Plücker embedding of the Grassmannian $G(k, n)$ into the $\binom{n}{k} - 1$ -dimensional projective space $P(\bigwedge^k \mathbb{C}^n)$. The homogeneous coordinates in this projective space will be denoted $p_I, I \subset \{1, \dots, n\}, |I| = k$. The H -action on $G(k, n)$ extends to the whole $P(\bigwedge^k \mathbb{C}^n)$. The **matroid polytope of a subspace $L \in G(k, n)$ is the image of the orbit closure $\overline{H.L} \subset G(k, n) \subset P(\bigwedge^k \mathbb{C}^n)$ under the momentum map $\mu : P(\bigwedge^k \mathbb{C}^n) \rightarrow \Delta(k, n)$ defined as follows [19, 21]:**

$$(1.2.8) \quad \mu(x) = \frac{\sum_{|I|=k} p_I(x) \cdot e_I}{\sum_{|I|=k} p_I(x)}.$$

(1.2.9) Since $G(k, n)$ is embedded equivariantly into $P(\bigwedge^k \mathbb{C}^n)$, we obtain the embedding of Chow quotients

$$(1.2.10) \quad G(k, n) // H \hookrightarrow P(\bigwedge^k \mathbb{C}^n) // H.$$

The latter quotient is, according to Example 0.2, a toric variety of dimension $\binom{n}{k} - n$ and the corresponding polytope is the secondary polytope of the hypersimplex $\Delta(k, n)$. By comparing the two Chow quotients we deduce from [30, 31] the following proposition.

(1.2.11) Proposition. *Let $Z = \sum c_i Z_i$ be a cycle from $G(k, n) // H$. Then :*

- a) *Each component Z_i is a closure of some $(n-1)$ -dimensional H -orbit Z_i^0 .*
- b) *Let $M(Z_i) = \mu(Z_i)$ be the matroid polytope of any subspace $L \in Z_i^0$ (or, what is the same, the image of Z_i under the momentum map). Then the polytopes $M(Z_i)$ form a polyhedral decomposition of $\Delta(k, n)$.*

(1.2.12) Example. Consider the case $k = 2, n = 4$. The hypersimplex $\Delta(2, 4)$ is the 3-dimensional octohedron. Each of the three generalized Lie complexes from (1.1.14) gives a decomposition of this octohedron into a union of two pyramids with a common quadrangular face.

We have the embedding (1.2.10) of $G(2,4)//H = P^1$ into $P(\bigwedge^2(\mathbf{C}^4))//H$ which is a toric variety of dimension 2. This variety is isomorphic to the projective plane P^2 . To see this, let us show that the secondary polytope (polygon, in our case) Σ of the octohedron $\Delta(2,4)$ is in fact a triangle. Indeed, by definition (0.2.7) vertices of Σ are in bijection with regular triangulations of $\Delta(2,4)$. Each triangulation of $\Delta(2,4)$ can be obtained as follows. Take any decomposition of $\Delta(2,4)$ into two pyramids as above and then decompose each of these pyramids into two tetrahedra in a compatible way:

(1.2.13)

Thus there are 3 triangulations which correspond to vertices of Σ and three pyramidal decompositions which correspond to edges of Σ and hence Σ is a triangle.

Since the symmetry group of the octohedron acts on Σ , it is a regular triangle. Hence the toric variety corresponding to Σ , has P^2 as its normalization. The fact that this variety is normal can be established by direct computation of vertices of Σ as points of the integer lattice \mathbf{Z}^6 (according to (0.2.6)) which we leave to the reader.

So the toric variety $P(\bigwedge^2(\mathbf{C}^4))//H$ is a projective plane. The subvariety $G(2,4)//H = P^1$ is a conic in this projective plane inscribed into the coordinate triangle:

(1.2.14)

(1.2.15) Proposition. *Let $Z = \sum c_i Z_i$ be a cycle from $G(k,n)//H$. Then all the multiplicities c_i equal 1 (or 0).*

Proof. The recipe for calculation of c_i given in [30,31] is the following. We consider the affine \mathbf{Z} -lattice Ξ_i generated by the vertices of the polytope $M(Z_i)$ which is imbedded into the affine lattice Ξ generated by all the vertices of $\Delta(k,n)$. Then $c_i = [\Xi : \Xi_i]$. Let us show that in fact $\Xi = \Xi_i$. Choose some vertex e_I of $M(Z_i)$. Subtracting it from points of Ξ , we identify Ξ with $\{(a_1, \dots, a_n) \in \mathbf{Z}^n : \sum a_i = 0\}$. Consider now all edges of $M(Z_i)$ containing e_I . By Proposition 1.3 b), they all have the form $e_j - e_I$, where $j \in I, l \notin I$. Since $M(Z_i)$ has full dimension, there are at least $n-1$ independent edges. However, any $n-1$ independent vectors of the form $e_j - e_I$ generate the lattice Ξ . \triangleleft

By the above proposition, generalized Lie complexes (= "infinite" points of the Chow quotient $G(k,n)//H$) can be thought of as usual reducible subvarieties (instead of cycles) in the Grassmannian, what further justifies their name. We will therefore denote these complexes by $Z = \bigcup Z_i$ to emphasize that they are varieties.

(1.2.16) Definition. *Two cycles $Z, Z' \in G(k,n)//H$ are called equivalent if the corresponding polyhedral decompositions of $\Delta(k,n)$ coincide. Equivalence classes under this relation are called Chow strata.*

By considering again the Plücker embedding we see that our stratification of $G(k,n)//H$ is induced from the stratification of the toric variety $P(\bigwedge^k \mathbf{C}^n)//H$ given by the torus orbits. Each Chow stratum can be specified by a finite list of usual strata corresponding to individual matroid polytopes from the polyhedral decomposition.

(1.2.17) **Definition.** A polyhedral decomposition \mathcal{P} of the hypersimplex $\Delta(k, n)$ is called a matroid decomposition if all the polytopes from \mathcal{P} are matroid polytopes (1.2.7). A matroid decomposition is called realizable if it comes from a generalized Lie complex (1.2.11).

Thus matroid decompositions of $\Delta(k, n)$ are precisely the labels by which **Chow strata** are labelled. The notion of a matroid polytope being equivalent to that of matroid (1.2.1), a matroid decomposition represents a new kind of combinatorial structure — a collection of usual matroids with certain properties (that the corresponding polytopes form a decomposition of the hypersimplex).

(1.3) Example: matroid decompositions of $\Delta(2, n)$.

Presumably our case??

In this section we give a complete description of matroid decompositions of the hypersimplex $\Delta(2, n)$. The structure involved will turn out to be identical to those in the description of stable n -pointed curves of Grothendieck [12] and Knudsen [37].

Recall that vertices of $\Delta(2, n)$ are of the form $e_{ij} := e_i + e_j, i \neq j, 1 \leq i, j \leq n$, where e_i are the standard basis vectors of \mathbb{R}^n .

(1.3.1) **Proposition.** a) Matroid polytopes in $\Delta(2, n)$ are in bijection with pairs (J, R) where $J \subset \{1, \dots, n\}$ is a non-empty subset and R is an equivalence relation on J with at least 2 equivalence classes. The matroid polytope $M(J, R)$ corresponding to (J, R) above has vertices e_{ij} where $i, j \in J$ are such that iRj does not hold.

b) The dimension of $M(J, R)$ equals $|J| - 1$ if R has ≥ 3 equivalence classes and equals $|J| - 2$ if R has exactly 2 equivalence classes.

Proof: This follows from ([21], Example 1.10 and Proposition 4, §2).

(1.3.2) **Corollary.** Matroid polytopes in $\Delta(2, n)$ which have full dimension $(n - 1)$, are in bijection with equivalence relations on $\{1, \dots, n\}$ with ≥ 3 equivalence classes.

Thus matroid decompositions of $\Delta(2, n)$ are certain "compatible" systems of equivalence relations on the same set $\{1, \dots, n\}$. We are going to describe them.

(1.3.3) By a graph we mean a finite 1-dimensional simplicial complex. So a graph Γ is defined by its set of vertices Γ_0 and the set of edges Γ_1 together with the incidence relation connecting these sets. If v is a vertex of a graph Γ , the *valency* of v is, by definition, the number of edges containing v .

By a *tree* we mean a connected graph T without loops such that every vertex of T has the valency either 1 or ≥ 3 . The vertices of valency 1 will be called *endpoints* of T . For any two vertices v, w of a tree T there is a unique edge path without repetitions joining these vertices. This path will be denoted $[v, w]$.

Let A_1, \dots, A_n be formal symbols. By a tree bounding the endpoints A_1, \dots, A_n , we shall mean a tree T with exactly n endpoints which are put into bijection (or just identified) with symbols A_i . Two such trees T, T' are called isomorphic if there is an isomorphism of graphs $T \rightarrow T'$ preserving A_i .

(1.3.4) Let T be a tree bounding endpoints A_1, \dots, A_n . Any vertex of T which is not an endpoint will be called interior. Let $v \in T_0$ be an interior vertex. We define an equivalence relation \cong_v on $\{1, \dots, n\}$ by setting $i \cong_v j$ if the edge path $[A_i, A_j]$ does not contain the vertex v .

In other words, the deletion of the vertex v splits the tree into several connected components and $i \cong_v j$ if the endpoints A_i, A_j are situated in the same component. The equivalence classes under \cong_v are in bijection with edges of T containing v .

(1.3.5) Proposition (1.2.5) implies that $\Delta(2, n)$ has n facets (faces of codimension 1) $\Gamma_i^+ = \text{Conv}\{e_{ij}, j \neq i\}$ which are $(n - 2)$ -dimensional simplices. It is clear that in any polyhedral decomposition of $\Delta(2, n)$ each Γ_i^+ is a facet of exactly one polytope from decomposition.

(1.3.6) **Theorem.** Matroid decompositions of the hypersimplex $\Delta(2, n)$ are in bijection with isomorphism classes of trees bounding endpoints A_1, \dots, A_n .

- Explicitly, if T is such a tree, the corresponding decomposition $\mathcal{P}(T)$ consists of matroid polytopes $M(\cong_v)$ (Proposition (1.3.1)) for all interior vertices v of T .
- Conversely, the tree T can be recovered from the corresponding matroid decomposition \mathcal{P} as follows. Internal vertices of T are barycenters of polytopes (of maximal dimension) from \mathcal{P} . Endpoints of T are barycenters of facets Γ_i^+ . The barycenter of each Γ_i^+ is joined to the barycenter of the unique polytope from \mathcal{P} containing Γ_i ; the barycenters of two polytopes from \mathcal{P} are joined if and only if these polytopes have a common facet.

(1.3.7) Remark. Let v be an interior vertex of the tree T . The vertices of the polytope $M(\cong_v)$ are those vertices $e_{ij} = e_i + e_j$ of $\Delta(2, n)$ for which the edge path $[A_i, A_j]$ does contain v .

Proof of (1.3.6): Let T be any tree bounding A_1, \dots, A_n . Let us show that the collection of polytopes $M(\cong_v)$ forms a polyhedral decomposition of $\Delta(2, n)$. By definition, this means that the two properties hold:

(1.3.8) Intersection of any two polytopes $M(\cong_v), M(\cong_w)$ is a common face of both of them.

(1.3.9) The union of the polytopes $M(\cong_v)$ is the whole hypersimplex $\Delta(2, n)$.

(1.3.10) Proof of (1.3.8). We shall prove a stronger statement: that $M(\cong_v) \cap M(\cong_w)$ is the convex hull of vertices common to $M(\cong_v), M(\cong_w)$. By (1.3.7), a vertex e_{ij} is common to $M(\cong_v), M(\cong_w)$ if the edge path $[A_i, A_j]$ contains $[v, w]$ as a sub-path. Let us subdivide the set $\{1, \dots, n\}$ into three parts: X_+, X_-, X_0 . We set $i \in X_+$ if the edge path $[A_i, v]$ does not contain points on $[v, w]$ other than v . We set $i \in X_-$ if the edge path $[A_i, w]$ does not contain points of $[v, w]$ other than w . We set $i \in X_0$ in all other cases:

(1.3.11)

Recall that $\Delta(2, n)$ lies in \mathbf{R}^n as the convex hull of sums $e_{ij} = e_i + e_j$ of two distinct basis vectors. Consider the linear function f on \mathbf{R}^n such that $f(e_i) = +1, -1$ or 0 if $i \in X_+, X_-$ or X_0 respectively. Then f is non-negative on all vertices of $M(\cong_v)$ and non-positive on all vertices of $M(\cong_w)$. The only vertices of $M(\cong_v), M(\cong_w)$ for which $f = 0$ are the vertices common to both these polytopes. Assertion (1.3.8) is proven.

(1.3.12) Proof of (1.3.9). It suffices to show that any face of codimension 1 of any $M(\cong_v)$ lies either on the boundary of $\Delta(2, n)$ or is a face of another polytope $M(\cong_w)$. The following description of facets (=faces of codimension 1) of matroid polytopes follows from ([21], §2, Theorem 5).

(1.3.12.1) Proposition. Let $J = \{1, \dots, n\}$ and $M = M(J, R)$ be the matroid polytope of full dimension corresponding to an equivalence relation R on J . Its facets are the following:

- (1) Facets $\Gamma_j^+(M)$ defined for any j unless $\{j\}$ is an equivalence class in itself and the total number of classes is 3. This facet is the matroid polytope $M(J', R')$, where $J' = \{1, \dots, n\} - \{j\}$ and R' is the equivalence relation induced by R . It lies entirely in the boundary of $\Delta(2, n)$.
- (2) Facet $\Gamma_K^-(M)$ defined for any equivalence class $K \in J/R$. This is the matroid polytope $M(J, R'')$, where R'' is the equivalence relation with only two classes of which one is K and the other is formed by all elements not in K .

The notation Γ^\pm is compatible with the notation for the facet of the full hypersimplex $\Delta(2, n)$ introduced in Proposition 1.2.5.

(1.3.12.2) Corollary. Let T be a tree bounding A_1, \dots, A_n and $v \in T$ be an interior vertex. The facets of

the matroid polytope $M(\cong_v)$ not lying in the boundary of $\Delta(2, n)$ are in bijection with edges of T containing v whose second end is also an interior vertex. The facet corresponding to such an edge e is of the form Γ_K^- where K is the \cong_v -equivalence class corresponding to e .

Now the assertion (1.3.9) follows from Corollary 1.3.12.2 since every edge of the tree T joins two vertices and the two matroid polytopes corresponding to these vertices have a common facet.

(1.3.15) We have proven that any tree bounding A_1, \dots, A_n gives a matroid decomposition of $\Delta(2, n)$. Conversely, let \mathcal{P} be any matroid decomposition. By taking barycenters of polytopes from \mathcal{P} and joining them as prescribed in Theorem 1.3.6, we obtain a certain graph T . Let us show that T is a tree which generates the decomposition \mathcal{P} . The fact that T is a tree follows from the next lemma

(1.3.15.1) Lemma. *Let $M \subset \Delta(2, n)$ be a matroid polytope of full dimension and $\Gamma \subset M$ be a facet not lying on the boundary of $\Delta(2, n)$. Then Γ is equal to the intersection of the whole $\Delta(2, n)$ with a hyperplane.*

Proof: According to Proposition 1.3.12.1, the polytope Γ has the form $M(J, R)$, where R is an equivalence relation on $J = \{1, \dots, n\}$ with only two equivalence classes, say A and B . Define a linear function g on \mathbf{R}^n whose value on the basis vector e_i equals 1, if $i \in A$ and equals (-1) , if $i \in B$. Then Γ is the intersection of $\Delta(2, n)$ with the kernel of g . Lemma (1.3.15.1) is proven.

(1.3.15.2) To finish the proof of Theorem 1.3.6, it remains to show that the tree obtained from the matroid decomposition \mathcal{P} in (1.3.15), generates \mathcal{P} . This checking is left to the reader.

(1.3.16) Example. There are four matroid decomposition of the octohedron $\Delta(2, 4)$: one consists of $\Delta(2, 4)$ itself and each of the others decomposes the octohedron into two pyramids (Example 1.2.12). These decompositions correspond to the following trees bounding endpoints A_1, \dots, A_4 :

(1.3.17)

(1.3.18) We shall show in §4 that all matroid decompositions of $\Delta(2, n)$ are realizable.

(1.4) Relation to the secondary variety for the product of two simplices.

In this section we shall compare the Chow quotient $G(k, n)/H$ with a toric variety of the same dimension. This toric variety will correspond to the convex polytope which is the secondary polytope for the product of two simplices.

(1.4.1) Let $P \subset \mathbf{R}^m$ be any convex polytope and $x \in P$ be any vertex. Denote by $N_x P$ the union of all half-lines drawn from x through all the points of P . This is an affine cone which we call the *normal cone* to P at x . The base of this cone i.e. a transversal section of $N_x P$ by an affine hyperplane will be called the *vertex figure* of P at x . Thus vertices of the vertex figure correspond to edges of P containing x .

(1.4.2) Let A be any finite set. Denote by Δ^A the simplex (of dimension $|A| - 1$) whose set of vertices is A . By definition, Δ^A is the subset in the space \mathbf{R}^A of functions $A \rightarrow \mathbf{R}$ consisting of functions $f(a)$ such that $f(a) \geq 0, \forall a$ and $\sum f(a) = 1$. To any $a \in A$ there corresponds a vertex δ_a of Δ^A . This is the function $A \rightarrow \mathbf{R}$ taking a to 1 and other elements to 0.

(1.4.3) Let e_I be a vertex of the hypersimplex $\Delta(k, n)$. The corresponding vertex figure is the product of two simplices $\Delta^{k-1} \times \Delta^{n-k-1}$ or, in the more invariant notation of (1.4.2), $\Delta^I \times \Delta^{\bar{I}}$, where \bar{I} is the complement to I .

Indeed, edges of $\Delta(k, n)$ containing e_I , are $[e_I, e_I + e_j - e_i]$ where $i \in I, j \notin I$. The required isomorphism takes such an edge to the vertex (δ_i, δ_j) of $\Delta^I \times \Delta^{\bar{I}}$.

The toric variety associated to the polytope $\Delta^{k-1} \times \Delta^{n-k-1}$ is the product of projective spaces $P^{k-1} \times P^{n-k-1}$ and the structure of $\Delta(k, n)$ near a vertex corresponds to the structure of a Lie complex near its singular point (Proposition 1.1.5).

(1.4.4) Proposition-Definition. Let \mathcal{P} be a matroid decomposition of $\Delta(k, n)$ and $e_I \in \Delta(k, n)$ - a vertex. Let \mathcal{P}_I be the induced polyhedral decomposition of the vertex figure $\Delta^I \times \Delta^{\bar{I}}$. Then all the vertices of polytopes constituting \mathcal{P}_I lie among the vertices of $\Delta^I \times \Delta^{\bar{I}}$. If \mathcal{P} is a realizable matroid decomposition then \mathcal{P}_I is a regular polyhedral subdivision of $\Delta^I \times \Delta^{\bar{I}}$.

Recall [22] that a polyhedral subdivision is called regular if it admits a strictly convex piecewise- linear function.

Proof: Vertices of polytopes from \mathcal{P}_I correspond to edges of polytopes from \mathcal{P} containing e_I . Since all these polytopes are matroid polytopes, the edges in question correspond to vertices of $\Delta^I \times \Delta^{\bar{I}}$. If \mathcal{P} is realizable then it is regular as a polyhedral subdivision of $\Delta(k, n)$ and so is \mathcal{P}_I . \triangleleft

(1.4.5) Let $I \subset \{1, \dots, n\}$ be a k -element subset. The coordinate subspace $\mathbf{C}^I \in G(k, n)$ is a fixed point under the action of the torus H (1.1.1). Therefore we have the action of H on the tangent space $T_I := T_{\mathbf{C}^I} G(k, n)$.

The tangent space to $G(k, n)$ at any point L is canonically identified, see [47], with $\text{Hom}(L, \mathbf{C}^n/L)$. Therefore we have the isomorphism of H -modules

$$(1.4.6) \quad T_I = \text{Hom}(\mathbf{C}^I, \mathbf{C}^{\bar{I}})$$

In other words, T_I is decomposed into $k(n-k)$ one-dimensional weight subspaces $V_{ij}, i \in I, j \notin I$ such that for any $t = (t_1, \dots, t_n) \in H$ and any $v \in V_{ij}$ one has

$$(t_1, \dots, t_n).v = (t_i/t_j)v.$$

(1.4.7) The character lattice of the torus H is the sublattice in \mathbf{Z}^n consisting of vectors with the sum of coordinates equal to 0. The character corresponding to the subspace V_{ij} is the vector $e_i - e_j \in \mathbf{Z}^n$. The collection of all vectors $e_i - e_j, i \in I, j \notin I$, forms the set of vertices of the simplex $\Delta^I \times \Delta^{\bar{I}}$.

(1.4.8) Call a point v of the tangent space T_I (and the corresponding point of the projectivization $P(T_I)$) *generic* if all the weight components of v are non-zero. By a generic H -orbit in T_I or $P(T_I)$ we shall mean the orbit of a generic point.

(1.4.9) For the torus orbits in the Grassmannian $G(k, n)$ we also have a notion of genericity introduced in (1.1.2). Closures of generic orbits in $G(k, n)$ were called Lie complexes. Let $Z = \overline{H \cdot L}$, $L \in G^0(k, n)$ be any Lie complex and $TC_I Z := TC_{\mathbf{C}^I} Z \subset T_I$ – its tangent cone at the point \mathbf{C}^I . It follows from (1.1.5) that $TC_I Z$ is the closure of a generic H - orbit in T_I . So we obtain the following proposition.

(1.4.10) Proposition. *Let us identify the quotient $G^0(k, n)/H$ (i.e. the set of generic (1.1.2) H - orbits on $G(k, n)$) with the set of Lie complexes. Then the correspondence $Z \mapsto$ (the projectivization of $TC_{\mathbf{C}^I} Z$) defines an open embedding of $G^0(k, n)/H$ into the set of generic H - orbits in $P(T_I)$.*

(1.4.11) We are going to compare the Chow quotient $G(k, n)//H$ with $P(T_I)//H$. The latter variety is, according to results recalled in §(0.2), a projective toric variety of the same dimension $k(n - k) - n + 1$ as $G(k, n)//H$. Due to (1.4.7) and Theorem 0.2.8, the toric variety $P(T_I)//H$ corresponds to the secondary polytope of $\Delta^I \times \Delta^I$. We shall therefore call this variety the *secondary variety* of $\Delta^I \times \Delta^I$.

(1.4.12) Proposition. *The open embedding from (1.4.10) extends to a regular birational (in particular, surjective) morphism*

$$(1.4.13) \quad f_I : G(k, n)//H \rightarrow P(T_I)//H.$$

This morphism takes any generalized Lie complex \mathcal{L} to the projectivization of its tangent cone at \mathbf{C}^I (which is regarded as an algebraic cycle with multiplicities 0 or 1 in $P(T_I)$).

Proof: Let $U_I \subset G(k, n)$ be the affine chart consisting of k -dimensional subspace L such that $L \oplus \mathbf{C}^I = \mathbf{C}^n$. Each such subspace can be regarded as the graph of a linear operator from \mathbf{C}^I to \mathbf{C}^I . This correspondence identifies U_I with the space of $k \times (n - k)$ -matrices thus introducing coordinates z_{ij} , $i \in I, j \notin I$ in U_I . The tangent vector space T_I becomes identified with U_I , cf. (1.4.6). Consider the action of \mathbf{C}^* on U_I by homotheties (simultaneous multiplication of the coordinates z_{ij} by a scalar). Since this action is a part of the torus action on $G(k, n)$, we deduce that for any Lie complex Z the intersection $Z \cap U_I$ will be a conic (i.e. \mathbf{C}^* -invariant) subvariety of U_I . The same will hold, by continuity, for any generalized Lie complex. The map f_I , therefore, takes any generalized Lie complex Z (which is a subvariety in $G(k, n)$) into the subvariety in $P(T_I) = P(U_I)$ represented by the conic subvariety $Z \cap U_I$ in U_I . We need to show that f_I is a regular morphism of algebraic varieties.

Since both $G(k, n)//H$ and $P(U_I)//H$ are projective, it suffices, by Serre's GAGA theorem [24], to show that f_I is a holomorphic map of complex analytic spaces corresponding to these varieties. However, this follows from the description of the Chow varieties given by D.Barlet [3]. More specifically, Barlet gave a condition for a family of p -dimensional cycles $Z(s) \subset X$ parametrized by a reduced analytic space S , to be analytic near a point $s_0 \in S$. This condition is essentially that for any codimension p analytic subvariety $Y \subset X$ which intersects $Z(s_0)$ properly, the 0-cycle $Z(s) \cap Y$ depends analytically on s near s_0 . To prove that $f_I(Z)$ depends analytically on $Z \in G(k, n)//H$ we note that any analytic subvariety Y of codimension $(n - 1)$ in $P(U_I)$ can be lifted to U_I (by setting one of the coordinates to be 1) to a subvariety \tilde{Y} . If Y intersected some $f_I(Z)$ properly then so does \tilde{Y} with respect to Z due to the conic property. Analytic dependence of $\tilde{Y} \cap Z$ on Z implies the analytic dependence of $Y \cap f_I(Z)$ which is just the image of $\tilde{Y} \cap Z$ in the projectivization. Proposition (1.4.11) is proven.

Later we shall make use of morphisms f_I to construct "coordinate charts" on the Chow quotient. The following fact is an immediate consequence of (1.4.11).

(1.4.14) Corollary. *Each regular polyhedral decomposition (in particular, triangulation) of the product of simplices $\Delta^I \times \Delta^I$ has the form \mathcal{P}_I for some realizable matroid decomposition of $\Delta(k, n)$.*

Thus the problem of classification of all realizable matroid decompositions of hypersimplex contains the classification problem for triangulations of the product of two simplices. *

(1.5) Relation to the Hilbert quotient.

(1.5.1) The Hilbert quotients were defined in n. (0.5). We want to compare the Chow quotient $G(k, n)//H$ with the Hilbert quotient $G(k, n)///H$. Recall that there is a regular birational morphism $\pi : G(k, n)///H \rightarrow G(k, n)//H$ to the Chow quotient, see (0.5.8).

(1.5.2) Theorem. *The morphism π is an isomorphism.*

Proof: By definition, points of $G(k, n)///H$ are subschemes which are limit positions of Lie complexes. By Proposition 1.11 every such subscheme Z is reduced at a generic point of every its component.

(1.5.3) Lemma. *Any subscheme Z from $G(k, n)///H$ is reduced.*

Proof: Consider the intersection of Z with some coordinate Schubert chart U_I . It suffices to prove, for every I , that $Z \cap U_I$ is reduced. The action of H in U_I is a linear one. This is the action corresponding to the products of simplices and it follows from the result of B.Sturmfels ([48], Theorem 6.1) that any limit position of generic H -orbits in U_I is reduced.

Lemma (1.5.3) implies that the morphism π is bijective on \mathbf{C} -points. To show that it is an isomorphism of algebraic varieties, it suffices to show that for any $Z \in G(k, n)///H$ and any non-zero Zariski tangent vector ξ to $G(k, n)///H$ at Z the vector $d\pi(\xi)$, the image of ξ under the differential of π , is non-zero. The tangent space $T_Z \mathcal{H}_G$ to the whole Hilbert scheme at Z equals $H^0(Z, \mathcal{N})$ where $\mathcal{N}_Z = \underline{\text{Hom}}_{\mathcal{O}_G}(J_Z/J_Z^2, \mathcal{O}_Z)$ is the normal sheaf of Z (see §0E). Let Z_{reg} be the smooth part of Z . The restriction of \mathcal{N}_Z to Z_{reg} is the sheaf of section of the normal bundle of Z_{reg} in the usual sense. Hence our vector $\xi \in T_Z \mathcal{H}_G$ gives a normal vector field on Z_{reg} . Since Z is reduced, this field is non-zero if ξ is non-zero, so the assertion follows. Theorem (1.5.2) is proven.

(1.6) The (hyper-) simplicial structure on the collection of $G(k, n)///H$.

(1.6.1) Recall (1.1.2) that $G^0(k, n)$ denotes the generic stratum in the Grassmannian $G(k, n)$. For any $i \in \{1, \dots, n\}$ there are the *intersection and projection maps*

$$(1.6.2) \quad G^0(k, n-1) \xleftarrow{B_i} G^0(k, n) \xrightarrow{A_i} G^0(k-1, n-1)$$

defined as follows. Let $J_i : \mathbf{C}^{n-1} \hookrightarrow \mathbf{C}^n$ be the embedding taking (x_1, \dots, x_{n-1}) to $(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_n)$. The intersection map A_i sends a k -dimensional subspace $L \subset \mathbf{C}^n$ to $J_i^{-1}L$. The projection map B_i is induced by the projection $\mathbf{C}^n \rightarrow \mathbf{C}^{n-1}$ forgetting the i -th coordinate.

The formal structure of these maps is analogous to that of faces of the hypersimplex $\Delta(k, n)$ (Proposition 1.2.4). The existence of such a system of "face" maps was the original reason for introducing hypersimplexes and then Grassmannians into the problem of combinatorial calculation of characteristic classes [17]. More recently, these maps were used by A.A.Beilinson, R.D. MacPherson and V.V.Schectman in [7] to give a "constructible" approximation to K-theory.

(1.6.3) As was noted in [7], the maps (1.6.2) descend to maps of the quotients

$$(1.6.4) \quad G^0(k, n-1)/(\mathbf{C}^*)^{n-1} \xleftarrow{b_i} G^0(k, n)/(\mathbf{C}^*)^n \xrightarrow{a_i} G^0(k-1, n-1)/(\mathbf{C}^*)^{n-1}$$

of the generic strata by their respective tori.

(1.6.5) Clearly there is no way to extend maps (1.6.2) to whole Grassmannians: if the subspace L is contained in the hyperplane $\{x_i = 0\}$ then $A_i(L)$ will have wrong dimension and similarly for B_i . However, it turns out that for Chow quotients the situation is different.

(1.6.6) Theorem. *The maps a_i, b_i in (1.6.4) can be extended to regular morphisms of projective algebraic varieties*

$$(1.6.7) \quad G(k, n-1)/(\mathbf{C}^*)^{n-1} \xleftarrow{\bar{b}_i} G(k, n)/(\mathbf{C}^*)^n \xrightarrow{\bar{a}_i} G(k-1, n-1)/(\mathbf{C}^*)^{n-1}$$

Proof of Theorem (1.6.6) will occupy the rest of this section.

(1.6.8) Let $e_1, \dots, e_n \in \mathbb{C}^n$ be the standard basis vectors and \mathbb{C}_i^{n-1} be the coordinate hyperplane spanned by $e_j, j \neq i$. For any $i \in \{1, \dots, n\}$ we consider the varieties

$$(1.6.9) \quad G_i^+ = \{L \in G(k, n) : e_i \in L\}, \quad G_i^- = \{L \in G(k, n) : L \subset \mathbb{C}_i^{n-1}\}.$$

They are the analogs of the family of coordinate hyperplanes in P^{n-1} . The next proposition is immediate.

(1.6.10) Proposition. a) As abstract varieties, G_i^+ are isomorphic to $G(k-1, n-1)$ and G_i^- to $G(k, n-1)$. b) Both G_i^+ and G_i^- are linear sections of $G(k, n)$ in the Plücker embedding. More precisely, let Π_i^+ and Π_i^- be projective subspaces in $P(\bigwedge^k \mathbb{C}^n)$ given by vanishing of the Plücker coordinates $p_I, i \notin I$ or, respectively, $p_I, i \in I$. Then $G_i^\pm = G(k, n) \cap \Pi_i^\pm$. c) The image of the subvarieties G_i^\pm under the moment map μ from (1.1) is the facet Γ_i^\pm of $\Delta(k, n)$ introduced in (1.2.5). Moreover, we have $G_i^\pm = \mu^{-1}(\Gamma_i^\pm)$.

(1.6.11) The isomorphisms in part a) of (1.6.10) are as follows. the isomorphism $u_i : G(k-1, n-1) \rightarrow G_i^+$ takes a $(k-1)$ -dimensional $\Lambda \subset \mathbb{C}^{n-1}$ to $J_i(\Lambda) \oplus \mathbb{C}e_i$, where the embedding $J_i : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ was defined in (1.6.1). The isomorphism $v_i : G(k, n-1) \rightarrow G_i^-$ takes a k -dimensional $M \subset \mathbb{C}^{n-1}$ into $J_i(M)$.

(1.6.12) Let us turn to the construction of \tilde{a}_i, \tilde{b}_i . Note that the maps a_i, b_i of the quotients of generic strata have the following transparent description in terms of Lie complexes (closures of generic orbits).

(1.6.13) Lemma. Let $Z = \overline{H \cdot L}$ be a Lie complex in $G(k, n)$. Then the Lie complex $a_i(Z)$ in $G(k-1, n-1)$ representing the orbit of $A_i(L)$, is equal to $u_i^{-1}(Z \cap G_i^+) = u_i^{-1}(Z \cap \Pi_i^+)$. Similarly, the Lie complex $b_i(Z)$ in $G(k, n-1)$, representing the orbit of $B_i(L)$, is equal to $v_i^{-1}(Z \cap G_i^-) = v_i^{-1}(Z \cap \Pi_i^-)$.

Proof: Denote by $g_i(t)$ the diagonal matrix $(1, \dots, 1, t, 1, \dots, 1) \in (\mathbb{C}^*)^n$ (where t is on the i -th place). Let $pr_i : \mathbb{C}^n \rightarrow \mathbb{C}_i^{n-1}$ be the coordinate projection. Let $L \in G^0(k, n)$ be a generic subspace. Then we have

$$(1.6.14) \quad (L \cap \mathbb{C}_i^{n-1}) \oplus \mathbb{C}e_i = \lim_{t \rightarrow \infty} g_i(t) \cdot L, \quad pr_i(L) = \lim_{t \rightarrow 0} g_i(t) \cdot L.$$

This shows that $Z \cap G_i^+$ (resp. $Z \cap G_i^-$) contains the orbit of $A_i(L)$ (resp. $B_i(L)$).

On the other hand, the vanishing of Plücker coordinates p_I , for which $e_I \notin \Gamma_i^\pm$, forms a system of equations for $Z \cap G_i^\pm$, as follows from the general theory of toric varieties [49]. Lemma (1.6.13) is proven.

The lemma just proven shows that a_i and b_i are given by intersecting Lie complexes with projective subspaces Π_i^\pm . However, $\dim(Z \cap \Pi_i^\pm) = \dim(Z) - 1$ whereas Π_i^\pm have high codimension. Thus to prove that the intersection gives a regular morphism of Chow quotients, extra work is needed.

(1.6.15) Proof that \tilde{a}_i, \tilde{b}_i are regular morphisms. For any face $\Gamma \subset \Delta(k, n)$ and any Lie complex Z we shall denote by $Z(\Gamma)$ the closure of the H -orbit in Z corresponding to Γ . In particular, the codimension 1 faces of $\Delta(k, n)$ are Γ_i^\pm from Proposition 1.8 and the corresponding orbit closures $Z(\Gamma_i^\pm) = Z \cap G_i^\pm = Z \cap \Pi_i^\pm$ are the varieties we are studying. Our aim is to show that the Chow form of $Z(\Gamma_i^\pm)$ can be polynomially expressed via that of Z .

(1.6.15.1) Lemma. Consider any coordinate hyperplane $\{p_I = 0\}$ in $P(\bigwedge^k \mathbb{C}^n)$ given by vanishing of a Plücker coordinate p_I . Then we have the equality of cycles

$$Z \cap \{p_I = 0\} = \sum_{\Gamma: e_I \notin \Gamma} Z(\Gamma),$$

where Γ runs over codimension 1 faces of $\Delta(k, n)$ (i.e. over Γ_i^\pm) not containing the vertex e_I .

Proof: The lemma says that the order of vanishing of p_I on $Z(\Gamma)$ equals 0 if $e_I \in \Gamma$ and 1 if $e_I \notin \Gamma$. According to the general rule (valid for any toric variety in an equivariant projective embedding [49]) this order equals the distance from e_I to the affine hyperplane spanned by Γ , the distance being measured in natural integer units indexed by the lattice. In our case, if $e_I \notin \Gamma$, then the said distance equals one.

(1.6.15.2) **Corollary.** Let $R_Z(l_1, \dots, l_n), l_i \in \bigwedge^k(\mathbb{C}^n)^*$, be the Chow form of a Lie complex Z . Let π_I be the coordinate projection to the coordinate hyperplane $\text{Ker}(p_I) \subset \bigwedge^k(\mathbb{C}^n)$. Then for any linear functionals $\lambda_1, \dots, \lambda_{n-1} \in (\text{Ker}(p_I))^*$ we have

$$R_Z(p_I, \pi_I^* \lambda_1, \dots, \pi_I^* \lambda_{n-1}) = \prod_{\Gamma: e_I \notin \Gamma} R_{Z(\Gamma)}(\lambda_1, \dots, \lambda_{n-1})$$

where on the right hand side stand Chow forms of subvarieties $Z(\Gamma) \subset P(\text{Ker}(p_I))$.

(1.6.15.3) End of the proof that \tilde{a}_i, \tilde{b}_i are regular. Consider some facet of $\Delta(k, n)$, say, Γ_i^+ and let $e_I \notin \Gamma_i^+$ be some vertex (i.e., $i \in I$). Consider the coordinate projection $\text{Ker}(p_I) \rightarrow \Pi_i^+$ of coordinate subspaces in $\bigwedge^k(\mathbb{C}^n)$. Here Π_i^+ is defined, as above, by vanishing of all p_I with $i \in I$. The projection of $\bigcup_{\Gamma: e_i \notin \Gamma} Z(\Gamma)$ to Π_i^+ is the component $Z(\Gamma_i^+)$ we are interested in, plus the union of some coordinate $(n-2)$ -dimensional subspaces (which are images of other components). Let $\mu_1, \dots, \mu_{n-1} \in (\Pi_i^+)^*$ be linear forms. Denote by $\tilde{\mu}_i$ the extension of μ_i to all of $\text{Ker}(p_I)$ by means of the coordinate projection $\text{Ker}(p_I) \rightarrow \Pi_i^+$. We obtain the equality

$$\prod_{\Gamma: e_I \notin \Gamma} R_{Z(\Gamma)}(\tilde{\mu}_1, \dots, \tilde{\mu}_{n-1}) = R_{Z(\Gamma_i^+)}(\mu_1, \dots, \mu_{n-1}) \cdot \text{prod}_S R_S(\mu_1, \dots, \mu_{n-1})^{\nu_S},$$

where S runs over coordinate $(k-1)$ -dimensional subspaces in π_i^+ , R_S is the Chow form of the subspace S and ν_S are some exponents. From this equality we obtain that $R_{Z(\Gamma_i^+)}$ can be obtained from the right hand side by division to a fixed polynomial. Since the left hand side itself depends polynomially on R_Z , Theorem (1.6.6) is completely proven.

Chapter 2. PROJECTIVE CONFIGURATIONS AND THE GELFAND-MACPHERSON ISOMORPHISM.

I.M.Gelfand and R.W.MacPherson have established in [20] an important correspondence between torus orbits in $G(k, n)$ and projective configurations i.e. $GL(k)$ -orbits on $(P^{k-1})^n$. In this section shall show that this correspondence extends to an isomorphism of Chow quotients.

(2.1) Projective configurations and their Chow quotient.

(2.1.1) Consider the $(k-1)$ -dimensional projective space P^{k-1} . By a *configuration* we shall mean an ordered collection $M = (x_1, \dots, x_n)$ of n points in P^{k-1} . The general linear group $GL(k)$ acts on P^{k-1} by projective transformations. This induces an action on the space $(P^{k-1})^n$ of configurations.

The study of orbits of this action is a classical problem of projective geometry. See [14] for investigations from the standpoint of Mumford's geometric invariant theory.

(2.1.2) The elements of a configuration M can also be visualized as hyperplanes (in the dual projective space). This point of view will be useful later. In this subsection we shall just consider elements of M as points.

(2.1.3) We will be interested in the Chow quotient $(P^{k-1})^n // GL(k)$. To apply Definition 0.1.7 it is first desirable to know which configurations are "generic enough". The answer, of course, is the following.

(2.1.4) A configuration $M = (x_1, \dots, x_n)$ of points in P^{k-1} will be said to be *in general position* if any i of these points, $i \leq k$, span a projective subspace of dimension exactly $i-1$. The set of all such configurations will be denoted by $(P^{k-1})_{gen}^n$. Orbits of configurations in general position will be referred to as *generic orbits* in $(P^{k-1})^n$.

(2.1.5) Generic $GL(k)$ -orbits on $(P^{k-1})^n$ depend on continuous parameters only when $n \geq k+2$. We shall assume in the sequel that this condition holds. In this case generic orbits have dimension $k^2 - 1$ since the stabilizer of a generic configuration consists only of homotheties.

(2.1.6) For any $0 \leq m \leq k-1$ denote by $[m]$ the $2m$ -dimensional homology class of P^{k-1} represented by P^m . By Künneth formula, the graded homology space of $(P^{k-1})^n$ is the n -fold tensor power of the graded homology space $H_*(P^{k-1})$. Therefore, the basis for the $2p$ -th homology group $H_{2p}((P^{k-1})^n)$ is given by tensor products $[m_1] \otimes \dots \otimes [m_n]$, $\sum m_i = p$.

(2.1.7) Proposition. *The homology class of the closure of any generic $GL(k)$ -orbit in $(P^{k-1})^n$, $n \geq k+2$, is a variety of dimension $k^2 - 1$ and of homology class*

$$\delta = \sum_{\substack{m_1 + \dots + m_n = k^2 - 1 \\ m_i \leq k-1}} [m_1] \otimes \dots \otimes [m_n].$$

The set of closures of generic orbits is a subvariety in the Chow variety $C_{k^2-1}((P^{k-1})^n, \delta)$ isomorphic to the quotient $(P^{k-1})_{gen}^n / GL(k)$.

Proof: Let $Z = \overline{GL(k) \cdot M}$ be the closure of any $k^2 - 1$ -dimensional orbit and $\delta \in H_{2(k^2-1)}((P^{k-1})^n, \mathbb{Z})$ its homology class. The coefficient in δ at $[m_1] \otimes \dots \otimes [m_n]$ can be calculated as follows. Take generic projective subspaces $L_i \subset P^{k-1}$ of codimension m_i . Our coefficient is just the intersection number of Z with $L_1 \times \dots \times L_n$. In other words, this is the number of projective transformations which take each point x_i of our configuration $M = (x_1, \dots, x_n)$ inside L_i . The condition $g(x_i) \in L_i$ is a linear condition on matrix element of a matrix $g \in GL(k)$ of codimension m_i . Taking into account all L_i , we obtain a system of $k^2 - 1$ linear equations on matrix elements of g . By Bertini's theorem applied to Z , if (L_1, \dots, L_n) are generic enough, the

intersection $Z \cap (L_1 \times \dots \times L_n)$ consists of finitely many points. For our linear system this implies that for generic L_j just one of two cases holds:

(2.1.7.1) The space of solutions of the system is 1-dimensional and consists of multiples of a non-degenerate matrix.

(2.1.7.2) The space of solutions is contained in the variety of degenerate matrices.

In the first case the coefficient equals one, in the second it equals 0. We need to show that for x_1, \dots, x_n in general position, the case a) always holds.

(2.1.7.3) Consider the product of Grassmannians $\Pi = \prod G(k - m_i, k)$ i.e. the variety of all tuples (L_1, \dots, L_n) as above. Let $\Pi_Y \subset \Pi$ be the subvariety of those tuples for which $x_i \subset L_i$ for all i .

(2.1.7.4) Lemma. *Let $Y = (x_1, \dots, x_n)$ be any configuration with $k^2 - 1$ - dimensional orbit. Then the truth of the case (2.1.7.2) above (or, equivalently, the vanishing of the coefficient at $[m_1] \otimes \dots \otimes [m_n]$) is equivalent to the following fact: For generic tuple of subspaces $(L_1, \dots, L_n) \subset \Pi_Y$ its stabilizer in $PGL(k - 1)$ has positive dimension.*

Proof: Case (2.1.7.2) means that the union of $GL(k)$ - orbits of points from Π_Y is not dense in Π . The codimension of Π_Y in Π equals $k^2 - 1$. Therefore case (2.1.7.2) means that for any $Y \in \Pi_Y$ its orbit has dimension smaller than $k^2 - 1$.

Now it is clear that if x_1, \dots, x_n are in general position then for any m_1, \dots, m_n summing to $k^2 - 1$ it is possible to choose codimension m_i subspace L_i through x_i such that the whole collection (L_1, \dots, L_n) has trivial stabilizer in $PGL(k)$. Proposition 2.1.7 is proven.

(2.1.8) Example. Consider the action of $GL(2)$ in $(P^1)^4$. The closure of the orbit of a 4-tuple of distinct points is a 3-dimensional variety. It contains four 2-dimensional orbits W_i where W_i is the set of points (x_1, \dots, x_4) such that all three x_j with $j \neq i$, coincide with each other but differ from x_i . The closure of each W_i is isomorphic to $P^1 \times P^1$. These closure intersect along the 1-dimensional orbit which is the set of coinciding tuples.

(2.1.9) We are interested in the Chow quotient $(P^{k-1})^n // GL(k)$. By definitions, points of this quotient are certain algebraic cycles in $(P^{k-1})^n$ of dimension $k^2 - 1$ and homology class δ given by Proposition 2.1.7. Moreover, since the stabilizer of a configuration can not be a unipotent subgroup in $PGL(k)$, we can apply Theorem 0.3.1 to conclude that components of any cycle from $(P^{k-1})^n // GL(k)$ are closures of $k^2 - 1$ -dimensional orbits.

(2.1.9) Example. For 4 distinct points on P^1 the only invariant is the cross-ratio which identifies $(P^1)_{gen}^4 / GL(2)$ with $P^1 - \{0, 1, \infty\}$. Denote by Z_λ the closure of the orbit given by 4-tuples with cross-ratio λ . When $\lambda \rightarrow 0, 1, \infty$, the variety Z_λ degenerates into one of three cycles in $(P^1)^4$. Namely, let Δ_{ij} to be the subset in $(P^1)^4$ given by $\{x_i = x_j\}$. Then the three cycles in question are

$$\Delta_{12} + \Delta_{34}, \Delta_{13} + \Delta_{24}, \Delta_{14} + \Delta_{23}.$$

For example, suppose that our four points x_i depends on a parameter t and degenerate in such a way that $x_1(0) = x_2(0)$ but $x_3(0)$ and $x_4(0)$ are different from them (see Fig. 2.1.10). Let $Z(t), t \neq 0$ be the closure of the orbit of $M(t) = ((x_i(t))_{i=1, \dots, 4})$. Then, of course, the orbit of the limit position $((x_i(0)))$, i.e. Z_{12} , will be a part of the cycle $Z(0) = \lim_{t \rightarrow 0} Z(t)$, but not the only part! Indeed, we can perform, for each t , a projective transformation $g(t)$ which stretches $x_1(t)$ and $x_2(t)$ back to some fixed distance. This transformation shrinks the remaining points $x_3(t)$ and $x_4(t)$ close to each other. The limit of the point $g(t)Y$ will lie on the second component Z_{34} .

(2.1.10)

(2.1.11) Similarly, if we have a degeneration $M(t) = (x_1(t), \dots, x_n(t))$ of a family of n points on P^1 such that just two points merge, e.g. $x_1(0) = x_2(0)$ and all the other $x_i(0)$ remain distinct, then $\lim_{t \rightarrow 0} \overline{GL(2)M(t)}$ will consist of two components. The first is the orbit of the limit configuration $(x_1(0) = x_2(0), x_3(0), \dots, x_n(0))$. The second component is the set of (x_1, \dots, x_n) such that $x_3 = \dots = x_n$ and x_1, x_2 are arbitrary. We shall see later (§4) that this phenomenon exactly corresponds to the degeneration of $(P^1, x_1(t), \dots, x_n(t))$ in the Knudsen's moduli space $\overline{M}_{0,n}$ of stable n -punctured curves of genus 0.

(2.2) The Gelfand-MacPherson correspondence.

(2.2.0) Let us recall the original idea of [20,40] how to construct a configuration from a point in Grassmannian. It will be more convenient for us to speak in this section about configurations of *hyperplanes* instead of points.

(2.2.1) Let $L \subset \mathbb{C}^n$ be a k -dimensional subspace not lying in any coordinate hyperplane $H_i = \{x_i = 0\}$. Then $(L \cap H_i)$ form a configuration of hyperplanes in L i.e. a point in $(P(L^*))^n$. If a subspace L' is obtained from L by the action of a torus element, we shall obtain a projectively isomorphic configuration of hyperplanes in L' . A class of projective isomorphism of configurations of n hyperplanes in $(k-1)$ -dimensional projective spaces is the same as a $GL(k)$ -orbit in the Cartesian power of a fixed projective space $(P^{k-1})^n$. Note that not every configuration of hyperplanes can be obtained, up to an isomorphism, from L as above. To make the assertion precise, denote by $G_{max}(k, n) \subset G(k, n)$ the set of L such that $\dim(H.L) = n-1$. Similarly denote by $((P^{k-1})^n)_{max}$ the set of configurations $\Pi = (\Pi_1, \dots, \Pi_n)$ such that $\dim(GL(k). \Pi) = k^2 - 1$. The Gelfand-MacPherson correspondence induces the bijection of orbit sets

$$(2.2.2) \quad \Lambda : G_{max}(k, n)/H \rightarrow ((P^{k-1})^n)_{max}/GL(k)$$

see [20].

(2.2.3) Note that sets in both sides of (2.2.2) are not, in general algebraic varieties since $G_{max}(k, n)$ and $((P^{k-1})^n)_{max}$ contain unstable points. For comparison of Mumford's quotients of both sides in (2.2.2) see section (2.4) below.

The main result of this section is the following theorem

(2.2.4) Theorem. *The Gelfand-MacPherson correspondence (2.2.2) extends to an isomorphism of Chow quotients*

$$(2.2.5) \quad \Lambda : G(k, n)//H \longrightarrow (P^{k-1})^n//GL(k).$$

This fact permits one to apply the information about behaviour of $(n-1)$ -dimensional torus orbits (which may be obtained by techniques of toric varieties and A-resultants [30]), to the study of (k^2-1) -dimensional orbits of $GL(k)$ which are at first glance harder to understand.

(2.2.6) Corollary. *Every cycle in $(P^{k-1})^n//GL(k)$ is a sum of closures of some (k^2-1) -dimensional orbits with multiplicities 0 or 1.*

Before starting to prove Theorem 2.2.5, let us give a simple matrix interpretation of the correspondence in question.

(2.2.7) Let $M(k, n)$ be the vector space of all complex k by n matrices, $M_0(k, n) \subset M(k, n)$ - the space of matrices of rank k , and $M'(k, n) \subset M(k, n)$ - the space of matrices whose every row is a non-zero vector in \mathbf{C}^k . The group $GL(k)$ acts on $M(k, n)$ from the right, and $(\mathbf{C}^*)^n \subset GL(n)$ - from the left and we have the identifications

$$(2.2.8) \quad GL(k) \backslash M_0(k, n) = G(k, n), \quad M'(k, n) / (\mathbf{C}^*)^n = (P^{k-1})^n.$$

The Gelfand- MacPherson correspondence comes from consideration of both types of orbits in (2.2.5) as double $(GL(k), (\mathbf{C}^*)^n)$ - orbits in $Mat(k, n)$.

(2.2.9) Let us carry on these considerations for Chow quotients. Note that each $(GL(k), (\mathbf{C}^*)^n)$ -orbit in the vector space $M(k, n)$ is invariant under multiplications by scalars (in this vector space) and thus may be identified with a subvariety in the projectivization $P(M(k, n))$. Instead of double orbits we can speak about left orbits of the product $GL(k) \times (\mathbf{C}^*)^n$. Consider the Chow quotient $P(M(k, n)) // GL(k) \times (\mathbf{C}^*)^n$. To prove Theorem 2.2.4 it suffices to construct isomorphisms

$$(2.2.10) \quad G(k, n) // (\mathbf{C}^*)^n \xrightarrow{\alpha} P(M(k, n)) // GL(k) \times (\mathbf{C}^*)^n \xleftarrow{\beta} (P^{k-1})^n // GL(k).$$

(2.2.11) The existence of these morphisms does not present any problem.

The morphism α associates to any cycle $Z = \sum c_i Z_i$ in $G(k, n) // (\mathbf{C}^*)^n$ the cycle $\sum \overline{p^{-1}(Z_i)}$ where $p : P(M_0(k, n)) \rightarrow G(k, n)$ is the projection from (2.2.8) (The multiplicities c_i all are equal to 1 by Proposition 1.2.15). Similarly, the morphism β associates to any cycle $W = \sum m_i W_i$ in $(P^{k-1})^n // GL(k)$ the cycle $\sum m_i q^{-1}(\overline{W_i})$, where $q : P(M'(k, n)) \rightarrow (P^{k-1})^n$ is the other projection arising from (2.3). To show that α and β thus defined are regular maps, it suffices to apply Barlet's criterion of analytic dependence of a cycle on a parameter [3]. Since α and β are both given by inverse images in fibrations, this criterion is trivially applicable.

(2.2.12) Let us show that α is an isomorphism. To do this, note that any generic $GL(k) \times (\mathbf{C}^*)^n$ - orbit in $P(M(k, n))$ has dimension $(n-1)(k^2-1)$. Each component of a cycle Z from $P(M(k, n)) // GL(k) \times (\mathbf{C}^*)^n$ is the closure of a single orbit which therefore should be the inverse image of an orbit of maximal dimension in $G(k, n)$. The algebraic cycle W formed by these orbits lies clearly in the Chow quotient $G(k, n) // (\mathbf{C}^*)^n$ and this is the unique element of this Chow quotient such that $\alpha(W) = Z$. This proves that α is bijective on \mathbf{C} -points. Denote by α^{-1} the inverse map. To prove that α is indeed an isomorphism of algebraic varieties we need to prove that α^{-1} is regular too (which need not necessarily be the case if the varieties involved are not normal). However, this again follows from Barlet's criterion similarly to the proof of Theorem 1.4.12.

Similarly we prove that β is an isomorphism. Theorem 2.2.4 is proven.

(2.3) Duality (or association).

It is known classically (since A.B.Coble [11]) that projective equivalence classes of configurations of n ordered points in P^{k-1} are in bijection with projective equivalence classes of configurations of n points in P^{n-k-1} . This correspondence is known as the *association* [11,14] and was used in the context of matroid theory (see [21], §2.3).

The most transparent way to define the association is via the Gelfand-MacPherson correspondence.

(2.3.1) Let us identify the dual subspace to the coordinate space \mathbf{C}^n with \mathbf{C}^n by means of the standard pairing. By considering orthogonal complements to k -dimensional subspaces we obtain an isomorphism $G(k, n) \cong G(n-k, n)$. The torus $H = (\mathbf{C}^*)^n$ acts in both Grassmannians and the said isomorphism is H -equivariant. Hence it induces the isomorphism of coset spaces $G(k, n)/H \rightarrow G(n-k, n)/H$. Taking into account the Gelfand - MacPherson isomorphism (2.2.2), we obtain the following isomorphism

$$(2.3.2) \quad A_{k,n} : (P^{k-1})_{max}^n / GL(k) \rightarrow (P^{n-k-1})_{max}^n / GL(n-k)$$

where the subscript "max" means the set of points whose orbits have the maximal dimension. The isomorphism (2.3.2) will be called the *association isomorphism*. By construction, this system of isomorphisms is involutive i.e. $A_{k,n} \circ A_{n-k,n} = Id$.

(2.3.3) If $(x_1, \dots, x_n), (y_1, \dots, y_n)$ are n -tuples of points in P^{k-1} and P^{n-k-1} respectively then we shall say that (y_i) is associated to (x_i) (and vice versa) if their orbits under projective transformations have maximal dimensions and are taken into each other by the association isomorphism.

(2.3.4) Explicitly the configuration associated to (x_i) can be calculated as follows. Let $\mathbf{x}_i \in \mathbb{C}^k$ be vectors whose projectivizations are x_i . By definition, we have to find a k -dimensional subspace $L \subset \mathbb{C}^n$ and an isomorphism $\mathbb{C}^k \rightarrow L^*$ which takes \mathbf{x}_i into the restriction of the i -th coordinate function to L . Then the vectors $\mathbf{y}_i \in \mathbb{C}^n/L$ defined as the projections of the standard basis vectors, will represent the associated configuration. In other words, we have to find a complete $(n-k)$ -dimensional system of linear relations between \mathbf{x}_i , namely $y_{j1}\mathbf{x}_1 + \dots + y_{jn}\mathbf{x}_n = 0, j = 1, \dots, n-k$. Vectors \mathbf{y}_i representing the associated configuration (y_i) are given by columns of the matrix $||y_{ji}||$. This gives the following criterion.

(2.3.5) Proposition. *Let $\mathbf{x}_i \in \mathbb{C}^k, \mathbf{y}_i \in \mathbb{C}^{n-k}$ are n -tuples of vectors such that the corresponding configurations of points $x_i \in P^{k-1}, y_i \in P^{n-k-1}$ have orbits of maximal dimension. Then (y_i) is associated to (x_i) if and only if there is a unique, up to constant, linear relation in $\mathbb{C}^k \otimes \mathbb{C}^{n-k}$:*

$$\sum_i \lambda_i (\mathbf{x}_i \otimes \mathbf{y}_i) = 0$$

which is such that all $\lambda_i \neq 0$.

This can be reformulated as follows.

(2.3.6) Let P be some projective space and $C \subset P$ be some finite subset. We shall say that C is a *circuit* (in the sense of matroid theory, see [21]) if C is projectively dependent but any its proper subset is projectively independent.

(2.3.8) Reformulation. *Let $x = (x_1, \dots, x_n) \in (P^{k-1})^n$ and $y = (y_1, \dots, y_n) \in (P^{n-k-1})^n$ be two n -tuples whose orbits with respect to projective transformations have maximal dimensions (i.e. k^2-1 and $(n-k)^2-1$). Consider the Segre embedding $P^{k-1} \times P^{n-k-1} \hookrightarrow P^{k(n-k)-1}$. Then y is associated to x if and only if the points $(x_i, y_i) \in P^{k(n-k)-1}$ form a circuit.*

"Normally" one would expect that points (x_i, y_i) are projectively independent.

(2.3.9) For the case of $2k$ points in P^{k-1} the source and the target of the association isomorphism are the same, so it is possible to speak about a configuration being self-associated. The following characterization of self-associated configurations due to A.Coble [11] is a corollary of Reformulation 2.3.8.

(2.3.10) Corollary. *Let $x = (x_1, \dots, x_{2k}) \in (P^{k-1})^{2k}$ be a $2k$ -tuple. Consider the 2-fold Veronese embedding $P^{k-1} \hookrightarrow P(S^2 \mathbb{C}^k)$. Then the configuration x is self-associated if and only if the images of x_i in the Veronese embedding form a circuit.*

(2.3.11) Example. Let $k = 2$. The association induces an isomorphism between the set of projective equivalence classes of n -tuples of distinct points on P^1 and the set of projective equivalence classes of n -tuples of distinct points in P^{n-3} . This correspondence can be seen geometrically as follows ([14], Ch.III, §2, Proposition 2).

Given n points $y_1, \dots, y_n \in P^{n-2}$ in general position, there is a unique rational normal curve (Veronese curve, for short) in P^{n-3} through these points. This curve is isomorphic to P^1 and hence y_i represent on it a configuration of n points in P^1 , which is the associated configuration to that of y_i in P^{n-3} . Conversely, given n distinct points on P^1 , we consider the $n-3$ -fold Veronese embedding of P^1 . It identifies P^1 with a Veronese curve in P^{n-3} . The images y_i of x_i in this embedding are in general position as it may be seen by calculating the Vandermonde determinant. These points represent the configuration associated to that of x_i on P^1 .

(2.3.12) Example. Let x_1, \dots, x_6 be a configuration of 6 points in P^2 in general position. Corollary 2.3.10 means in this case that the configuration (x_i) is self associated if and only if the six points x_i lie on a conic. Further examples can be found in [11,14].

(2.3.13) Theorem 2.2.4 implies that the association isomorphism extends to the Chow quotients of the spaces of projective configurations. In other words, we have the following fact.

(2.3.14) Corollary. *There is an isomorphism of Chow quotients*

$$(P^{k-1})^n // GL(k) \rightarrow (P^{n-k-1})^n // GL(n-k).$$

(2.4) Gelfand-MacPherson correspondence and Mumford's quotients.

For completeness sake we include here the comparison of Mumford's quotients of $G(k, n)$ modulo torus and of P^{k-1} modulo projective transformations.

(2.4.1) First of all, the theory of Mumford is sensitive not only to the structure of orbits but also to the choice of group generating these orbits. In order that things behave well, we should consider the subgroup $H_1 = \{(t_1, \dots, t_n) \in (\mathbb{C}^*)^n : \prod t_i = 1\}$ acting on $G(k, n)$ and the subgroup $SL(k) \subset GL(k)$ acting on $(P^{k-1})^n$.

(2.4.2) Recall (see n.0.4) that to define Mumford's quotient by any group G acting on any variety X we should fix two things: an ample line bundle \mathcal{L} on X and a linearization i.e. an extension α of G -action to \mathcal{L} .

(2.4.3) First consider the H_1 -action on the Grassmannian $G(k, n)$. The Picard group of $G(k, n)$ is generated by the sheaf $\mathcal{O}(1)$ in the Plücker embedding. so there is essentially no freedom in choosing \mathcal{L} . We set $\mathcal{L} := \mathcal{O}(1)$. For this choice of \mathcal{L} a linearization is given by an integral vector $a = (a_1, \dots, a_n)$ defined modulo multiples of $(1, \dots, 1)$. Denote $t^a = t_1^{a_1} \dots t_n^{a_n}$ the character of H_1 corresponding to a . The H_1 -action on \mathbb{C}^n corresponding to a , has the form

$$(t_1, \dots, t_n) \mapsto \text{diag}(t^a \cdot t_1, \dots, t^a \cdot t_n).$$

This action induces an H_1 -action on $\bigwedge^k \mathbb{C}^n$ which is the linearization corresponding to a .

(2.4.4) Denote by $A(k, n)$ the coordinate ring of $G(k, n)$ in the Plücker embedding. It is well-known [14] that $A(k, n)$ can be identified with the ring of polynomials $\Phi(M)$ in entries of an indeterminate $(k \times n)$ -matrix $M = ||v_{ij}||$ which satisfy the condition $\Phi(gM) = \Phi(M)$ for any $g \in SL(k)$. In particular, the Plücker coordinate p_I corresponds to the polynomial in v_{ij} given by the $(k \times k)$ -minor of M on columns from I .

The Mumford quotient $(G(k, n)/H_1)_{\mathcal{O}(1), a}$ is, by definition, the projective spectrum $\text{Proj}(A(k, n)^{H_1})$ of the invariant subring in $A(k, n)$.

(2.4.5) Consider now the $SL(k)$ -action on $(P^{k-1})^n$. For any integral vector $a = (a_1, \dots, a_k)$ denote by $\mathcal{O}(a) = \mathcal{O}(a_1, \dots, a_n)$ the line bundle on $(P^{k-1})^n$ whose local sections are functions multihomogeneous of degrees (a_1, \dots, a_k) . It is well-known [24,25] that bundles $\mathcal{O}(a)$ exhaust the Picard group of $(P^{k-1})^n$. For any $a \in \mathbb{Z}^n$ the bundle $\mathcal{O}(a)$ has exactly one $SL(k)$ -linearization since the center of $SL(k)$ has dimension 0. This linearization will be denoted by λ .

(2.4.6) The bundle $\mathcal{O}(a)$ is ample if and only if all $a_i > 0$. Assuming that this is the case, let $B(k, n, a) = \bigoplus_d B(k, n, a)_d$ be the homogeneous coordinate ring of $(P^{k-1})^n$ in the projective embedding given by $\mathcal{O}(a)$. The degree d homogeneous component $B(k, n, a)_d$ of this ring consists of polynomials $F(w_1, \dots, w_n)$ in coordinates on n vectors $w_i \in \mathbb{C}^k$ such that $F(t_1 w_1, \dots, t_n w_n) = t^{da} F(w_1, \dots, w_n)$ for any $t_i \in \mathbb{C}^*$. Writing the vectors in coordinate form as columns $w_i = (v_{1i}, \dots, v_{ki})^t$, we realize elements of $B(k, n, a)_d$ as polynomials $F(M)$ in entries of an indeterminate $(k \times n)$ -matrix $M = ||v_{ij}||$ such that $F(M \cdot t) = t^{da} F(M)$. The Mumford's quotient $((P^{k-1})^n / SL(k))_{\mathcal{O}(a), \lambda}$ is, by definition, the projective spectrum $\text{Proj}(B(k, n, a)^{SL(k)})$.

(2.4.7) Theorem. Let $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ be an integer vector. If at least one $a_i \leq 0$ then the Grassmannian $G(k, n)$ does not contain a -stable orbits. If all a_i are positive then we have an isomorphism of Mumford quotients

$$(G(k, n)/H_1)_{\mathcal{O}(1), a} \cong ((P^{k-1})^n/SL(k))_{\mathcal{O}(a), \lambda}.$$

Proof: Both varieties are projective spectra of the same ring $R = \bigoplus R_d$ where R_d consists of polynomials $\Phi(M)$, $M \in \text{Mat}(k \times n)$ such that

$$\Phi(M.t) = t^{da}\Phi(M), t \in (\mathbb{C}^*)^n, \quad \Phi(gM) = \Phi(M), g \in SL(k).$$

(2.4.9) Remark. The algebra generated by $k \times k$ -minors of an indeterminate $k \times n$ matrix is known as the bracket algebra. It traditionally makes its appearance in two seemingly different contexts. The first appearance is as the coordinate ring of the Grassmannian $G(k, n)$ in the Plücker embedding. The other is in the study in the (semi-) invariants of system of vectors by symbolic method (see [14]). However, the idea of serious use of Grassmannians for the study of projective configurations appeared only fairly recently in the papers of Gelfand and MacPherson.

Chapter 3. VISIBLE CONTOURS OF (GENERALIZED) LIE COMPLEXES AND VERONESE VARIETIES.

The Grassmannian point of view on projective configurations (i.e. the Gelfand- MacPherson isomorphism, see §3.2) simplifies considerably the study of the Chow quotient. Indeed, instead of working with $(k^2 - 1)$ -dimensional subvarieties on $(P^{k-1})^n$ which are closures of $PGL(k)$ -orbits, we have to deal with Lie complexes in $G(k, n)$ which are $(n - 1)$ -dimensional toric varieties.

(3.1) Visible contours and the logarithmic Gauss map.

(3.1.1) There is a classical method (see e.g., [2,24,27]) to analyze any complex of projective subspaces in P^{n-1} i.e. an $(n - 1)$ -dimensional subvariety $Z \subset G(k, n)$. Namely, take any point $p \in P^{n-1}$ and consider the subvariety

$$Z_p = \{L \in Z : p \in L\}$$

of subspaces in Z which contain p . This subvariety will be called the *visible contour* of Z at p .

Let $G(k - 1, n - 1)_p \subset G(k, n)$ be the variety of all $(k - 1)$ -dimensional projective subspaces containing p . It is clear that $Z_p = Z \cap G(k - 1, n - 1)_p$.

(3.1.2) Still another step towards a visualization of the complex Z at a point p is done as follows. Let P_p^{n-1} be the space of lines in P^{n-1} through p . Then $G(k - 1, n - 1)_p$ is identified with the variety of all $(k - 2)$ -dimensional projective subspaces in P_p^{n-2} . We define the *visible sweep* of Z at p to be the subvariety $Sw_p(Z) \subset P_p^{n-2}$ which is the union of all the projective subspaces corresponding to elements of Z_p .

(3.1.3) **Remarks.** a) If $k = 2$ then Z consists of lines in P^{n-1} . The lines belonging to the complex Z can be thought of as rays of light piercing the space, so Z_p is the contour which is seen by an observer at a point p . In this case the visible contour is the same as the visible sweep.

b) Although the consideration of the locus Z_p is classical, there seems to be no good name in the literature for it. The term "complex cone" which is used sometimes [27] for the union of subspaces from Z_p (i.e. the cone over the visible sweep, in our terminology) is obviously unsuitable for modern usage.

c) Dually, one can take any hyperplane $\Pi \subset P^{n-1}$ and consider the locus of subspaces from Z which lie in Π .

(3.1.4) Since $\text{codim} G(k - 1, n - 1)_p = n - k$, we find that for any complex $Z \subset G(k, n)$ and a generic p the variety Z_p will have dimension $p - 1$. Thus Z_p will be a curve if Z consists of lines, a surface if Z consists of planes etc.

(3.1.5) We shall use the approach of visible contours to study Lie complexes and, more generally, closures of arbitrary $(n - 1)$ -dimensional torus orbits in $G(k, n)$ (such closures can be components of generalized Lie complexes). Visible sweeps of Lie complexes will be studied in §(3.6).

(3.1.6) Let us realize our torus $H = (\mathbf{C}^*)^n / \mathbf{C}^*$ as an open subset in P^{n-1} consisting of points with all homogeneous coordinates non-zero. The point $e = (1 : \dots : 1) \in P^{n-1}$ becomes the unit in H . Denote by \mathbf{h} the Lie algebra of H . It is identified with the tangent space to P^{n-1} at e . Explicitly, $\mathbf{h} = \mathbf{C}^n / \{(a, \dots, a)\}$. For any $x \in H$ let $\mu_x : H \rightarrow H$ be the operator of multiplication by x .

Any subvariety $L \subset P^{n-1}$ not lying inside a coordinate hyperplane gives a subvariety $L \cap H$ in the algebraic group H .

(3.1.7) **Definition.** Let $X \subset H$ be a p -dimensional algebraic subvariety. The logarithmic Gauss map of X is the (rational) map $\gamma_X : X \rightarrow G(p, \mathbf{h})$ which takes a smooth point $x \in X$ to the p -dimensional subspace $d(\mu_x^{-1})(T_x X) \subset T_e H = \mathbf{h}$ — the translation to the unity of the tangent space $T_x X \subset T_x H$.

The name "logarithmic" comes from the fact that explicit formula for γ_X involves logarithmic derivatives (see below).

(3.1.8) Let $L \subset P^{n-1}$ be a $(k-1)$ -dimensional projective subspace not lying in a coordinate hyperplane. The orbit closure $\overline{H.L}$ has dimension $n-1$ i.e. this is a complex. Since this complex is H -invariant, its visible contour $(\overline{H.L})_p$ at any point $p \in P^{n-1}$ with all coordinates non-zero, will be isomorphic to the visible contour at the point $e = (1 : \dots : 1)$.

Before stating the next proposition let us note that the Grassmannian $G(k-1, n-1)_e$, where visible contours lie, is canonically identified with $G(k-1, \mathbf{h})$. (Correspondingly, the space P_e^{n-2} where visible sweeps lie, is $P(\mathbf{h})$.)

(3.1.9) Proposition. a) If the subspace L does not lie in a coordinate hyperplane then the visible contour $(\overline{H.L})_e$ coincides with the closure of the image of $L \cap H$ under the logarithmic Gauss map. In particular, this visible contour is a rational variety.

b) The intersection of $\overline{H.L}$ with the sub-Grassmannian $G(k-1, n-1)_e$ is proper and transversal at its generic point.

Proof: a) Neither the complex $\overline{H.L}$ nor the image of $L \cap H$ under the logarithmic Gauss map will change if we translate L by the H -action. So we can (and will) assume that L contains the point $e = (1 : \dots : 1)$. For $h \in H$ the translated subspace $h^{-1}L$ contains e if and only if $h \in L$. Thus the variety $(H.L)_e = (H.L) \cap G(k-1, n-1)_e$ consists of subspaces $h^{-1}L, h \in L$. In other words, $(H.L)_e$ is the image of the map $L \cap H \rightarrow G(k-1, n-1)_e$ taking $h \in L \cap H$ to the subspace $h^{-1}L$. This map clearly coincides with the logarithmic Gauss map.

b) If Z is any complex in $G(k, n)$ then the assertion will be true for the intersection of Z with $G(k-1, n-1)_p$, where $p \in P^{n-1}$ is a generic point. In our case, due to the invariance under the torus action, the situation at any $(p = (p_1 : \dots : p_n))$ with all $p_i \neq 0$ is the same as at e .

(3.1.10) Theorem. Suppose that $L \subset P^{n-1}$ is a $(k-1)$ -dimensional subvariety belonging to the generic stratum $G^0(k, n)$. Then the logarithmic Gauss map $\gamma_{L \cap H}$ extends to a regular embedding $L \hookrightarrow G(k-1, \mathbf{h}) = G(k-1, n-1)_e$. In other words, the visible contour $(\overline{H.L})_e$ is identified with L itself.

Proof: First let us show that the logarithmic Gauss map γ extends to all of L as a regular map. This will be done by calculation in coordinates which we shall also use on other occasions.

(3.1.11) Let x_1, \dots, x_n be homogeneous coordinates in P^{n-1} . Let $p = (y_1 : \dots : y_n) \in L$ be any point. Since L lies in the generic stratum, there are $1 \leq i_1 < \dots < i_{k-1} \leq n$ such that $y_j \neq 0$ for $j \notin \{i_1, \dots, i_{k-1}\}$. After renumbering variables we can (and will) assume that y_k, \dots, y_n are non-zero. Consider the affine space $L - \{x_n = 0\}$ which contains our point p . Introduce in this space affine coordinates z_1, \dots, z_{k-1} where $z_i = x_i/x_n$. We can set x_n to be 1 on $L - \{x_n = 0\}$ and express all the other coordinates as affine-linear functions in z_i i.e.

$$x_i = z_i, i = 1, \dots, k-1, \quad x_i = f_i(z) = \sum_{\nu=1}^{k-1} a_{i\nu} z_\nu + a_k, i = k, \dots, n-1.$$

We also set $f_i(z) = z_i$ for $i = 1, \dots, k-1$.

(3.1.12) We identify the torus $H = (\mathbf{C}^*)^n / \mathbf{C}^* \subset P^{n-1}$ with the set $\{(t_1, \dots, t_{n-1}, 1) \in (\mathbf{C}^*)^n\}$ i.e. with $(\mathbf{C}^*)^{n-1}$. Its Lie algebra is therefore identified with \mathbf{C}^{n-1} . In this notation the map γ takes a point $z = (z_1, \dots, z_{k-1})$ to the $(k-1)$ -dimensional subspace in \mathbf{C}^{n-1} spanned by the rows of $(k-1)$ by $(n-1)$ -matrix $\|\partial \log f_i / \partial z_j\|, i = 1, \dots, n-1, j = 1, \dots, k-1$. We can multiply the j -th row by z_j without changing this subspace. After this the matrix takes the form

$$(3.1.13) \quad \begin{pmatrix} 1 & 0 & \dots & 0 & \frac{a_{k1}z_1}{f_k(z)} & \dots & \dots & \frac{a_{n-1,1}z_1}{f_{n-1}(z)} \\ 0 & 1 & \dots & 0 & \frac{a_{k2}z_2}{f_k(z)} & \dots & \dots & \frac{a_{n-1,2}z_2}{f_{n-1}(z)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \frac{a_{k,k-1}z_{k-1}}{f_k(z)} & \dots & \dots & \frac{a_{n-1,k-1}z_{k-1}}{f_{n-1}(z)} \end{pmatrix}$$

This matrix is clearly regular near our point p since $f_k(p), \dots, f_{n-1}(p)$ are non-zero. The rank of the matrix (3.1.13) being equal k , we deduce that γ is regular at p . We have proven that γ extends to a regular morphism $L \rightarrow G(k-1, n-1)$.

(3.1.14) Let us finish the proof of Theorem 3.1.10 by showing that the logarithmic Gauss map γ is an embedding. Consider the set of all $(k-1)$ by $(n-1)$ matrices of which the first $(k-1)$ columns form the unit $(k-1)$ by $(k-1)$ matrix. The entries of the remaining $(n-k)$ columns are independent affine coordinates in the open Schubert cell $C^{(k-1)(n-k)} \subset G(k-1, n-1)$. Let us show that entries of any given column of (3.1.13) whose number is greater than k , alone suffice to separate all points of L . Indeed, consider, say, the p -th column, $p > k$ and regard its entries as defining a transformation $(z_1, \dots, z_{k-1}) \mapsto (s_1, \dots, s_{k-1})$ where $s_i = \frac{a_{pi} z_i}{\sum_{\nu=1}^{k-1} a_{p\nu} z_\nu}$. This is a projective transformation corresponding to the k by k matrix

$$(3.1.15) \quad T_p = \begin{pmatrix} a_{p1} & 0 & 0 & \dots & 0 \\ 0 & a_{p2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & a_{p3} & \dots & a_{pk} \end{pmatrix}$$

Since our subspace $L \subset P^{n-1}$ belongs to the generic stratum, every entry of the matrix $a_{ij}, i = 1, \dots, k-1, j = 1, \dots, n-k$, is non-zero. Hence the matrix (3.1.15) defines a non-degenerate projective transformation and separates the points (as well as the tangent vectors). Theorem 3.1.10 is proven.

(3.2) Bundles of logarithmic forms on P^{k-1} and visible contours.

(3.2.1) It is well known [24] that maps from any projective variety X to Grassmannians of the form $G(r, V), \dim(V) > r$, are in correspondence with rank r vector bundles on X . More precisely, given such a bundle E we consider the vector space $V = H^0(X, E)^*$. Suppose that E is generated by global sections and let N be the dimension of V . Define a map $\phi_E : X \rightarrow G(r, V) = G(N-r, H^0(X, E))$ as follows. For a point $x \in X$ the value $\phi_E(x)$ is the codimension r subspace in $H^0(X, E)$ consisting of all the section vanishing at x . Conversely, suppose given a map $\phi : X \rightarrow G(r, V)$. Let S be the tautological rank r bundle on $G(r, V)$ (whose fiber at a subspace L is L itself). Associate to ϕ the bundle $\phi^* S^*$ on X .

The bundles on P^{k-1} corresponding to visible contours of Lie complexes have the following description.

(3.2.2) Let $M = (M_1, \dots, M_n)$ be a configuration of n hyperplanes in P^{k-1} which are in general position. Then M is a divisor with normal crossings and we can define the sheaf $\Omega_{P^{k-1}}^1(\log M)$ of differential 1-forms on P^{k-1} with logarithmic poles along M , see [13]. By definition, the space of sections of this sheaf near a point $x \in P^{k-1}$ is generated (over $\mathcal{O}_{P^{k-1}, x}$) by 1-forms regular at x and also by forms $d \log f_i$ where f_i are local equations of hyperplanes from M containing x .

An important property of the sheaf $\Omega_{P^{k-1}}^1(\log M)$ is that it is locally free i.e. can be seen as a rank $(k-1)$ vector bundle over P^{k-1} .

(3.2.3) Proposition. *Let $M = (M_1, \dots, M_n)$ be a configuration of n hyperplanes in the projective space $L := P^{k-1}$ which are in general position. and let f_i be a linear form defining M_i . Then:*

a) *The space $W = H^0(L, \Omega_L^1(\log M))$ has dimension $n-1$ and consists of forms*

$$\sum_i \alpha_i d \log f_i = d \log \prod_i f_i^{\alpha_i}, \alpha_i \in \mathbb{C}, \sum \alpha_i = 0.$$

Higher cohomology groups of $\Omega_L^1(\log M)$ vanish.

b) *The vector bundle $E = \Omega_L^1(\log M)$ defines a regular embedding $\phi_E : L \hookrightarrow G(k-1, W^*)$.*

c) *Suppose that L is realized as a subspace in the coordinate P^{n-1} so that M_i is given by the vanishing of the i -th coordinate. Then ϕ_E coincides with the (extension of) the logarithmic Gauss map $\gamma_{L \cap H}$, and the image $\phi_E(L)$ coincides with the visible contour of the Lie complex $\overline{H.L}$.*

(3.2.4) *Proof of (3.2.3), a):* The sheaf Ω_L^1 of regular 1-forms is obviously a subsheaf of $\Omega_L^1(\log M)$. To describe the quotient, we shall, following P. Deligne [13], denote by \tilde{M} the disjoint union of hyperplanes in M and let $\epsilon : \tilde{M} \rightarrow L$ be the natural map. Then we have the exact sequence

$$(3.2.5) \quad 0 \rightarrow \Omega_L^1 \rightarrow \Omega_L^1(\log M) \xrightarrow{Res} \epsilon_* \mathcal{O}_{\tilde{M}} \rightarrow 0$$

where Res is the Poincaré residue morphism, see [13]. Consider the corresponding long exact sequence of cohomology. The equality $H^0(L, \epsilon_* \mathcal{O}_{\tilde{M}}) = \mathbb{C}^n$ means that the residue of a global logarithmic form along each M_i is constant. The sum of the residues is given by the boundary map $H^0(L, \epsilon_* \mathcal{O}_{\tilde{M}}) \rightarrow H^1(L, \Omega^1) = \mathbb{C}$ should be zero. Since the forms exhibited in the formulation are indeed global sections of our sheaf, we obtain the statement about H^0 . The vanishing of higher H^i follows from known information about the cohomology of the sheaf \mathcal{O} on P^{k-2} and Ω^1 on P^{k-1} .

(3.2.6) *Proof of (3.2.3) b) and c):* We can assume that M is given by the intersection of an embedded $L = P^{k-1} \subset P^{N-1}$ with coordinate hyperplanes $\{x_i = 0\}$. Then, by n.a), the basis of $H^0(L, \Omega_L^1(\log M))$ is given by 1-forms $d \log(x_i/x_n)$, $i = 1, \dots, n-1$. We identify the space of section with \mathbb{C}^{n-1} by using this basis. Now looking at explicit formula (3.1.12), we find that the map $\phi_E : L \rightarrow G(k-1, n-1)$ is defined by the formula identical to that of the logarithmic Gauss map.

(3.2.7) **Proposition.** *The Chern classes of $E = \Omega_{P^{k-1}}^1(\log(M_1 + \dots + m_n))$ have the form*

$$c_i(E) = \binom{n-k+i}{i} \in H^{2i}(P^{k-1}, \mathbb{Z}) = \mathbb{Z}.$$

In particular, the determinant (= top exterior power) of E is isomorphic to $\mathcal{O}_{P^{k-1}}(n-k)$.

Proof: This follows at once from the exact sequence (3.2.5).

(3.2.8) **Example.** Consider a Lie complex in $G(2, n)$, the Grassmannian of lines in P^{n-1} . Let this complex have the form $Z = \overline{H, l}$, where l is a line belonging to the generic stratum. The visible contour Z_e lies in the projective space P_e^{n-2} of all lines in P^{n-1} through the point e . Proposition 3.2.3 means that Z_e is a rational normal curve (Veronese curve, for short) in P_e^{n-2} . More precisely, it is the embedding of $l = P^1$ defined by the invertible sheaf $\Omega_l^1(\log(m_1 + \dots + m_n)) \cong \mathcal{O}_l(n-2)$. Here $m_i \in l$ is the point of intersection of l with the coordinate hyperplane $\{x_i = 0\}$.

(3.3) Visible contour as a Veronese variety in the Grassmannian.

(3.3.1) Recall [24,25] the d -fold Veronese embedding

$$(3.3.2) \quad P^{k-1} = P(\mathbb{C}^k) \hookrightarrow P(S^d \mathbb{C}^k), \quad x \mapsto x^d$$

of P^{k-1} into the projectivization of $(S^d \mathbb{C}^k)$, the space of homogeneous degree d polynomials in k variables. This is the embedding corresponding to the line bundle $\mathcal{O}(d)$. We shall say that a $(k-1)$ -dimensional subvariety $X \subset P^N$ is a d -fold Veronese variety if there is a projective equivalence $P^N \cong P(S^d \mathbb{C}^k)$ taking X into the image of (3.3.2). A Veronese curve in P^N is the same as a rational normal curve of degree N .

(3.3.3) The dimension of the projective space $P(S^d \mathbb{C}^k)$ of the d -fold Veronese embedding (3.3.2) equals $N = \binom{d+k-1}{d-1} - 1$. Note that the same dimension is attained by the projective space of the Plücker embedding of the Grassmannian $G(k-1, d+k-1)$. Therefore it makes sense to look for Veronese subvarieties in $P(\binom{d+k-1}{d-1} - 1) = P(\bigwedge^{k-1} \mathbb{C}^{d+k-1})$ which lie on the Grassmannian.

(3.3.4) We shall say that a $(k-1)$ -dimensional subvariety $X \subset G(k-1, d+k-1)$ is a d -fold Veronese variety if it becomes such after the Plücker embedding of $G(k-1, d+k-1)$.

(3.3.5) Proposition. Let $M = (M_1, \dots, M_n)$ be a configuration of hyperplanes in P^{k-1} in general position, $E = \Omega_{P^{k-1}}^1(\log M)$ -the corresponding logarithmic bundle and $\phi_E : P^{k-1} \rightarrow G(k-1, n-1)$ - the embedding corresponding to E . Then $\phi_E(P^{k-1})$ is an $(n-k)$ -fold Veronese variety in $G(k-1, n-1)$.

Proof: Let us construct an isomorphism of linear spaces $\bigwedge^{k-1} \mathbf{C}^{n-1} \rightarrow S^{n-k} \mathbf{C}^k$ taking $\phi_E(P^{k-1})$ into the standard Veronese variety. Let S be the tautological rank $(k-1)$ bundle on $G(k-1, n-1)$. Then $E = \phi_E^*(S^*)$ and hence

$$\phi_E^*\left(\bigwedge^{k-1} S^*\right) = \bigwedge^{k-1} E = \mathcal{O}_{P^{k-1}}(n-k)$$

by Proposition 3.2.7. Thus we obtain a linear map of restriction

$$(3.3.6) \quad \bigwedge^{k-1} \mathbf{C}^{n-1} = H^0(G(k-1, n-1), \bigwedge^{k-1} S^*) \xrightarrow{r} H^0(P^{k-1}, \phi_E^*\left(\bigwedge^{k-1} S^*\right)) \cong \\ \cong H^0(P^{k-1}, \mathcal{O}(n-k)) = S^{n-k} \mathbf{C}^k.$$

(3.3.7) Let us show that the restriction map r in (3.3.6) is an isomorphism.

Since spaces in both sides have the same dimension, it suffices to show that r is injective i.e. that the variety $X = \phi_E(P^{k-1})$ does not lie in any hyperplane in $P(\bigwedge^{k-1} \mathbf{C}^{n-1})$. Take an affine chart in P^{k-1} in which the last hyperplane M_n is the infinite one. All the other hyperplanes M_i are then defined by vanishing of affine-linear functions $f_i, i = 1, \dots, n-1$ on $\mathbf{C}^{k-1} = P^{k-1} - M_n$. The fact that the variety X lies in a hyperplane means that there is a collection of numbers $a_{i_1, \dots, i_{k-1}}$, not all of them zero, such that the meromorphic $(k-1)$ -form

$$\Omega = \sum_{1 \leq i_1 < \dots < i_{k-1} \leq n-1} a_{i_1, \dots, i_{k-1}} d\log f_{i_1} \wedge \dots \wedge d\log f_{i_{k-1}}$$

on \mathbf{C}^{k-1} vanishes identically. However, the coefficient $a_{i_1, \dots, i_{k-1}}$ can be read off Ω as the residue at the intersection point $M_{i_1} \cap \dots \cap M_{i_{k-1}}$ so all the coefficients should be zero. Proposition 3.3.5. is proven.

(3.3.8) Let \mathbf{h} be the quotient of \mathbf{C}^n by the subspace of $(a, \dots, a), a \in \mathbf{C}$. Note that this subspace is canonically identified with the Lie algebra of the torus H and with the tangent space to P^{n-1} at the point $e = (1, \dots, 1)$. We shall denote, therefore, by $G(k-1, n-1)_e$ the Grassmannian of $(k-1)$ -dimensional subspaces in \mathbf{h} .

(3.3.9) Definition. By a special Veronese subvariety in $G(k-1, n-1)_e$ we shall mean a subvariety of the form $\phi_E(P^{k-1})$, where:

- a) $E = \Omega_{P^{k-1}}^1(\log M)$ is the logarithmic bundle corresponding to some configuration $M = (M_1, \dots, M_n)$ of hyperplanes in general position;
- b) The space $H^0(E)$ is identified with $\{(a_1, \dots, a_n) \in \mathbf{C}^n : \sum a_i = 0\}$ as in Proposition 3.2.3, and its dual - with \mathbf{h} .

Thus the notion of a special Veronese variety makes an explicit appeal to a choice of coordinate system.

(3.3.10) Note that by Proposition 3.2.3 special Veronese varieties are precisely the visible contours of Lie complexes in $G(k, n)$. In particular, these variety define, around a generic point of $G(k-1, n-1)_e$, a foliation with $k-1$ - dimensional fibers which is just the intersection of $G(k-1, n-1)_e$ with the foliation given by the orbits of H . Let us note also the following corollary.

(3.3.11) Corollary. The set $G^0(k, n)/H = (P^{k-1})_{gen}^n/GL(k)$ of projective equivalence classes of configuration of n hyperplanes in P^{k-1} in general position is in one-to-one correspondence with the set of special Veronese varieties in $G(k-1, n-1)_e$. This correspondence takes a configuration $M = (M_1, \dots, M_n)$ into the subvariety $\phi_E(P^{k-1})$, where $E = \Omega^1(\log M)$.

(3.3.12) Clearly all special Veronese varieties in $G(k-1, n-1)_e$ represent the same homology class $\Delta \in H_{2k-2}(G(k-1, n-1), \mathbf{Z})$. A precise determination of Δ will be given in §3.9 below. By Corollary 3.3.11

we obtain an embedding of $G^0(k, n)/H$ into the Chow variety $\mathcal{C}_{k-1}(G(k-1, n-1), \Delta)$. Denote by V the closure of $G^0(k, n)/H$ in this Chow variety. So it is the variety of cycles in $\mathcal{C}_{k-1}(G(k-1, n-1), \Delta)$ which are limit positions of special Veronese varieties.

(3.3.13) Similarly, all special Veronese varieties in $G(k-1, n-1)_e$ form a flat family. Let \mathcal{H} be the Hilbert scheme parametrizing all subschemes in $G(k-1, n-1)$, cf. (0.5.1). Define the variety W to be the closure of $G^0(k, n)/H$ in the Hilbert scheme \mathcal{H} . So it is the variety of subschemes in $\mathcal{C}_{k-1}(G(k-1, n-1), \Delta)$ which are limit positions of special Veronese varieties.

Our next result shows that all the information about the Chow quotient $G(k, n)//H$ is contained in visible contours.

(3.3.14) Theorem. *The correspondence $Z \mapsto Z_e$ extends to an isomorphism of the variety $G(k, n)//H$ with V and W .*

Proof: Since, by Proposition 3.1.9 b), every orbit closure $\overline{H.L}$ intersects the variety $G(k-1, n-1)_e$ properly, we can conclude, by using the result of Barlet [4] that the map $Z \mapsto Z_e = Z \cap G(k-1, n-1)_e$ defines a regular morphism $\psi : G(k, n)//H \rightarrow V$. Proposition 3.1.9 implies that ψ is set-theoretically a bijection. To show that this is an isomorphism of algebraic varieties it suffices to apply once again the reasoning with normal vector fields on a generalized Lie complex Z used in the proof of Theorem 1.5.2. Similarly for the Hilbert scheme compactification.

(3.3.15) Remark. A natural problem would be to study all Veronese subvarieties in Grassmannians. In general, not every such variety is projectively equivalent to a special one. This is because there are rank $(k-1)$ vector bundles on P^{k-1} which have the same Chern classes as $\Omega^1(\log M)$ but not having this form. A study of bundles $\Omega^1(\log M)$ from the point of view of stable vector bundles on projective spaces will be undertaken in a subsequent paper of I. Dolgachev and the author.

(3.4) Properties of special Veronese varieties.

As has been recalled in §2, Lie complexes in $G(k, n)$ correspond to projective equivalence classes of configurations of n hyperplanes in P^{k-1} in general position. We have seen in the previous subsection that the space P^{k-1} can be recovered from the corresponding Lie complex Z as its visible contour Z_e . Let us recover the configuration too.

(3.4.1) For any points $x_1, \dots, x_m \in P^{n-1}$ let $\langle x_1, \dots, x_m \rangle$ denote their projective span. We shall define also by $G_{\langle x_1, \dots, x_m \rangle}$ the subvariety in $G(k, n)$ formed by P^{k-1} 's containing $\langle x_1, \dots, x_m \rangle$. As an abstract variety, it is isomorphic to $G(k-p, n-p)$, where $p = \dim \langle x_1, \dots, x_m \rangle + 1$. Let $e_i \in P^{n-1}$ be the images of the standard basis vectors of \mathbb{C}^n .

(3.4.2) Proposition. *Let $M = (M_1, \dots, M_n)$ be a configuration of n hyperplanes in P^{k-1} in general position, $E = \Omega_{P^{k-1}}^1(\log M)$, and $X = \phi_E(P^{k-1}) \subset G(k-1, n-1)_e$ -the corresponding special Veronese variety (i.e. the visible contour of the Lie complex corresponding to M). Then $\phi_E(M_j) = X \cap G_{\langle e, e_j \rangle}$.*

Proof: Let us give a coordinate description of ϕ_E which, unlike the description given in (3.1.11), is symmetric with respect to permutation of hyperplanes.

(3.4.3) Let z_1, \dots, z_k be homogeneous coordinates in P^{k-1} and $g_j(z) = \sum_i a_{ij} z_j$ -the linear equations of $M_j, j = 1, \dots, n$. Let \mathbf{h} denote, as before, the quotient $\mathbb{C}^n / \{(a, \dots, a)\}$. The map $\phi_E : P^{k-1} \rightarrow G(k-1, \mathbf{h}) = G(k-1, n-1)_e$ is defined as follows.

(3.4.5) Let $z = (z_1 : \dots : z_k) \in P^{k-1}$ be generic. Consider the Jacobian $(k \times n)$ -matrix $N(z) = \|\partial \log f_i / \partial z_j\|$. Due to the identity $\sum_j z_j \frac{\partial \log f_i}{\partial z_j} = 1, \forall i$, the k -dimensional subspace spanned by rows of this matrix contains the vector $(1, \dots, 1)$ and hence defines a $(k-1)$ -dimensional subspace in \mathbf{h} which is precisely $\phi_E(z)$. For any subset $I \subset \{1, \dots, n\}, |I| = k$, denote by $p_I(N(z))$ the $k \times k$ -minor of $N(z)$ on columns from I .

(3.4.6) We can (and will) assume, by renumbering the coordinates, that the number j in the formulation of Proposition 3.4.2 equals 1. The subspace generated by rows of $N(z)$ lies in the sub-Grassmannian $G_{\langle e, e_1 \rangle}$ if and only if all minors $p_I(N(z))$, $1 \notin I$, vanish. To evaluate the limit of this subspace for $x \rightarrow M_i$, multiply $N(z)$ by $\text{diag}(f_1(z), \dots, f_k(z))$ from the left. Then any minor p_I of $\text{diag}(f_1(z), \dots, f_k(z)).N(z)$ with $1 \notin I$, will contain a row vanishing on M_i and hence will vanish on M_i itself. On the other hand, the minor $p_{1,2,\dots,k}$ of $\text{diag}(f_1(z), \dots, f_k(z)).N(z)$ is constant. Since we can write instead of (f_1, \dots, f_k) any $(f_1, f_{i_2}, \dots, f_{i_k})$, $1 < i_2 < \dots < i_k \leq n$, this proves that $\phi_E(M_1) = \phi_E(P^{k-1}) \cap G_{\langle e, e_1 \rangle}$. Proposition 3.4.2 is proven.

(3.4.7) **Corollary.** Any special Veronese variety contains the $\binom{n}{k-1}$ points

$$\langle e, e_{i_1}, \dots, e_{i_{k-1}} \rangle \in G(k-1, n-1)_e, 1 \leq i_1 < \dots < i_{k-1} \leq n.$$

(3.4.8) **Proposition.** The intersection $\phi_E(P^{k-1}) \cap G_{\langle e, e_j \rangle}$ is itself a special Veronese variety corresponding to the projective space M_j and the configuration of hyperplanes $(M_i \cap M_j), i \neq j$.

Proof: Straightforward. Left to the reader.

(3.4.9) **Example.** Consider the case $k = 2$ when $G(k, n)$ consists of lines in P^{n-1} . The variety of Lie complexes in $G(2, n)$ is, by §2, the same as the quotient $((P^1)^n - \bigcup \{x_i = x_j\})/GL(2)$ i.e. the set of projective equivalence classes of n -tuples of distinct points on P^1 . As we have seen in Example 3.2.8 that the visible contour of any Lie complex in $G(2, n)$ is a Veronese curve in P_e^{n-2} , the variety of lines in P^{n-1} through e . Corollary 3.4.7 means that every special Veronese curve in P_e^{n-2} contains n points $\langle e, e_i \rangle$ which are in general position.

It is a classical fact that for any points $p_1, \dots, p_n \in P^{n-2}$ in general position the set $V_0(p_1, \dots, p_n)$ of all Veronese curves through p_i is in bijection with $(P^1)_{gen}^n/GL(2)$ (see [14], Ch.III, §2, Proposition 3).

(3.4.10) **Example (continued).** As a transparent particular case, consider the case of 4 points p_1, \dots, p_4 in P^2 . Veronese curves in P^2 are just smooth conics. Conics through p_1, \dots, p_4 form a 1-dimensional pencil $\mathcal{L} = P^1$. There are exactly three degenerate conics in this pencil namely unions of lines

$$\langle p_1, p_2 \rangle \cup \langle p_3, p_4 \rangle, \langle p_1, p_3 \rangle \cup \langle p_2, p_4 \rangle, \langle p_1, p_4 \rangle \cup \langle p_2, p_3 \rangle.$$

The set of cross-ratios of p_1, \dots, p_4 regarded on conics from \mathcal{L} is in bijection with the set of non-degenerate conics from \mathcal{L} i.e. with P^1 minus 3 points.

(3.5) Steiner constructions of Veronese varieties in Grassmannians.

Veronese curves in projective spaces possess a lot of remarkable properties, see [2,14,46]. Most of these properties do not generalize to higher-dimensional Veronese varieties. It is our opinion that the "right" class of ambient spaces for p -dimensional Veronese varieties is formed not by projective spaces but by Grassmannians of the form $G(p, V)$. In this section we shall show that (special) Veronese varieties admit a "synthetic" construction in the spirit of Steiner.

(3.5.1) Consider some projective space P^{m-1} . Let $L \subset P^{m-1}$ be a projective subspace of codimension d . Denote by $]L[$ the space of all hyperplanes in P^{m-1} containing L . We shall call it the *star* of L . This is a projective space of dimension $d-1$.

(3.5.2) We may like to have a *parametrization* of the star $]L[$ i.e. an identification $f : P^{d-1} \rightarrow]L[$ of $]L[$ with the standard P^{d-1} . Such an identification is the same as a linear operator $E : \mathbb{C}^m \rightarrow \mathbb{C}^d$ whose kernel is \mathbf{L} , the linear subspace corresponding to L . Indeed, given such an E , we obtain a bijection $\Pi \mapsto E^{-1}(\Pi)$ between hyperplanes in \mathbb{C}^d and hyperplanes in \mathbb{C}^m containing \mathbf{L} i.e. hyperplanes from $]L[$.

In coordinate notation, we write E as a row of linear functions $g_i : \mathbb{C}^m \rightarrow \mathbb{C}$. Then to any $(\lambda_1 : \dots : \lambda_d) \in P^{d-1}$ we associate the hyperplane $\text{Ker}(\sum \lambda_i g_i) \in]L[$.

(3.5.3) It will be convenient for us to view a parametrization f above as a linear form $\sum \lambda_i g_i$ on \mathbf{C}^m whose entries are linear forms in $\lambda_1, \dots, \lambda_k$. This is tantamount to viewing a linear operator $E : \mathbf{C}^m \rightarrow \mathbf{C}^d$ as an element of $\mathbf{C}^d \otimes (\mathbf{C}^m)^*$.

(3.5.4) Recall Steiner's construction of Veronese curves in P^m [24]. Take m projective subspaces of codimension 2, $L_1, \dots, L_m \subset P^m$. The star $]L_i[$ of each L_i is just a pencil of hyperplanes i.e. it is isomorphic to the projective line P^1 . Let us identify these pencils with each other, e.g. by choosing projective equivalence $f_i : P^1 \rightarrow]L_i[$. Consider the curve in P^m which is the image of P^1 under the map

$$t \mapsto (f_1(t) \cap \dots \cap f_m(t)).$$

This is a Veronese curve. It depends on the choice of subspaces L_i and of identifications f_i .

In classical terminology, one would say that a Veronese curve can be obtained as the locus of intersections of corresponding hyperplanes from m pencils in correspondence.

(3.5.5) Construction. (The Grassmannian Steiner construction.) Take $n - k$ projective subspaces in P^{n-2} , say, L_1, \dots, L_{n-k} , of codimension k . Put the stars $]L_i[$ into 1-1 correspondence with each other, e.g., by choosing projective isomorphisms $f_i : P^{k-1} \rightarrow]L_i[$. Then consider the subvariety in $G(k-1, n-1)$ given by the parametrization

$$t \mapsto (f_1(t) \cap \dots \cap f_{n-k}(t)), \quad t \in P^{k-1}.$$

This is a direct generalization of the construction in (3.5.4). Using the fact (3.5.3) that parametrized stars are the same as linear forms with coefficients linearly depending on parameters, we get the following reformulation of the construction.

(3.5.6) Reformulation. Take a linear operator $A : \mathbf{C}^k \rightarrow \text{Hom}(\mathbf{C}^{n-1}, \mathbf{C}^{n-k})$ such that for any non-zero $z \in \mathbf{C}^k$ the operator $A(z) : \mathbf{C}^{n-1} \rightarrow \mathbf{C}^{n-k}$ is surjective. The Grassmannian Steiner construction is the subvariety in $G(k-1, n-1)$ consisting of points $\text{Ker} A(z)$, $z \in \mathbf{C}^k - \{0\}$.

(3.5.7) Theorem. Any special $(k-1)$ -dimensional Veronese variety in $G(k-1, n-1)$ can be obtained by the Grassmannian Steiner construction.

Proof: Let X be a special Veronese variety coming from a configuration (M_1, \dots, M_n) of hyperplanes in P^{n-1} in general position. As in (3.1.11) we can assume that M_n is the infinite hyperplane and choose affine coordinates z_1, \dots, z_{k-1} in $\mathbf{C}^{k-1} = P^{k-1} - M_n$ such that M_i is given by the equation $z_i = 0$ for $i = 1, \dots, k-1$.

(3.5.8) Consider the coordinate space \mathbf{C}^{n-1} with coordinates y_1, \dots, y_n and basis vectors e_1, \dots, e_{n-1} . Decompose it into the direct sum $\mathbf{C}^{k-1} \oplus \mathbf{C}^{n-k}$ where \mathbf{C}^{k-1} is spanned by e_1, \dots, e_{k-1} and \mathbf{C}^{n-k} by e_k, \dots, e_{n-1} .

(3.5.9) By definition, the variety $X \subset G(k-1, n-1)$ has the rational parametrization $z \mapsto \gamma(z)$, where $z \in P^{k-1}$ and γ is the logarithmic Gauss map. Explicit formula (3.1.13) gives that for generic $z \in \mathbf{C}^{k-1}$ the subspace $\gamma(z)$ is the graph of the linear operator $\mathbf{C}^{k-1} \rightarrow \mathbf{C}^{n-k}$ given by the matrix

$$(3.5.10) \quad B(z) = \begin{pmatrix} \frac{a_{k1}z_1}{f_k(z)} & \frac{a_{k2}z_2}{f_k(z)} & \dots & \frac{a_{k,k-1}z_{k-1}}{f_k(z)} \\ \frac{a_{k+1,1}z_1}{f_k(z)} & \frac{a_{k+1,2}z_2}{f_k(z)} & \dots & \frac{a_{k+1,k-1}z_{k-1}}{f_k(z)} \\ \dots & \dots & \dots & \dots \\ \frac{a_{n-1,1}z_1}{f_{n-1}(z)} & \frac{a_{n-1,2}z_2}{f_{n-1}(z)} & \dots & \frac{a_{n-1,k-1}z_{k-1}}{f_{n-1}(z)} \end{pmatrix}$$

where $f_j(z) = \sum_{\nu=1}^{k-1} a_{j\nu}z_\nu + a_{jk}$ is the equation of the hyperplane M_j , $j = k, \dots, n-1$. In other words, the subspace $\gamma(z)$ is spanned by the $(k-1)$ vectors $e_i + \sum_{j=k}^{n-1} \frac{a_{ji}z_i}{f_j(z)} e_j$. It is immediate to see that $\gamma(z)$ is the intersection of $(n-k)$ hyperplanes given, in the standard coordinates y_1, \dots, y_{n-1} , by linear equations

$$(3.5.11) \quad f_j(z)y_j - (a_{j1}z_1)y_1 - \dots - (a_{j,k-1}z_{k-1})y_{k-1} = 0, \quad j = 1, \dots, n-k.$$

The linear functions in (3.5.11), considered together, define a linear operator $a(z) : \mathbf{C}^{n-1} \rightarrow \mathbf{C}^{n-k}$ whose matrix elements are affine functions of z_1, \dots, z_{k-1} .

Let us complete the affine coordinates z_1, \dots, z_{k-1} in \mathbf{C}^{k-1} to homogeneous coordinates z_1, \dots, z_k in P^{k-1} so that the vanishing of z_k defines the infinite hyperplane. Then affine-linear functions $f_j(z_1, \dots, z_{k-1})$ will become linear functions $F_j(z) = F_j(z_1, \dots, z_k) = \sum_{\nu=1}^k a_{j\nu} z_\nu$. The $(n-k)$ linear functions in (3.5.11) give rise to a family of linear operators $A(z_1, \dots, z_k) : \mathbf{C}^{n-1} \rightarrow \mathbf{C}^{n-k}$ given by the matrix

$$(3.5.12) \quad \begin{pmatrix} -a_{k1}z_1 & -a_{k2}z_2 & \dots & -a_{k,k-1}z_{k-1} & F_k(z) & 0 & \dots & 0 \\ -a_{k+1,1}z_1 & -a_{k+1,2}z_2 & \dots & -a_{k+1,k-1}z_{k-1} & 0 & F_{k+1}(z) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -a_{n-1,1}z_1 & -a_{n-1,2}z_2 & \dots & -a_{n-1,k-1}z_{k-1} & 0 & 0 & \dots & F_{n-1}(z) \end{pmatrix}$$

whose entries are linear functions in z_1, \dots, z_k . Theorem 3.5.7 is proven.

(3.5.13) Remark. It is immediate to extract from the formula (3.5.11) the $(n-k)$ subspaces $L_i \subset \mathbf{C}^{n-1}$ whose stars $]L_i[$ are identified (in the synthetic version (3.5.5) of the Grassmannian Steiner construction). Namely, $L_i, i = k, k+1, \dots, n-1$ is the span of the vectors e_1, \dots, e_{k-1}, e_i .

We observe that the position of these subspaces is rather special. The identifications of the stars are also very special. The extremely interesting question of possibility of Steiner construction of more general Veronese varieties in Grassmannians will be treated elsewhere.

(3.5.14) Varieties in projective spaces defined by various generalizations of the Steiner's construction (3.5.4) were studied in detail in the book [46] by T.G.Room. To obtain such a generalization, one takes r subspaces $L_1, \dots, L_r \subset P^{m-1}$ of codimension d , identifies all the stars $]L_i[$ with each other and considers the codimension r subspaces in P^{m-1} which are the intersections of corresponding hyperplanes from these stars. If $d < r$ then the union of these subspaces is a proper subvariety in P^{m-1} which will be called a *projectively generated variety* [46]. The fundamental remark of Room is that any projectively generated variety in P^{m-1} can be given by a system of equations of which each equation has the form of a determinant with entries - linear forms on P^{m-1} .

We shall use this idea in the next section to get a better hold of Veronese varieties in Grassmannians.

(3.6) The sweep of a Veronese variety in Grassmannian.

(3.6.1) Let X be any subvariety in the Grassmannian $G(k-1, n-1)$. So X is a family of $(k-2)$ -dimensional projective subspaces in P^{n-2} . The *sweep* of X is, by definition, the subvariety $Sw(X) \subset P^{n-2}$ defined as the union of the subspaces from X .

(3.6.2) We shall be mostly interested in the case when $X \subset G(k-1, n-1)$ be a $(k-1)$ -dimensional special Veronese variety, see (3.3.9). In other words (3.3.10), X is the visible contour of a Lie complex $Z = \overline{H}\overline{L}$ i.e. the locus of P^{k-1} 's in P^{n-1} which belong to the complex Z and contain the chosen point $e = (1, \dots, 1)$. The sweep of X is just what we called in (3.1.2) the visible sweep of the complex Z at e . So it is the projectivization of the cone in P^{n-1} with vertex e given by the union of all P^{k-1} 's from the complex Z which contain e .

(3.6.3) Let $\mathfrak{h} = \mathbf{C}^n / \{(a, \dots, a)\}$ be the Lie algebra of the maximal torus $H \subset PGL(n)$. Recall (3.1.8) that the Grassmannian $G(k-1, n-1)$ in which the visible contours (and hence special Veronese varieties) lie, is in fact $G(k-1, \mathfrak{h})$. Therefore the sweep of any special Veronese variety lies naturally in the projective space $P(\mathfrak{h})$.

(3.6.4) Let t_1, \dots, t_n be standard coordinate functions on \mathbf{C}^n . A linear form $\sum c_i t_i$ descends to a linear form on \mathfrak{h} if $\sum c_i = 0$. In particular, the *roots* i.e. the linear forms $t_i - t_j$ are forms on \mathfrak{h} .

(3.6.5) The possibility of defining X by the Grassmannian Steiner construction (3.5.5) implies that $Sw(X)$ is always projectively generated variety in the sense of (3.5.14).

(3.6.6) Theorem. Suppose that $k \leq n-k$. Let z_1, \dots, z_k be homogeneous coordinates in P^{k-1} . Suppose that a configuration $M = (M_1, \dots, M_n)$ of hyperplanes in P^{k-1} consists of k coordinate hyperplanes $M_i =$

$\{z_i = 0\}, i = 1, \dots, k$ and $(n - k)$ other hyperplanes $M_j = \{\sum_{i=1}^k a_{ji} z_i = 0\}$. Then the sweep of the Veronese variety in $P^{n-2} = P(\mathbf{h})$ corresponding to M is given by vanishing of all $k \times k$ -minors of the following $k \times (n - k)$ -matrix of linear forms on \mathbf{h} :

$$(3.6.7) \quad A^\dagger(t_1, \dots, t_n) = \|a_{ji}(t_j - t_i)\|, i = 1, \dots, k, j = k + 1, \dots, n.$$

Proof: Let $A : \mathbf{C}^k \rightarrow \text{Hom}(\mathbf{C}^{n-1}, \mathbf{C}^{n-k})$ be the linear system of linear operators such that X consists of kernels of $A(z), z \in \mathbf{C}^k - \{0\}$. An explicit formula for A is given in (3.5.12). Using partial dualization, let us associate to A a linear operator

$$A^\dagger : \mathbf{C}^{n-1} \rightarrow \text{Hom}(\mathbf{C}^k, \mathbf{C}^{n-k})$$

A point $t \in \mathbf{C}^{n-1}$ lies in the kernel of $A(z)$ for some non-zero $z \in \mathbf{C}^k$ if and only if the linear operator $A^\dagger(t) : \mathbf{C}^k \rightarrow \mathbf{C}^{n-k}$ has non-trivial kernel i.e. the rank of $A^\dagger(t)$ is less than k . Thus the sweep $Sw(X)$ is defined by vanishing of all $k \times k$ minors of the matrix $A^\dagger(t)$ of linear forms on P^{n-2} .

To see that A^\dagger has the claimed form, we use the formula (3.5.12). This formula was written with respect to the non-symmetric system of coordinates y_1, \dots, y_{n-1} in \mathbf{h} . In the language of (3.6.4) we have $y_i = t_i - t_n$. Substituting this to (3.5.12) and transposing the matrix, we arrive at the formula (3.6.7). Theorem 3.6.6 is proven.

(3.6.8) Let us describe a more geometric construction for the sweep of the Veronese variety corresponding to a projective configuration.

Let $\text{Mat}(k, n - k)$ be the vector space of all k by $(n - k)$ matrices and $P(\text{Mat}(k, n - k))$ be the projectivization of this space. The projectivization of the set of matrices of rank 1 is just the Segre embedding $P^{k-1} \times P^{n-k-1} \subset P(\text{Mat}(k, n - k))$. Let $\nabla \subset P(\text{Mat}(k, n - k))$ be the projectivization of the space of matrices of rank $< k$. This is an algebraic subvariety of codimension $n - 2k + 1$.

(3.6.9) Let now (x_1, \dots, x_n) be a configuration of points in P^{k-1} in general position. (Recall that modulo projective isomorphism, configurations of points give the same orbit space as configurations of hyperplanes.) Let (y_1, \dots, y_n) be the configuration of points in P^{n-k-1} associated to x_1, \dots, x_n (see §2.3 about association). By Reformulation (2.3.8), the points $z_i = (x_i, y_i) \in P^{k-1} \times P^{n-k-1} \subset P(\text{Mat}(k, n))$ form a circuit i.e. span a projective space, say, L whose dimension is $n - 2$ and are in general position as points of L . The space $P^{n-2} = P(\mathbf{h})$ also comes with a circuit given by points \bar{e}_i , - projectivizations of images of the basis vectors $e_i \in \mathbf{C}^n$ in $\mathbf{h} = \mathbf{C}^n / \mathbf{C}$. Hence there is a unique projective transformation $\phi : L \rightarrow P(\mathbf{h})$ taking z_i to \bar{e}_i . We shall be interested in the intersection $\nabla \cap L \subset L$ where ∇ is the determinantal variety in (3.6.8).

(3.6.10) Proposition. *The map ϕ identifies the subvariety $\nabla \cap L \subset L$ with the sweep $Sw(X(x_1, \dots, x_n)) \subset P(\mathbf{h})$ of the Veronese variety corresponding to the configuration (x_1, \dots, x_n) .*

Proof: Denote the sweep $Sw(X(x_1, \dots, x_n))$ shortly by S . Let M_1, \dots, M_n be the hyperplanes in the dual P^{k-1} corresponding to x_i . After choosing suitable homogeneous coordinates we can apply Theorem (3.6.6) which gives a representation of S as the inverse image of ∇ under the linear embedding $A^\dagger : \mathbf{h} \rightarrow \text{Mat}(k, n - k)$. We regard A^\dagger as a map $\mathbf{C}^n \rightarrow \text{Mat}(k, n - k)$ using the isomorphism $\mathbf{h} = \mathbf{C}^n / \mathbf{C}$. Let $e_i \in \mathbf{C}^n$ be the standard basis vectors. Proposition (3.6.10) is a consequence of the following statements:

(3.6.11) The matrices $A^\dagger(e_i)$ lie in the Segre embedding $P^{k-1} \times P^{n-k-1} \subset P(\text{Mat}(k, n - k))$ i.e., $\text{rank } A^\dagger(e_i) = 1$.

(3.6.12) The configuration of hyperplanes $\text{Ker}(A^\dagger(e_i)) \subset P^{k-1}$ is projectively isomorphic to (M_1, \dots, M_n) and the configuration of points $\text{Im}(A^\dagger(e_i)) \subset P^{n-k-1}$ is associated to (M_1, \dots, M_n) .

Both these statements are immediate from the explicit form (3.6.7) of the matrix A^\dagger .

(3.6.14) Corollary. *Let $n = 2k$. Then Veronese varieties corresponding to a configuration $(M_1, \dots, M_{2k}) \subset P^{k-1}$ and to the associated configuration, have the same sweep.*

(3.6.15) Any determinantal variety i.e. variety defined by vanishing of minors of a matrix of linear forms, bears two canonical families of projective subspaces, so-called α - and β - families [46]. Let us recall their construction and explain their relevance to our situation.

Let $k \leq n - k$ and $\nabla \subset P(\text{Mat}(k, n - k))$ denote, as before, the projectivization of the space of matrices of rank $< k$. For any 1-dimensional subspace $\lambda \subset \mathbb{C}^k$ set

$$(3.6.16) \quad \Pi_\alpha(\lambda) = P(\{M : \mathbb{C}^k \rightarrow \mathbb{C}^{n-k} : M(\lambda) = 0\}) \subset \nabla.$$

This is a projective subspace in ∇ of codimension $k - 1$. Thus we get a family of projective subspaces in ∇ (called α -subspaces) of codimension $k - 1$, parametrized by $P^{k-1} = P(\mathbb{C}^k)$.

Similarly, for any hyperplane $\Lambda \subset \mathbb{C}^{n-k}$ set

$$(3.6.17) \quad \Pi_\beta(\Lambda) = P(\{M : \mathbb{C}^k \rightarrow \mathbb{C}^{n-k} : \text{Im}(M) \subset \Lambda\}) \subset \nabla.$$

This is a projective subspace in ∇ of codimension $n - k - 1$. We get a family of projective subspaces in ∇ (called β -subspaces) of codimension $n - k - 1$ parametrized by $P^{n-k-1} = P((\mathbb{C}^{n-k})^*)$.

(3.6.18) Let $L \subset P(\text{Mat}(k, n - k))$ be a projective subspace of dimension $n - 2$. Consider the variety $S = L \cap \nabla$. It contains projective subspaces $\Pi_\alpha(\lambda) \cap L$ whose dimension is at least $n - k - 1$ (they will be called the α -subspaces in S) and subspaces $\Pi_\beta(\Lambda) \cap L$ whose dimension is at least $k - 1$ (they will be called β -subspaces in S). The role of these subspaces in our situation is as follows.

(3.6.19) Proposition. *Let (x_1, \dots, x_n) be a configuration of points in P^{k-1} in general position and (y_1, \dots, y_n) - the associated configuration in P^{n-k-1} . Let $X \subset G(k - 1, n - 1)$, $X' \subset G(n - k, n - 1)$ be the Veronese varieties corresponding to (x_i) and (y_i) (their dimensions equal, respectively, $k - 1$ and $n - k - 1$). Let $S, S' \subset P_e^{n-2} = P(\mathbf{h})$ be the sweeps of these varieties. Then $S = S'$ and the subspaces from X (resp. from X') lying on $S = S'$ are precisely the β - (resp. α -) subspaces on S defined in (3.6.18).*

Proof: This is a reformulation of Theorem 3.5.7 about Steiner construction of X .

(3.7) An example: Visible contours and sweeps of Lie complexes in $G(3, 6)$.

In this section we study in detail the construction of §3.6 in the particular case corresponding to configuration of 6 points on P^2 . In other words, we consider the case $k = 3, n = 6$.

(3.7.1) To any sextuple $(x_1, \dots, x_6) \in (P^2)_{gen}^6$ a Veronese surface $X(x_1, \dots, x_6) \subset G(2, 5)$ is associated. Its sweep, denoted $S(x_1, \dots, x_6) \subset P^4$ is a cubic hypersurface since by Theorem 3.6.6 it is given by vanishing of the determinant of a 3 by 3 matrix of linear forms. We shall study such hypersurfaces.

(3.7.2) We are interested in configurations modulo projective isomorphism. So we can consider equally well the sextuple of lines $M_i \subset \tilde{P}^2$ dual to x_i . This sextuple represents the same element of $(P^2)^6/GL(3)$.

(3.7.3) We can always assume that lines M_i have the particular form considered in Theorem 3.6.6: for $i=1, 2, 3$ the line M_i is given by the equation $z_i = 0$ and M_j for $j = 4, 5, 6$ is given by the equation $\sum a_{ij} z_j = 0$. The 3×3 matrix $||a_{ij}||$ is defined by a projective isomorphism class of (M_1, \dots, M_6) not uniquely but only up to multiplication of rows and columns by non-zero scalars. Generic position of lines L_ν implies that all $a_{ij} \neq 0$. Hence by multiplication of rows and columns by scalars we can take $||a_{ij}||$ into a unique matrix of the form

$$(3.7.4) \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix}.$$

In this way the quotient $(P^2)_{gen}^6/GL(3)$ becomes identified with the space of (a, b, c, d) such that all the minors of the matrix (3.7.4) are non-zero.

(3.7.5) Proposition. *The points x_1, \dots, x_6 lie on a conic (or, equivalently, the lines M_1, \dots, M_6 are tangent to a conic) if and only if the matrix elements a, b, c, d of the matrix (3.7.4) satisfy the equation $\Psi(a, b, c, d) = 0$, where*

$$(3.7.6) \quad \Psi(a, b, c, d) = \det \begin{pmatrix} a(1-c) & b(1-d) \\ c(1-a) & d(1-b) \end{pmatrix} = ad - bc + abc + bcd - acd - abd.$$

Proof: This is Proposition 2.13.1 of [41].

(3.7.7) The Veronese variety $X = X(x_1, \dots, x_6)$ lies in $G(2, 5)$, the space of lines in $P^4 = P(\mathbf{h})$, $\mathbf{h} = \mathbf{C}^6/\mathbf{C}$. Let $p_i \in P^4$, $i = 1, \dots, 6$ be the point corresponding to the standard basis vector $e_i \in \mathbf{C}^6$ (In the realization of P^4 as the space of lines in P^5 through $e = (1, \dots, 1)$, the point p_i corresponds to the line $\langle e, e_i \rangle$. By corollary 3.4.7, the variety X contains all the lines $\langle p_i, p_j \rangle$. This implies that points p_i are singular points of the sweep $S = S(x_1, \dots, x_6)$ of X . Indeed, the tangent directions at any p_i to lines $\langle p_i, p_j \rangle$, $j \neq i$, span the whole tangent space $T_{p_i}P^4$ which will therefore coincide with the Zariski tangent space of S at p_i .

A more detailed information about the singularities of S is given in the next proposition which is the main result of this section.

(3.7.8) Proposition. a) If $x_1, \dots, x_6 \in P^2$ are in general position and do not lie on a conic then the sweep $S(x_1, \dots, x_6)$ has only 6 singular points namely p_i and these points are simple quadratic singularities.
b) If x_1, \dots, x_6 are in general position and lie on a conic $K \subset P^2$ then $S(x_1, \dots, x_6)$ has a curve C of singular points which is a Veronese curve in P^4 containing p_1, \dots, p_6 . In this case the configuration of x_i on $K \cong P^1$ is isomorphic to that of p_i on C . The variety $S(x_1, \dots, x_6)$ is the union of all chords of C .

Proof: Consider the varieties

$$(3.7.9) \quad P^2 \times P^2 \subset \nabla \subset P^8 = P(\text{Mat}(3, 3))$$

where ∇ consists of degenerate matrices and $P^2 \times P^2$ - of matrices of rank 1. It is well-known that $P^2 \times P^2 = \text{Sing}(\nabla)$.

Assume that the configuration (M_1, \dots, M_6) of lines dual to x_i has the form specified in Theorem 3.6.6 with the 3×3 matrix $\|a_{ij}\|$ given in the normal form (3.7.4). Let $A^\dagger : P^4 = P(\mathbf{h}) \rightarrow P(\text{Mat}(3, 3))$ be the embedding given by formula (3.6.7). In other words (taking into account the normal form of $\|a_{ij}\|$) we have

$$(3.7.10) \quad A^\dagger(t_1, \dots, t_6) = \begin{pmatrix} t_1 - t_4 & t_1 - t_5 & t_1 - t_6 \\ t_2 - t_4 & a(t_2 - t_5) & b(t_2 - t_6) \\ t_3 - t_4 & c(t_3 - t_5) & d(t_3 - t_6) \end{pmatrix}$$

Theorem 3.6.6 implies that our sweep S equals $(A^\dagger)^{-1}(\nabla)$.

(3.7.11) Let $L \subset P(\text{Mat}(3, 3))$ be the image of A^\dagger . It is immediate to check that the degree of Segre variety $P^2 \times P^2 \subset P^8$ equals 6. On the other hand, the subspace $L \subset P^4$ already intersects $P^2 \times P^2$ in 6 points $q_i = A^\dagger(e_i)$, $i = 1, \dots, 6$. Hence there remains one of two possibilities:

Case 1. L intersects $P^2 \times P^2$ transversally in 6 points $q_i = A^\dagger(e_i)$.

Case 2. The intersection $L \cap (P^2 \times P^2)$ contains a component of positive dimension.

Part a) of Proposition 3.7.8 will follow from the next two lemmas.

(3.7.12) Lemma. If x_1, \dots, x_6 are in general position and do not lie on a conic then for $L = A^\dagger(P(\mathbf{h}))$ the Case 1 holds.

(3.7.13) Lemma. If for L the Case 1 holds then $L \cap \nabla$ has q_i as the only singular points and the singularities at q_i are simple quadratic.

(3.7.14) Proof of Lemma 3.7.13: The only possibility which we have to exclude is that there is a point $q \in L \cap \nabla$ which is smooth on ∇ (i.e. $\text{rank}(q) = 2$) and such that $T_q L \subset T_q \nabla$. Do rule out this possibility, let $\lambda \in P^2$ be the point corresponding to $\text{Ker}(q)$ and let $\Pi = \Pi_\alpha(\lambda)$ be the corresponding α -subspace i.e. the projectivization of the space of all 3×3 matrices annihilating λ . Then $\dim(\Pi) = 5$. Since L is connected in the embedded tangent space to ∇ at q (which is 7-dimensional), we have $\dim(L \cap \Pi) \geq 2$. However, by Proposition 3.6.19, the intersection of L with all α -subspaces should be 1-dimensional.

(3.7.15) Proof of Lemma 3.7.12: It suffices to show that each q_i is a isolated singular point of the intersection $L \cap (P^2 \times P^2)$. To this end, we prove that the tangent spaces $T_{q_i} L, T_{q_i}(P^2 \times P^2) \subset T_{q_i} P^8$ intersect only in 0.

Since the roles of q_i are symmetric, it is enough to consider $i = 1$. The point $q_1 = A^\dagger(e_1)$ is given, in virtue of the normal form (3.6.27) of A^\dagger , by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The tangent space at q_1 to the locus of rank 1 matrices is easily seen to consist of matrices of the form

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_5 & \lambda_5 & \lambda_5 \end{pmatrix}.$$

Therefore the intersection $T_{q_1}L \cap T_{q_1}(P^2 \times P^2)$ is obtained from the space of solutions (t_1, \dots, t_6) of the linear system

$$(3.7.16) \quad t_2 - t_4 = a(t_2 - t_5) = b(t_2 - t_6), \quad t_3 - t_4 = c(t_3 - t_5) = d(t_3 - t_6)$$

by factorization by the 2-dimensional subspace spanned by $e_1 = (1, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. Hence we are reduced to the following statement:

(3.7.18) Lemma. *If x_i do not lie on a conic (i.e. if the polynomial $\Psi(a, b, c, d)$ given by (3.7.6) does not vanish) then the linear system (3.7.16) has 2-dimensional space of solutions.*

Proof: This is a system of four equations on 6 variables whose matrix of coefficients has the form

$$\begin{pmatrix} 0 & 1-a & 0 & -1 & a & 0 \\ 0 & 1-b & 0 & -1 & 0 & b \\ 0 & 0 & 1-c & -1 & c & 0 \\ 0 & 0 & 1-d & -1 & 0 & d \end{pmatrix}.$$

Let us disregard the first column, then move the column with (-1) 's to the left and then subtract the first row from all the other rows. We obtain the matrix

$$\begin{pmatrix} -1 & 1-a & 0 & a & 0 \\ 0 & a-b & 0 & -a & b \\ 0 & -1+a & 1-c & c-a & 0 \\ 0 & -1+a & 1-d & -a & d \end{pmatrix}.$$

It is immediate to see that all 4 by 4 minors of this matrix have the form $\pm\Psi(a, b, c, d)$.

We have proven part a) of Proposition 3.7.8.

(3.7.19) Let us prove part b) of Proposition 3.7.8. So assume that $x_1, \dots, x_6 \in P^2$ are points in general position lying on a conic K . Consider the 2-fold Veronese embedding

$$(3.7.20) \quad v_2 : P^2 \hookrightarrow P^5 = P(S^2(\mathbf{C}^3)) \subset P(Mat(3, 3))$$

where $P(S^2(\mathbf{C}^3))$ is embedded into $P(Mat(3, 3))$ as the space of symmetric matrices. Let $L \subset P(S^2(\mathbf{C}^3))$ be the projective envelope of $v_2(x_i)$. Let also $\nabla_{sym} \subset P(S^2(\mathbf{C}^3))$ be the space of degenerate quadratic forms i.e. $\nabla_{sym} = \nabla \cap P(S^2(\mathbf{C}^3))$. Note that $P^2 = \text{Sing}(\nabla_{sym})$ has codimension 2 in ∇_{sym} .

Since x_i lie on a conic, their configuration is self-associated (Example 2.3.12). Now the interpretation of $S(x_1, \dots, x_6)$ given in (3.6.9), (3.6.10) implies that $S(x_1, \dots, x_6) = L \cap \nabla_{sym}$.

The conic K is equal to $v_2^{-1}(L \cap v_2(P^2))$. Since x_i are in general position, K is smooth. Now since $P^2 = \text{Sing}(\nabla_{sym})$, we find that $C = v_2(K) = \text{Sing}(L \cap \nabla_{sym})$ is the singular curve of $L \cap \nabla_{sym} = S(x_1, \dots, x_6)$. This is clearly a Veronese curve in $L = P^4$. The Veronese embedding v_2 identifies k with C and points $x_i \in K$ with our distinguished points $p_i \in C$. Finally, ∇_{sym} is the union of chords of $v_2(P^2)$ (every degenerate

quadratic form in 3 variables is a sum of two quadratic forms of rank 1). Hence $S(x_1, \dots, x_6)$ contains the union of chords of C . Since the latter is also a 3-dimensional variety, the two varieties in question coincide.

Proposition 3.7.8 is completely proven.

(3.7.21) Fix 6 points $p_1, \dots, p_6 \in P^4$ in general position. Any two choices of p_i can be taken to each other by a unique projective isomorphism. Let $\mathcal{L} \subset P(S^3(\mathbb{C}^5))$ be the linear system of all cubic hypersurfaces in P^4 which contain p_i as singular points. It is clear by dimension count that \mathcal{L} has dimension 4. On the other hand, taking p_i to be the standard points (images of e_i in $P(\mathbf{h})$) we have constructed a 4-dimensional family of sweeps $S(x_1, \dots, x_6)$ which all belong to \mathcal{L} . Hence we have the following corollary.

(3.7.22) Corollary. *Let $p_1, \dots, p_6 \in P^4$ be points in general position. A generic cubic hypersurface $S \subset P^4$ for which p_i are singular points, is projectively equivalent to the visible sweep of some Lie complex in $G(3, 6)$. In particular, S can be realized as $P^4 \cap \nabla$ for a suitable embedding $P^4 \subset P(\text{Mat}(3, 3))$. The variety S contains two families of lines (α and β -families, in the determinantal realization), whose parameter spaces P, P' are isomorphic to P^2 . These families give rise to two Veronese surfaces in $G(2, 5)$ which correspond to a pair of associated configurations $(M_1, \dots, M_6) \subset P, (M'_1, \dots, M'_6) \subset P'$ of lines. Explicitly, $M_i \subset P$ (resp. $M'_i \subset P'$) is the locus of all lines from the first (resp. second) family which contain the point p_i .*

(3.7.23) Remarks. a) Any generic intersection $P^4 \cap \nabla \subset P(\text{Mat}(3, 3))$ intersects the Segre variety $P^2 \times P^2 = \text{Sing}(\nabla)$ in 6 points (since $6 = \deg(P^2 \times P^2)$) and hence is a cubic hypersurface of the form studied in the above corollary.

b) The correspondence

$$(x_1, \dots, x_6) \bmod PGL(3) \longmapsto S(x_1, \dots, x_6)$$

is two-to-one. Hence it defines a two-sheeted covering of fourfolds $\pi : (P^2)_{gen}^6 / GL(3) \rightarrow \mathcal{L}$. This covering is well-known classically, see [14, 41]. It extends to a map of the Mumford quotient $((P^2)^6 / GL(3))_{Mumf} \rightarrow \mathcal{L}$ which is a double cover ramified along a hypersurface $W \subset \mathcal{L}$ of degree 2. This hypersurface is called the modular variety of level 2 (see [14]). The projective dual $\tilde{W} \subset \tilde{\mathcal{L}}$ is a so-called Segre cubic threefold i.e. a cubic hypersurface with 10 ordinary singular points. (It is known that all such threefolds are projectively isomorphic.)

(3.8) Chordal varieties of Veronese curves.

(3.8.1) Let $C \subset P^r$ be a Veronese curve. An $s-1$ -dimensional projective subspace $L \subset P^r$ is called *chordal* to C if it intersects C in s points (counted with multiplicities). Denote by $Ch_{s-1}(C)$ the variety of all chordal $(s-1)$ -dimensional subspaces of C . Clearly $Ch_{s-1}(C)$ is isomorphic to the s -fold symmetric power of $C \cong P^1$ i.e. to the projective space P^s . We obtain therefore a special class of embeddings $P^s \subset G(s, r+1)$. It turns out that these embeddings give particular cases of Veronese varieties in Grassmannians considered in §3.3 above.

(3.8.2) Proposition. *The chordal variety of any Veronese curve is a Veronese variety in the Grassmannian.*

This proposition is classical and due to L.M.Brown [9]. In modern language it is a consequence (or, rather, a reformulation) of the following fact.

(3.8.3) Proposition. *There is a unique isomorphism of $GL(2)$ -modules*

$$\xi : \bigwedge^k (S^n(\mathbb{C}^2)) \longrightarrow S^{n-k-1}(S^k(\mathbb{C}^2)) \otimes \left(\bigwedge^2 (\mathbb{C}^2) \right)^{\otimes k(k-1)/2}$$

such that for any $l_1, \dots, l_k \in \mathbb{C}^2$,

$$\xi(l_1^m \wedge \dots \wedge l_k^m) = (l_1 \dots l_k)^{n-k-1} \otimes \prod_{i < j} (l_i \wedge l_j). \quad \triangleleft$$

(3.8.4) Put now $s = k - 1, r = n - 2$ so that we obtain Veronese varieties in $G(k - 1, n - 1)$ i.e., are in the setting of sections (3.3) and (3.4). Recall that special Veronese varieties in $G(k - 1, n - 1)_e$ (i.e. in the space of P^{k-1} 's in P^{n-1} through $e = (1, \dots, 1)$) are in bijection with the set of projective equivalence classes of n -tuples of points in P^{k-1} in general position. Let us clarify the place in this picture of chordal varieties of Veronese curves.

Suppose that n distinct points $x_1, \dots, x_n \in P^{k-1}$ happen to lie on a Veronese curve D (of degree $k - 1$) in P^{k-1} . Then they are in general position in P^{k-1} , as it follows from the calculation of the Vandermonde determinant. Thus they represent an element of $(P^{k-1})_{gen}^n / GL(k)$. Such element, by Corollary 3.3.11, is represented by a unique special Veronese variety $X(x_1, \dots, x_n) \subset G(k - 1, n - 1)_e$ of dimension $k - 1$. On the other hand, the curve D being isomorphic to P^1 , the points x_i represent an element from $(P^1)_{gen}^n / GL(2)$. The latter set, as we have seen in Example 3.4.9, is identified with the set of Veronese curves in $P_e^{n-2} = G(1, n - 1)_e$ through n points $< e, e_i >$. Let $C(x_1, \dots, x_n)$ be the special Veronese curve representing the configuration of x_i on D .

(3.8.5) Theorem. *The special Veronese variety $X(x_1, \dots, x_n)$ coincides with the chordal variety of the Veronese curve $C(x_1, \dots, x_n)$.*

Using the language of hyperplane configurations, this can be reformulated as follows.

(3.8.6) Reformulation. *Let D be a Veronese curve in P^{k-1} , $M = (M_1, \dots, M_n) \subset P^{k-1}$ - a configuration of hyperplanes which are osculating to D (i.e. each M_i intersects D in just one point x_i with multiplicity $(k - 1)$). Then the embedding $\phi_E : P^{k-1} \hookrightarrow G(k - 1, n - 1)$ defined by the vector bundle $E = \Omega_{P^{k-1}}^1(\log M)$ maps P^{k-1} isomorphically to the chordal variety of some other Veronese curve $C \subset P^{n-1}$. The curve C is the image of D in the projective embedding defined by the line bundle $\Omega_D^1(\log(x_1 + \dots + x_n))$.*

Proof: An easy calculation in coordinates shows that the restriction of 1-forms defines an isomorphism

$$H^0(P^{k-1}, \Omega_{P^{k-1}}^1(\log M)) \longrightarrow H^0(D, \Omega_D^1(\log(x_1 + \dots + x_n))). \quad (3.8.7)$$

Denote for short the bundle $\Omega_{P^{k-1}}^1(\log M)$ on P^{k-1} by E and the bundle $\Omega_D^1(\log(x_1 + \dots + x_n))$ on D - by F . Let ϕ_E and ϕ_F be the corresponding maps to the Grassmannian and the projective space respectively. We shall show that under the identification (3.8.7) the image of ϕ_E is the chordal variety of the image of ϕ_F . By definition of ϕ_E, ϕ_F (see section (3.2) above) this is equivalent to part a) the following statement:

(3.8.8) Lemma. *Let $p \in P^{k-1}$ be a generic point. Then there are $(k - 1)$ points $y_1(p), \dots, y_{k-1}(p) \in D$ such that a form $\omega \in H^0(P^{k-1}, E)$ vanishes at p (as a section of E) if and only if the restriction of ω to D (as a 1-form) vanishes at all y_i .*

b) Explicitly, points $y_i(p)$ are precisely the points of osculation of the $k - 1$ osculating hyperplanes to D passing through p

Note that through any generic point $p \in P^{k-1}$ there pass exactly $k - 1$ osculating hyperplanes to D . (Since osculating hyperplanes to D form a Veronese curve \hat{D} in the dual projective space, this just means that the degree of \hat{D} is also $k - 1$).

We shall verify Lemma 3.8.8 in coordinates.

(3.8.9) Let us regard the affine space \mathbb{C}^k as the space of polynomials $f(t) = \sum_{i=0}^{k-1} a_i t^i$ of degree $\leq k - 1$ in one variable. The hyperplanes M_i have the form $M_i = \{f : f(x_i) = 0\}$, where $x_i \in \mathbb{C}$ are distinct numbers. The equation of M_i is, therefore, the evaluation map $f \mapsto f(x_i)$. Any point $p \in P^{k-1}$ is represented as a polynomial $f(t)$. If α_i are the roots of $f(t)$ then the points $y_i(p)$ are the polynomials $(t - \alpha_i)^{k-1}$. We can normalize any polynomial f to have the form $f(t) = \prod (t - \alpha_i)$.

A section of the bundle E is given as the logarithmic differential of the function

$$f \mapsto \prod_{j=1}^n f(x_j)^{\lambda_j} = \prod_{i=1}^{k-1} \prod_{j=1}^n (x_j - \alpha_i)^{\lambda_j}, \quad \sum \lambda_i = 0.$$

Our assertion means that this function has a critical point at a given f if and only if the function $\prod (t - x_j)^{\lambda_j}$ has a critical value at each $t = \alpha_i$. But this follows from the equality

$$\frac{\partial}{\partial t} \log \prod_j (t - x_j)^{\lambda_j} \Big|_{t=\alpha_i} = - \frac{\partial}{\partial \alpha_r} \log \prod_j (x_j - \alpha_i)^{\lambda_j} = \sum_j \frac{\lambda_j}{\alpha_i - t_j}.$$

Lemma 3.8.8 and hence Theorem 3.8.5 are proven.

(3.8.10) Examples. a) Consider the case of 5 points in P^2 i.e. $k = 3, n = 5$. Such configurations lead to Veronese surfaces in $G(2, 4) \subset P^5$. Since every 5 points in P^2 lie on a unique conic, any special Veronese surface in $G(2, 4)$ will be the chordal variety of a Veronese curve (twisted cubic) in P^3 .

b) Similarly, the case of n points in P^{n-3} leads to Veronese $(n-3)$ -folds in $G(n-3, n-1)$ which are chordal varieties of Veronese curves in P^{n-2} . Note that $G^0(k, n)/H = G^0(n-k, n)/H$ by duality and hence the case of n points in P^{n-3} is equivalent to that of n points on P^1 (see section (2.3)). Any n points in P^{n-3} in general position lie on a unique Veronese curve which provides the dual configuration.

c) For the case $k = 3, n = 6$ (six points in P^2) we associate to sextuples $(x_1, \dots, x_6) \in (P^2)_{gen}^6$ Veronese surfaces $X(x_1, \dots, x_6)$ in $G(2, 5)$, the space of lines in P^4 . When x_1, \dots, x_6 lie on a conic, the surface $X(x_1, \dots, x_6)$ is the chordal surface of a Veronese curve in P^4 . We have seen this is proposition 3.7.8.

(3.8.11) Remark. It is a remarkable fact that chordal varieties of Veronese curves (regarded as subvarieties in Grassmannians) possess deformations which do not come from Veronese curves at all (and represent general projective configurations).

(3.9) The homology class of a (special) Veronese variety in Grassmannian.

We have associated to each isomorphism class of generic configurations of n points in P^{k-1} a certain embedding of P^{k-1} into the Grassmannian $G(k-1, n-1)$ — the Veronese variety. For instance, configurations of points on P^1 correspond to Veronese curves in P^{n-2} through a fixed set of n generic points. In this section we calculate the homology class Δ represented by these Veronese varieties in $G(k-1, n-1)$. It turns out that the coefficients of the expansion of Δ in the basis of Schubert cycles are exactly the dimensions of irreducible representations of the group $GL(n-k)$.

(3.9.1) Let us review the homology theory of the Grassmannian $G(p, q) = G(p, \mathbb{C}^q)$ (see [24] for more details). Let $\alpha = (\alpha_1 \geq \dots \geq \alpha_p \geq 0), \alpha_i \leq q-p$, be a decreasing sequence of non-negative integers. We visualize α as a Young diagram in which α_i are the lengths of rows. Because of inequalities $\alpha_i \leq q-p$ the diagram α lies inside the rectangle $[0, q-p] \times [0, p] \subset \mathbb{R}^2$:

(3.9.2)

We define $|\alpha| = \sum \alpha_i$ to be the number of cells in α .

(3.9.3) To each Young diagram α as above we associate the lattice path $\Lambda(\alpha) \subset [0, q-p] \times [0, p]$ going from the points $(0, 0)$ to the point $(q-p, p)$. This path is just the boundary of α , see Fig. (3.9.2). It consists of exactly q edges E_1, \dots, E_q which are horizontal or vertical segments of the lattice. We write E_i in such order that E_1 begins at $(0, 0)$ and E_q ends at $(q-p, p)$. The number of vertical edges is p and a path is completely determined by specifying which of the E_i are vertical. So the numbers of possible lattice paths

in $[0, q-p] \times [0, p]$ equals $\binom{q}{p}$. The same will be the number of all Young diagrams in $[0, q-p] \times [0, p]$, if we count also the "empty" diagram $(0, \dots, 0)$.

(3.9.4) Let $\alpha \subset [0, q-p] \times [0, p]$ be a Young diagram and $\Lambda(\alpha) = (E_1, \dots, E_q)$ be the corresponding lattice path. Associate to α the sequence $ht(\alpha) = (0 \leq ht_1(\alpha) \leq \dots \leq ht_q(\alpha) = p)$ by setting $ht_i(\alpha)$ to be equal the ordinate (height) of the end of the edge E_i .

(3.9.5) Let $V = (V_1 \subset V_2 \subset \dots \subset V_{q-1} \subset V_q = \mathbf{C}^q)$ be a complete flag of linear subspaces in \mathbf{C}^q so that $\dim(V_i) = i$. Let also $\alpha \subset [0, q-p] \times [0, p]$ be a Young diagram. We define the *Schubert variety* $S_\alpha(V) \subset G(p, q)$ to be the locus of subspaces $L \subset \mathbf{C}^q, \dim(L) = p$ such that $\dim(L \cap V_i) \geq ht_i(\alpha)$ where $ht_i(\alpha)$ was defined in (3.9.4).

It is well-known [24] that $S_\alpha(V)$ is an irreducible variety of complex dimension $|\alpha|$. The homology class in $H_{2|\alpha|}(G(p, q), \mathbf{Z})$ represented by $S_\alpha(V)$ is independent on V . This class is denoted by σ_α and called the *Schubert cycle*.

It is known that the homology group $H_{2r}(G(p, q), \mathbf{Z})$ is freely generated by cycles σ_α where α runs over all Young diagrams with r cells contained in the rectangle $[0, q-p] \times [0, p]$.

(3.9.6) Any Young diagram α (not necessarily contained in a given rectangle) defines the *Schur functor* Σ^α on the category of vector spaces [38]. By definition, for a vector space V the space $\Sigma^\alpha(V)$ is the space of irreducible representation of the group $GL(V)$ with the highest weight α . It can be defined, e.g., as the image of the Young symmetrizer h_α in the tensor space $V^{\otimes |\alpha|}$. In particular, for $\alpha = (m, 0, \dots, 0)$ (a horizontal strip of length m) the functor Σ^α is the symmetric power S^m . For a vertical strip $\alpha = (1^m) = (1, \dots, 1, 0, \dots, 0)$ (m units) the functor Σ^α is the exterior power \bigwedge^m .

(3.9.7) For a Young diagram α we denote by α^* the dual (or transposed) Young diagram defined by $\alpha_i^* = \text{Card}\{j : \alpha_j \geq i\}$. The rows on α^* correspond to columns of α and vice versa.

Now we can formulate the main result of this section.

(3.9.8) Theorem. *Let $X \subset G(k-1, n-1)$ be a special $(k-1)$ -dimensional Veronese variety and $\Delta \in H_{2k-2}(G(k-1, n-1), \mathbf{Z})$ be the homology class of X . Then the decomposition of Δ in the basis of Schubert cycles has the form*

$$\Delta = \sum_{|\alpha|=k-1} m_\alpha \cdot \sigma_\alpha, \quad \text{where} \quad m_\alpha = \dim(\Sigma^{\alpha^*}(\mathbf{C}^{n-k})).$$

(3.9.9) To prove Theorem 3.9.8, we first take X to be the variety of $(k-2)$ -dimensional chords of a Veronese curve $C \subset P^{n-2}$, which is a particular case of Veronese varieties, see section (3.8). Then we degenerate C into a union of lines.

More precisely, let $e_1, \dots, e_{n-1} \in P^{n-2}$ be the points corresponding to standard basis vectors of \mathbf{C}^{n-1} . Consider the reducible curve $D = D_1 \cup \dots \cup D_{n-2}$ where D_i is the line $\langle e_i, e_{i+1} \rangle$. Since D can be obtained as a limit position of Veronese curves, we find that Δ is equal to the homology class of the chordal variety $Ch_{k-2}(D)$, see (3.8.1) for the notation.

(3.9.10) The variety $Ch_{k-2}(D)$ is reducible and splits into $\binom{n-2}{k-1}$ components $X_{i_1, \dots, i_{k-1}}$ which correspond to sequences $1 \leq i_1 < \dots < i_{k-1} \leq n-2$. The component $X_{i_1, \dots, i_{k-1}}$ is the locus of chordal subspaces $\langle x_{i_1}, \dots, x_{i_{k-1}} \rangle$ where x_{i_ν} lies on the line $\langle e_{i_\nu}, e_{i_\nu+1} \rangle \subset D$. Therefore our homology class Δ is the sum of homology classes $[X_{i_1, \dots, i_{k-1}}]$ of all the components of $Ch_{k-2}(D)$.

(3.9.11) Let us introduce a different, more suitable for our purposes, combinatorial labeling of components of $Ch_{k-2}(D)$.

Denote by $W(n-k, k-1)$ the set of all (not necessarily decreasing) sequences $\lambda = (\lambda_1, \dots, \lambda_{n-k}) \in \mathbf{Z}_+^{n-k}$ of non-negative integers such that $\sum \lambda_i = k-1$. We shall call elements of $W(n-k, k-1)$ *weights* (in the sense of representation theory) in $n-k$ variables and of degree $k-1$.

To any sequence $1 \leq i_1 < \dots < i_{k-1} \leq n-2$ we associate a weight $\lambda(i_1, \dots, i_{k-1}) = (\lambda_1, \dots, \lambda_{n-k}) \in W(n-k, k-1)$ as follows. Let j_1, \dots, j_{n-k-1} be all elements of the set $\{1, \dots, n-2\} - \{i_1, \dots, i_{k-1}\}$, written

in the increasing order. Set also $j_{n-k} = n - 1$. Now define

$$(3.9.12) \quad \lambda(i_1, \dots, i_{k-1}) = (\lambda_1, \dots, \lambda_{n-k}), \quad \text{where } \lambda_\nu = j_\nu - j_{\nu-1} - 1.$$

The numbers λ_ν are just the lengths of arithmetic progressions with increment one into which the sequence (i_1, \dots, i_{k-1}) splits.

The correspondence $(i_1, \dots, i_{k-1}) \mapsto \lambda(i_1, \dots, i_{k-1})$ establishes a bijection between the set of all $(k-1)$ -element subsets in $\{1, \dots, n-2\}$ and the set $W(n-k, k-1)$. This bijection is the labeling we need.

We shall denote by $X(\lambda), \lambda \in W(n-k, k-1)$, the component $X(i_1, \dots, i_{k-1}) \subset Ch_{k-2}(D)$ where $\lambda(i_1, \dots, i_{k-1}) = \lambda$.

Now Theorem 3.9.8 will be a consequence of the following fact.

(3.9.13) Theorem. *Let $\alpha, |\alpha| = k-1$, be a Young diagram in the rectangle $[0, n-k-1] \times [0, k-1]$ and let $\lambda \in W(n-k, k-1)$ be any weight. Then the homology class of the component $X(\lambda) \subset G(k-1, n-1)$ has the form*

$$[X(\lambda)] = \sum_{|\alpha|=k-1} K_{\lambda, \alpha^*} \cdot \sigma_\alpha,$$

where K_{λ, α^*} is the multiplicity of weight λ in the irreducible representation $\Sigma^{\alpha^*}(\mathbb{C}^{n-k})$ (the Kostka number).

We shall concentrate on the proof of Theorem 3.9.13.

(3.9.14) Proposition. *Let $\lambda = (\lambda_1, \dots, \lambda_{n-k}) \in W(n-k, k-1)$ be a weight. The component $X(\lambda)$ is isomorphic to the product of projective spaces $\prod P^{\lambda_j}$. It is embedded into the Grassmannian as the image of the direct sum map*

$$\oplus : \prod P^{\lambda_j} = \prod G(\lambda_j, \lambda_j + 1) \hookrightarrow G(\sum \lambda_j, \sum (\lambda_j + 1)) = G(k-1, n-1).$$

Proof: Let $1 \leq i_1 < \dots, i_{k-1} \leq n-2$ be sequence of integers to which λ is associated, see (3.9.10). The component $X(\lambda)$ consists of chords which join points of lines $\langle e_{i_\nu}, e_{i_\nu+1} \rangle$. Let us split the sequence (i_1, \dots, i_{k-1}) into segments which are arithmetic progressions with increment 1. Then λ_ν are precisely lengths of these segments. Now let $i, i+1, \dots, i+\lambda_\nu$ be any such segment. The $(\lambda_\nu - 1)$ -dimensional chords of the subcurve $\langle e_i, e_{i+1} \rangle \cup \langle e_{i+1}, e_{i+2} \rangle \cup \dots \cup \langle e_{i+\lambda_\nu}, e_{i+\lambda_\nu+1} \rangle$ are just arbitrary hyperplanes in the projective subspace $\langle e_i, e_{i+1}, \dots, e_{i+\lambda_\nu+1} \rangle$. Any $k-2$ -dimensional chord from our component $X_{i_1, \dots, i_{k-1}} = X(\lambda)$ is therefore the projective span of hyperplanes in the independent projective subspaces $P^{\lambda_j} = P(\mathbb{C}^{\lambda_\nu+1})$, as required.

(3.9.15) Proposition. *The coefficient at the Schubert cycle σ_α in the decomposition of the class $[X(\lambda)]$ equals the multiplicity of the irreducible representation $\Sigma^\alpha \mathbb{C}^{n-k}$ in the tensor product $\bigwedge^{\lambda_1}(\mathbb{C}^{n-k}) \otimes \dots \otimes \bigwedge^{\lambda_{n-k}}(\mathbb{C}^{n-k})$ of exterior powers.*

The proof is based on the following (known) fact.

(3.9.16) Lemma. *For three Young diagrams α, β, γ such that $|\gamma| = |\alpha| + |\beta|$ let $c_{\alpha\beta}^\gamma$ be the multiplicity of Σ^γ in $\Sigma^\alpha \otimes \Sigma^\beta$ (the Littlewood - Richardson number). Then:*

a) *The image of the cycle $\sigma_\alpha \otimes \sigma_\beta$ under the direct sum map*

$$\phi : G(p_1, V_1) \times G(p_2, V_2) \rightarrow G(p_1 + p_2, V_1 \oplus V_2)$$

equals $\sum_\gamma c_{\alpha\beta}^\gamma \sigma_\gamma$.

b) *If A and B are finite-dimensional vector spaces then for any Young diagram γ we have the isomorphism of $GL(A) \times GL(B)$ -modules*

$$\Sigma^\gamma(A \oplus B) \cong \bigoplus_{|\alpha|+|\beta|=|\gamma|} c_{\alpha\beta}^\gamma \Sigma^\alpha(A) \otimes \Sigma^\beta(B).$$

Proof of (3.9.16): In part a) it suffices to treat the "stable" case when V_i have infinite dimension. We shall assume that it is so.

Let $H^*(G(p, \infty), \mathbf{Z})$ be the cohomology ring of $G(p, \infty)$ and let $Rep(GL(p))$ be the Grothendieck ring of polynomial representations of $GL(p)$. Let also $\Lambda_p = \mathbf{Z}[x_1, \dots, x_p]^{S_p}$ denote the ring of symmetric polynomials in p variables x_1, \dots, x_p . There are isomorphisms of rings

$$\Lambda_p \cong H^*(G(p, \infty), \mathbf{Z}) \cong Rep(GL(p)),$$

which take the elementary symmetric function $e_j \in \Lambda_p$ into the j -th Chern class of the tautological bundle on $G(p, \infty)$ and into the representation $\bigwedge^j(\mathbf{C}^p) \in Rep(GL(p))$.

For any Young diagram α denote by $s_\alpha(x_1, \dots, x_p) \in \Lambda_p$ the *Schur polynomial* (see [38]). It corresponds to the following elements of the two above rings:

- The cocycle $\sigma^\alpha \in H^{2|\alpha|}(G(p, \infty), \mathbf{Z})$ dual to the Schubert cycle σ_α (i.e. $(\sigma^\alpha, \sigma_\beta) = \delta_{\alpha\beta}$).
- The irreducible representation $\Sigma^\alpha \mathbf{C}^p$ of which s_α is the character.

Consider the tensor product $\Lambda_{p_1} \otimes \Lambda_{p_2}$. It can be regarded as a ring of polynomials $f(x_1, \dots, x_{p_1}, y_1, \dots, y_{p_2})$ symmetric with respect to x_i and with respect to y_i . Therefore we have an embedding

$$\delta : \Lambda_{p_1+p_2} \rightarrow \Lambda_{p_1} \otimes \Lambda_{p_2}.$$

(it is a part of Hopf algebra structure on the limit $\Lambda = \lim \Lambda_p$, see [38]).

The homology space of $G(p, \infty)$ is dual to Λ_p . The map ϕ_* , induced on homology by the direct sum map ϕ from part a) of the lemma, is known to be dual to δ .

Similarly, if we assume in part b) of the lemma that $\dim(A) = p_1$, $\dim(B) = p_2$ then the restriction map

$$Rep(GL(p_1 + p_2)) \rightarrow Rep(GL(p_1)) \otimes Rep(GL(p_2))$$

is identified with δ .

Thus in part a) we have to find the matrix elements of the dual map $\delta^* : \Lambda^* \otimes \Lambda^* \rightarrow \Lambda^*$ in the basis dual to that of s_α . In part b) we have to find matrix elements of δ in the basis of s_α . So both parts follow from the equality

$$s_\gamma(x_1, \dots, x_{p_1}, y_1, \dots, y_{p_2}) = \sum_{|\alpha|+|\beta|=|\gamma|} c_{\alpha\beta}^\gamma s_\alpha(x_1, \dots, x_{p_1}) s_\beta(y_1, \dots, y_{p_2}),$$

which is a reformulation of ([38], formula 5.9. Ch.I).

(3.9.17) *Proof of Proposition 3.9.15:* Note that the fundamental class of the projective space P^{λ_i} considered as $G(\lambda_i, \mathbf{C}^{\lambda_i+1})$ is the Schubert cycle corresponding to the Young diagram (1^{λ_i}) i.e. to the vertical strip of length λ_i . The Schur functor corresponding to this diagram is the exterior power. Now the result follows from Lemma 3.9.14.

(3.9.18) *Proof of Theorem 3.9.13:* By Lemma 3.9.15, it suffices to show that the weight multiplicity $K_{\lambda, \alpha}$ equals the multiplicity of Σ^α in $\bigwedge^{\lambda_1}(\mathbf{C}^{n-k}) \otimes \dots \otimes \bigwedge^{\lambda_{n-k}}(\mathbf{C}^{n-k})$. By Young duality this is equivalent to the saying that $K_{\lambda, \alpha}$, for any α , is equal to the multiplicity of Σ^α in the product of symmetric powers $S^{\lambda_1}(\mathbf{C}^{n-k}) \otimes \dots \otimes S^{\lambda_{n-k}}(\mathbf{C}^{n-k})$.

To see the truth of this latter statement, decompose \mathbf{C}^{n-k} into a sum of 1-dimensional subspaces $L_1 \oplus \dots \oplus L_{n-k}$. Then decomposition of $\Sigma^\alpha(\mathbf{C}^{n-k})$ as a $GL(L_1) \times \dots \times GL(L_{n-k})$ -module is just the weight decomposition. On the other hand, applying repeatedly Lemma 3.9.16 b) we find that $K_{\lambda, \alpha}$ i.e. the multiplicity of $S^{\lambda_1}(L_1) \otimes \dots \otimes S^{\lambda_{n-k}}(L_{n-k})$ in $\Sigma^\alpha(L_1 \oplus \dots \oplus L_{n-k})$ equals the multiplicity of Σ^α in $S^{\lambda_1} \otimes \dots \otimes S^{\lambda_{n-k}}$.

Theorems 3.9.13 and 3.9.8 are completely proven.

(3.9.19) **Remark.** A different expression for the coefficients m_α in Theorem 3.9.8 can be obtained from Klyachko's formula (Proposition 1.1.8) for the homology class of the whole Lie complex $Z \subset G(k, n)$. Our Veronese variety is just the visible contour of Z i.e., the intersection $Z \cap G(k-1, n-1)_p$, see (3.1.1). The intersection map

$$H_r(G(k, n), \mathbf{Z}) \rightarrow H_{r-n+k}(G(k-1, n-1)_p, \mathbf{Z})$$

is easy to describe. It takes a Schubert cell $\sigma_{\beta_1, \dots, \beta_k}$ to $\sigma_{\beta_2, \dots, \beta_k}$ if $\beta_1 = n - k$ and to 0 otherwise. This leads to the formula

$$m_\alpha = \sum_{i=0}^k (-1)^i \binom{n}{i} \dim(\Sigma^{n-k, \alpha_1, \dots, \alpha_{k-1}}(\mathbf{C}^{k-i})).$$

According to Theorem 3.9.8 this expression equals just $\dim \Sigma^{\alpha^*}(\mathbf{C}^{n-k})$ but we do not know a straightforward proof of this fact.

(3.9.20) Remark. Let $\alpha = (\alpha_1, \dots, \alpha_{k-1})$, $|\alpha| = k-1$, be a Young diagram in the rectangle $[0, n-k] \times [0, k-1]$. Denote by $\bar{\alpha} = (n-k-\alpha_{k-1}, \dots, n-k-\alpha_1)$ the diagram complementary to α in this rectangle. It is known [24] that for any Young diagram β with $|\beta| = (k-1)(n-k-1)$ the intersection index $\sigma_\alpha \cdot \sigma_\beta$ equals 1 if $\beta = \bar{\alpha}$ and to 0 otherwise. Hence the coefficient m_α in the expansion, by Schubert cycles, of the cycle represented by the Veronese variety $S \subset G(k-1, n-1)$, equals $S \cdot \sigma_{\bar{\alpha}}$.

Let us realize $\sigma_{\bar{\alpha}}$ as the class of the Schubert variety $S_{\bar{\alpha}}(V)$ for a generic flag V . Let us take the Veronese variety S to be the chordal variety of Veronese curve. We obtain the following restatement of Theorem 3.9.8:

The number of chordal $(k-1)$ - dimensional subspaces of a Veronese curve in P^{n-2} satisfying any given Schubert condition, equals the dimension of some irreducible representation of $GL(n-k)!$ The dimension of any representation can be realized in this way.

It would be interesting to find a conceptual explanation of this fact e.g., define a $GL(n-k)$ -action on the vector space freely generated by points from $S \cap S_{\bar{\alpha}}(V)$. Let us also point out to a series of papers of A.N. Kirillov and N.Yu. Reshetikhin (see [33,34] and references therein) on new combinatorial formulas for weight multiplicities $K_{\lambda, \alpha}$. Their construction is based on an interpretation of any $K_{\lambda, \alpha}$ as the number of solutions of some special system of algebraic equations (the equations of Bethe - Ansatz). This interpretation seems to be connected with the one given above.

(3.9.21) Example. It is well known that the number of nodes of a plane rational curve of degree d equals $(d-1)(d-2)/2$. We can obtain this as a particular case of Theorem 3.9.8. Let $C \subset P^d$ be a Veronese curve, $L \subset P^d$ - a projective subspace of dimension $d-3$ and $\pi : P^d - L \rightarrow P^2$ - the projection with center L . Nodes of the plane curve $\pi(C)$ correspond to 1-dimensional chords of C intersecting L . Let $X = Ch_1(C) \subset G(2, d+1)$ be the surface of chordal lines of C . By Theorem 3.9.8, its homology class has the form

$$[X] = \dim(S^2 \mathbf{C}^{d-1}) \cdot \sigma_{1,1} + \dim(\bigwedge^2 \mathbf{C}^{d-1}) \cdot \sigma_{2,0}.$$

The coefficient at $\sigma_{2,0}$ equals the intersection index of S with the Schubert cycle $\sigma_{d-1, \dots, d-1, d-3}$. The corresponding Schubert variety is the locus of all lines intersecting a given $(d-3)$ -dimensional subspace in P^d , for example, L . So we find the number of nodes of $\pi(C)$ to be $\dim(\bigwedge^2 \mathbf{C}^{d-1}) = (d-1)(d-2)/2$.

(3.9.22) Example. The number of 4-secant lines of a spatial rational curve $X \subset P^3$ of degree d can be found by reasoning similar to the above example. This number equals

$$(3.9.23) \quad \dim(\Sigma^{2,2} \mathbf{C}^{d-3}) = (d-2)(d-3)^2(d-4)/12.$$

The right hand side of (3.9.23) is a well-known formula for the number of quadrisecants, see e.g. [24, Ch.2, §5].

(3.9.24) Example. The number of trisecant lines of a rational curve of degree d in P^4 equals

$$(3.9.25) \quad \dim(\bigwedge^3 \mathbf{C}^{n-2}) = (n-2)(n-3)(n-4)/6.$$

**Chapter 4. CHOW QUOTIENT OF $G(2, n)$
AND GROTHENDIECK - KNUDSEN MODULI SPACE $\overline{M}_{0,n}$.**

In this section we study in detail the Chow quotient $G(2, n)//H$ of the Grassmannian $G(2, n)$ of lines in P^{n-1} . We establish the isomorphism of this Chow quotients with the moduli space $\overline{M}_{0,n}$ of stable n -punctured curves of genus 0 introduced by A.Grothendieck [12] and later by F.Knudsen [37]. In particular, $G(2, n)//H$ is a smooth variety and the complement to the open stratum is a divisor with normal crossing. The relation of the space $\overline{M}_{0,n}$ to the Grassmannian permits us to represent this space as an iterated blow-up of the projective space P^{N-3} .

(4.1) The space $G(2, n)//H$ and stable curves.

(4.1.1) According to Theorem 2.2.4, we have an isomorphism

$$G(2, n)//H = (P^1)^n // GL(2).$$

In other words, our Chow quotient compactifies the space

$$M_{0,n} = ((P^1)^n - \bigcup \{x_i = x_j\}) / GL(2)$$

of projective equivalence classes of n -tuples of distinct points on P^1 . The space $M_{0,n}$ can be considered as the moduli space of systems (C, x_1, \dots, x_n) where C is a smooth curve of genus 0 and x_i are distinct points on C .

(4.1.2) There is a well-known compactification of $M_{0,n}$ by means of so-called stable n -pointed curves of genus 0 introduced by Grothendieck and Knudsen [12][37]. Let us recall the definitions.

(4.1.3) Definition. A stable n -pointed curve of genus 0 is a connected (but possibly reducible) curve C over k together with n smooth distinct points $x_1, \dots, x_n \in C$, satisfying the following conditions:

- (1) C has only ordinary double points and every irreducible component of C is isomorphic to the projective line P^1 .
- (2) The arithmetic genus of C is equal to 0.
- (3) On each component of C there are at least three points which are either marked or double. Points of C which are either marked or double will be called special.

The condition (2) is equivalent to saying that the graph formed by components of C is a tree. We shall prefer the following "dual" point of view on this tree.

(4.1.4) Definition. Let (C, x_1, \dots, x_n) be a stable n -pointed curve of genus 0. Its tree $T(C, x_1, \dots, x_n)$ has the following vertices:

- (1) Endpoints (1-valent vertices) A_1, \dots, A_n corresponding to x_0, \dots, x_n .
- (2) Vertices corresponding to all the components of C .

Two vertices of type (2) are joined by an edge if the corresponding components intersect. An endpoint A_i is joined by a new edge to the vertex of type (2) corresponding to the unique component containing the point x_i .

Definition 4.1.4 is illustrated on Fig. 4.1.5.

(4.1.5)

Thus edges of $T(C, x_1, \dots, x_n)$ correspond to special points of C .

(4.1.6) F.Knudsen has constructed in [37] the moduli space $\overline{M}_{0,n}$ of stable n -pointed curves and proved that it is a smooth compact algebraic variety. To formulate Knudsen's result more precisely, let us introduce a notion of a stable n -pointed curve over an arbitrary base scheme S . By definition, it is a flat proper morphism $\pi : C \rightarrow S$ together with n distinguished sections $s_1, \dots, s_n : S \rightarrow C$ such that for any geometric point $s \in S$ the fiber $C_s = \pi^{-1}(s)$ is a reduced (i.e. without nilpotents) algebraic curve and $(C_s, s_1(s), \dots, s_n(s))$ is a stable n -pointed curve of genus 0. An isomorphism between two such objects $(\pi : C \rightarrow S, s_1, \dots, s_n)$ and $(\pi' : C' \rightarrow S, s'_1, \dots, s'_n)$ over the same base S is just an isomorphism $f : C \rightarrow C'$ commuting with projections and taking s_i to s'_i .

(4.1.7) Theorem. [37] *There exists a smooth projective complex algebraic variety $\overline{M}_{0,n}$ such that for any scheme S over \mathbb{C} the set of isomorphism classes of stable n -pointed curves of genus 0 over S is naturally identified with $\text{Hom}(S, \overline{M}_{0,n})$.*

An open subset $M_{0,n}$ in $\overline{M}_{0,n}$ is formed by n -pointed curves (C, x_1, \dots, x_n) such that C is smooth i.e. $C \cong P^1$.

Now we can formulate the complete description of the Chow quotient of $G(2, n)$.

(4.1.8) Theorem. *The Chow quotients $G(2, n)//H$ and $(P^1)^n//GL(2)$ are isomorphic to the moduli space $\overline{M}_{0,n}$.*

To prove Theorem 4.1.8 note that Theorem 3.3.14 together with Corollary 3.4.9 implies the following description of $G(2, n)//H$.

(4.1.9) Corollary. *Take n points p_1, \dots, p_n in P^{n-2} in general position. Let $V_0(p_1, \dots, p_n)$ be the space of all Veronese curves in P^{n-2} through p_i . Denote by $V(p_1, \dots, p_n)$ the closure of $V_0(p_1, \dots, p_n)$ in the Chow variety and by $W(p_1, \dots, p_n)$ -the closure in the Hilbert scheme. Then $V(p_1, \dots, p_n) \cong W(p_1, \dots, p_n) \cong G(2, n)//H$.*

It was proven in [29] (Theorem 0.1) that $V(p_1, \dots, p_n)$ and $W(p_1, \dots, p_n)$ are isomorphic to $\overline{M}_{0,n}$. More precisely, any subscheme from $W(p_1, \dots, p_n)$ is in fact reduced and being regarded together with p_i , is a stable n -pointed curve of genus 0. Theorem 4.1.8 is proven.

(4.1.10) Remark. It seems to be difficult to prove directly that the Chow quotient $(P^1)^n//GL(2)$ coincides with $\overline{M}_{0,n}$. However, the Grassmannian picture (i.e. the Gelfand- MacPherson isomorphism, see §2) leads to stable curves very naturally: these curves are just visible contours of generalized Lie complexes.

(4.1.11) We can now give a translation to the language of stable curves of general constructions of §1.

The combinatorial invariant of a stable n -pointed curve $(C, x_1, \dots, x_n) \in \overline{M}_{0,n}$ is its tree $T(C)$ (Definition 4.2.) For each tree T bounding the endpoints $1, \dots, n$ we define the stratum $M(T) \subset \overline{M}_{0,n}$ consisting of stable C curves with $T(C) = T$. In particular, to a 1-vertex tree corresponds the open stratum $M_{0,n} \subset \overline{M}_{0,n}$.

The combinatorial invariant of a generalized Lie complex $Z \subset G(k, n)//H$ is the corresponding matroid decomposition of the hypersimplex $\Delta(k, n)$ (Proposition 1.9). The Chow strata in $G(k, n)//H$ were defined (Definition 1.2.16) as the loci of Z for which the corresponding matroid decomposition is fixed.

It was proven in section 1.3 that matroid decomposition of the hypersimplex $\Delta(2, n)$ correspond exactly to trees bounding n endpoints A_1, \dots, A_n , see Theorem 1.3.6. Thus we obtain the following corollary.

(4.1.12) Corollary. *All matroid decompositions of the hypersimplex $\Delta(2, n)$ are realizable i.e. come from non-empty Chow strata in $G(2, n)//H$. These Chow strata have the form $M(T) \subset \overline{M_{0,n}}$. The stratum $M(T)$ is isomorphic to the product $\prod M_{0,e(v)}$, where e runs over v of T and $e(v)$ is the number of edges containing v .*

(4.1.13) Forgetting i -th point on any stable n -pointed curve $(C, x_1, \dots, x_n) \in \overline{M_{0,n}}$ gives a new n -pointed curve. This curve might be unstable i.e. the condition d) of Definition 4.1.3 might be violated. Clearly, this happens in the case when the component of C containing x_i contains only two other double or marked points). Blowing down this component defines a stable curve $\pi_i(C)$ pointed with images of $x_j, j \neq i$, see [37]. It was shown in [37] that π_i defines a morphism $\overline{M_{0,n}} \rightarrow \overline{M_{0,n-1}}$ which identifies $\overline{M_{0,n}}$ with the universal family of curves over $\overline{M_{0,n-1}}$.

It was shown in [29] that π_i corresponds to geometric projection of (degenerate) Veronese curves from $V(p_1, \dots, p_n)$ (Corollary 4.5) from the point p_i . In terms of generalized Lie complexes (= points of $G(2, n)//H$) the projection π_i is described as follows.

(4.1.14) Proposition. *Let $Z \subset G(2, n)$ be a generalized Lie complex. Let $G(2, n-1)^i$ be the space of lines in P^{n-1} which in fact lie in the $(n-2)$ -dimensional projective subspace spanned by basis vectors $e_j, j \neq i$. Then $Z \cap (G(2, n-1)^i)$ is a generalized Lie complex in $G(2, n-1)_i$. The operation of intersection with $G(2, n-1)^i$ corresponds, under the identification of Theorem 4.1.8, to the projection $\pi_i : \overline{M_{0,n}} \rightarrow \overline{M_{0,n-1}}$.*

(4.2) The birational maps $\sigma_i : \overline{M_{0,n}} \rightarrow P^{n-3}$.

(4.2.1) The Grothendieck - Knudsen space $\overline{M_{0,n}}$ can be seen as a "high-brow" compactification of the space $M_{0,n}$ of projective equivalence classes of n -tuples of distinct points on P^1 , see (4.1.1). On the other hand, every three distinct points on P^1 can be brought to the points $0, 1, \infty$ by a unique projective transformation. Doing this with the first three points of any n -tuple, we find that

$$M_{0,n} \cong \{(x_4, \dots, x_n) \in \mathbb{C}^{n-3} : x_i \neq 0, 1, \forall i \text{ and } x_i \neq x_j, \forall i \neq j\}.$$

So $M_{0,n}$ as an open subset in \mathbb{C}^{n-3} . This suggests a "naive" compactification of $M_{0,n}$ which is just the projective space P^{n-3} compactifying \mathbb{C}^{n-3} . One expects then that $\overline{M_{0,n}}$, being the finer compactification, maps to P^{n-3} by means of a regular birational map.

(4.2.2) As shown in [29], the regular map $\overline{M_{0,n}} \rightarrow P^{n-3}$ can be constructed as follows. Realize $\overline{M_{0,n}}$ as the space $V(p_1, \dots, p_n)$ of limit position of Veronese curves in P^{n-2} containing given generic points p_1, \dots, p_n . For any curve $C \in V(p_1, \dots, p_n)$ all p_i are smooth points of C . Fix some i and consider the projective space P_i^{n-3} of all lines in P^{n-2} through p_i . By associating to any curve $C \in V(p_1, \dots, p_n)$ its embedded tangent line $T_{p_i}C$ one gets a regular map

$$\sigma_i : \overline{M_{0,n}} \rightarrow P_i^{n-3}.$$

It was demonstrated in [29] that P_i^{n-3} is exactly the "naive" compactification of $M_{0,n}$ mentioned in (4.2.1). Its dependence on i is easy to explain: we need to specify which point of an n -tuple is set to be ∞ . So the construction of (4.2.1) corresponds to $i = 3$.

Here we are going to study the maps σ_i in more detail.

(4.2.3) Let $L_i, i = 1, \dots, n$, be the line bundle on $\overline{M_{0,n}}$ whose fiber at a pointed curve (C, x_1, \dots, x_n) is $T_{x_i}^*C$, the cotangent space to C at x_i . Clearly $L_i \cong \sigma_i^*(\mathcal{O}_{P_i^{n-3}}(1))$. The following fact was proven in [29].

(4.2.4) Proposition. *For any $i \in \{1, \dots, n\}$ the space $H^0(\overline{M_{0,n}}, L_i)$ has dimension $n-2$. The corresponding morphism γ_{L_i} is everywhere regular, birational and, moreover, one-to-one outside the subvariety in P_i^{n-3} formed by lines which lie in a hyperplane spanned by p_i . In the Veronese picture the space $P(H^0(\overline{M_{0,n}}, L_i)^*)$ is identified with P_i^{n-3} and γ_{L_i} is identified with σ_i .*

(4.2.5) Let us give a description of maps σ_i on the language of generalized Lie complexes. So we start with the standard coordinatized projective space $P^{n-1} = P(\mathbb{C}^n)$. For any point $x \in P^{n-1}$ we shall denote by $P^{n-2}(x) \subset G(2, n)$ the space of lines in P^{n-1} meeting x . Let $e_i \in P^{n-1}$ be the i -th basis vector.

(4.2.6) Proposition. *Any Lie complex (and hence any generalized Lie complex) in $G(2, n)$ contains each projective space $P^{n-2}(e_i)$.*

Proof: Let Z be a Lie complex. The intersection $Z \cap P_{e_i}^{n-2}$ contains the closure of a generic torus orbit in $P^{n-2}(e_i)$. Since this generic orbit is dense in $P^{n-2}(e_i)$, the assertion follows.

(4.2.7) Note that the dimension of a (generalized) Lie complex Z is just by one greater than that of $P^{n-2}(e_i)$. Hence at a generic point l of $P^{n-2}(e_i)$ the tangent space $T_l Z$ represents a line in the normal space $N_l(Z/G(2, n)) = T_l G(2, n)/T_l Z$. In the construction of the Veronese curve corresponding to Z (as the visible contour, see §(3.1)) we considered the set of all lines in Z meeting a given point u or, in other words, the intersection $Z \cap P^{n-2}(u)$. More precisely, we specialized to $u = e = (1, \dots, 1)$.

(4.2.8) Proposition. *The space P_i^{n-3} is naturally identified with the projectivization of the normal space to $P^{n-2}(e_i)$ in $G(2, n)$ at the point $p_i = \langle e_i, u \rangle$. The map σ_i is identified to the map taking any generalized Lie complex Z to the line in the above normal space given by the subspace $T_l Z$.*

Proof: The subvarieties $P^{n-2}(e_i)$ and $P^{n-2}(u)$ in $G(2, n)$ are of middle dimension and intersect transversely in the point $p_i = \langle e_i, u \rangle$. Therefore the normal space in question is naturally identified with the tangent space at p_i to $P^{n-2}(u)$. Since σ_i is defined by considering the tangent line to $Z \cap P^{n-2}(u)$ at p_i , the assertion follows.

(4.2.9) The advantage of the description in (4.1.20) is that it clearly states the dependence on the choice of a point u . It also shows how to obtain a more invariant description of σ_i . To do this, one should by consider all the 1-dimensional subspaces in the normal spaces to $P^{n-2}(e_i)$ in $G(2, n)$ at all the generic points or, in other words, the corresponding subbundle in the normal bundle. Let us describe these bundles.

(4.2.10) Proposition. *Let $x \in P^{n-1}$ be any point. Then the normal bundle of the subvariety $P^{n-2}(x) \subset G(2, n)$ is naturally isomorphic to the twisted tangent bundle $T_{P^{n-2}(x)} \otimes \mathcal{O}_{P^{n-2}(x)}(-1)$.*

Proof: Let $l \in P_x^{n-2}$ be any line in P^{n-1} containing x and let $N = N_{l/P^{n-1}}$ be the normal bundle of l . The tangent space $T_l G(2, n)$ is identified (by Kodaira-Spencer) with the space $H^0(l, N)$ i.e. with the space of normal vector fields on l . The subspace $T_l P_x^{n-2} \subset T_l G(2, n)$ consists of those fields v which vanish at x i.e. $v(x) = 0$. Hence we have a linear map

$$(N_{P^{n-2}(x)/G(2, n)})_l \mapsto (N_{l/P^{n-1}})_x, \quad v \mapsto v(x).$$

It is immediate to check that this map is in fact an isomorphism. Let now E be the vector bundle on $P^{n-2}(x)$ whose fiber over a line l is the normal space $(N_{l/P^{n-1}})_x$. We have proven that $N_{P^{n-2}(x)/G(2, n)}$ is isomorphic to E . Let us regard P_x^{n-2} as the projectivisation of the vector space $W = T_x P^{n-1}$. Then we have the following description of the bundle E on $P(W)$: the fiber of E at a 1-dimensional linear subspace $\Lambda \subset W$ is W/Λ . This is the standard description of $T_{P(W)} \otimes \mathcal{O}_{P(W)}(-1)$, the so-called Euler sequence, see [45].

(4.2.11) Proposition 4.2.10 implies that the projectivization of the normal bundle $N_{P^{n-2}(x)/G(2, n)}$ is the same as that of the tangent bundle $T_{P^{n-2}(x)}$. So any (generalized) Lie complex defines a 1-dimensional subbundle in the tangent bundle of $P^{n-2}(e_i)$ (this subbundle can be defined only over generic points; over some special points of $P^{n-2}(e_i)$ it may have singularities i.e. become a non-locally free coherent sheaf). In other words, we have a field of directions in $P^{n-2}(e_i)$. Let us denote this field by $\Sigma_i(Z)$ in $P^{n-2}(e_i)$. Let H_i be the coordinate hyperplane in P^{n-1} opposite to e_i . The projection from e_i identifies H_i with $P^{n-2}(e_i)$ so we can consider the direction field $\Sigma_i(Z)$ as being defined on H_i . It can be regarded as the choice-free materialization of $\sigma_i(C)$ where C is the stable curve corresponding to Z .

(4.2.12) Let us describe the direction field $\Sigma_i(Z)$ geometrically. Note that since Z is H -invariant the visible contour Z_p does not change, up to isomorphism, for points p lying in one torus orbit. We shall look how does Z_p split when p goes to a point on H_i not lying on other H_j .

(4.2.13) Proposition. *The field of directions $\Sigma_i(Z)$ is well defined at any point of H_i not lying on coordinate hyperplanes. For such a point x the visible cone $Z_x = Z \cap P^{n-2}(x)$ splits into a stable curve (family of lines) in $P^{n-2}(x)$, all whose lines lie in H_i and a plane pencil of lines containing $\langle x, e_i \rangle$. The direction $\Sigma_i(Z)(x)$ is given by the unique line of this pencil lying in H_i .*

Proof: Let C be the visible contour of Z at the point $e = (1, \dots, 1)$. Let K be the union of all lines from C . This is a cone with vertex e containing all the lines $\langle e, e_i \rangle$. All points of $\langle e, e_i \rangle$ except e are smooth points of K . Let L be the embedded tangent (2-) plane to K along $\langle e, e_i \rangle$. The L lies in $P^{n-2}(e_i)$ and is, by definition, equal to $\sigma_i(C)$. To prove the proposition, it suffices to consider the case when $x \in H_i$ is the barycenter of the coordinate simplex i.e. all homogeneous coordinates of x are equal to 1 except the i -th which is equal to 0. Consider the transformation $\gamma(t) = (1, \dots, 1, t, 1, \dots, 1) \in (\mathbb{C}^*)^n$ (t is on the i -th place). Then we have $x = \lim_{t \rightarrow 0} \gamma(t) \cdot e$. Hence the visible cone at x of the (torus - invariant) complex Z equals $\lim_{t \rightarrow 0} \gamma(t) \cdot K$. But the latter limit will be the union of the 2-plane L (which is preserved under all $\gamma(t)$ and some other part which will lie inside H_i . It remains to show that the intersection $L \cap H_i$ is precisely the line in H_i whose direction is the value at x of the direction field $\Sigma_i(Z)$. This checking is left to the reader.

(4.3) Representation of the space $\overline{M}_{0,n} = G(2, n)/H$ as a blow-up.

In the previous sections we have constructed regular birational morphisms σ_i from $\overline{M}_{0,n} = G(2, n)/H$ to projective spaces. When such a morphism is found, it is always desirable to decompose it to simpler ones. Standard examples of "simplest" regular birational morphisms are provided by blow-ups.

(4.3.1) Recall [24,25] that the *blow-up* (or monoidal transformation, or sigma- process) $Bl_Y X$ is defined for any smooth closed subvariety Y (which is called the *center*) in a smooth variety X . This is a new smooth variety equipped with a canonical morphism p to X . The morphism p is one-to one outside Y and for any $y \in Y$ the preimage $p^{-1}(y)$ is canonically identified with the projectivization of the normal space $T_y X / T_y Y$. If $Z \subset X$ is another submanifold not contained in Y then the *strict preimage* (or proper transform) of Z is the closure \tilde{Z} of $p^{-1}(Z - Y)$ in $Bl_Y X$. The subvariety $\tilde{Y} \subset Bl_Y X$ can be, in its turn blown up, thus giving an iterated blow-up which is abusively denoted $Bl_Z Bl_Y X$. This blow-up does not, in general, coincide with $Bl_Y Bl_Z(X)$ (it does, if Y and Z are disjoint). Similar construction can be performed for several subvarieties Y_1, \dots, Y_r .

(4.3.2) Our aim in this section is to decompose the morphism σ_i into a sequence of monoidal transformations and therefore to give a "constructive" definition of $\overline{M}_{0,n} = G(2, n)/H$ as a iterated blow-up of a projective space. Proposition 4.10 suggests that in order to obtain $\overline{M}_{0,n}$ we should blow up $n - 1$ generic points q_1, \dots, q_{n-1} in P^{n-3} , and all projective subspaces spanned by them. However, an iterated blow-up depends on the ordering of the centers, so the question is delicate.

(4.3.3) Theorem. *Choose $n - 1$ generic points q_1, \dots, q_{n-1} in P^{n-3} . The variety $\overline{M}_{0,n}$ can be obtained from P^{n-3} by a series of blow ups of all the projective spaces spanned by q_i . The order of these blow-ups can be taken as follows :*

- 1) Points q_1, \dots, q_{n-2} and all the projective subspaces spanned by them in order of the increasing dimension;
 - 2) The point q_{n-1} , all the lines $\langle q_1, q_{n-1} \rangle, \dots, \langle q_{n-3}, q_{n-1} \rangle$ and subspaces spanned by them in order of the increasing dimension;
 - 3) The line $\langle q_{n-2}, q_{n-1} \rangle$, the planes $\langle q_i, q_{n-2}, q_{n-1} \rangle, i \neq n - 3$ and all subspaces spanned by them in order of the increasing dimension.
- etc. etc

(4.3.4) Remark. A representation of $\overline{M}_{0,n}$ as an iterated blow-up of the Cartesian power $(P^1)^{n-3}$ was given by S.Keel. Still another representation of $\overline{M}_{0,n}$ as a blow-up of P^{n-3} , different from the one given here, can be deduced from a more general construction of W.Fulton and R.MacPherson [15]. In Fulton-MacPherson construction all the centers of blow-ups have codimension 2.

(4.3.5) The rest of this section will be devoted to the proof of Theorem 4.3.3.

In Proposition 1.4.12 we have constructed some regular birational morphisms f_I of general Chow quotient of Grassmannian $G(k, n)$ to the "secondary variety" of the product of two simplices $\Delta^{k-1} \times \Delta^{n-k-1}$. We want now to analyze these morphisms for the Chow quotient of $G(2, n)$ which is $\overline{M}_{0,n}$ in order to use them as a halfway approximation to a required sequence of blow-ups.

(4.3.6) Recall that for every two-element subset $I = \{i, j\} \subset \{1, \dots, n\}$ the coordinate subspace $\mathbf{C}^I = \mathbf{C}e_i \oplus \mathbf{C}e_j \subset \mathbf{C}^n$ is a fixed point for the torus action on $G(2, n)$ and hence our torus H acts in the tangent space $T_I = T_{\mathbf{C}^I}G(2, n)$ and on its projectivization. To each H - orbit in $G(2, n)$ whose closure contains \mathbf{C}^I is therefore associated an H - orbit in the projective space $P(T_I)$. The map f_I from $G(2, n)/H = \overline{M}_{0,n}$ to $P(T_I)/H$ is induced by this correspondence.

(4.3.7) As we explained in §0.2, the Chow quotient of a projective space by a torus H is a toric variety corresponding to the secondary polytope of the point configuration given by the characters of H defining the action. In our case the space T_I is identified with the space of 2 by $n-2$ matrices $\|a_{ij}\|, i \in I, j \in \bar{I}$ and the action of a torus element (t_1, \dots, t_n) on such a matrix gives a new matrix $\|t_i^{-1}t_j a_{ij}\|, i \in I, j \in \bar{I}$. The H - characters are therefore identified with vectors $e_j - e_i, i \in I, j \in \bar{I}$ of \mathbf{Z}^n . These vectors are vertices of the simplicial prism $\Delta^1 \times \Delta^{n-3}$ which we shall also denote $\Delta^I \times \Delta^{\bar{I}}$ to emphasize the dependence on I .

For the case of simplicial prisms $\Delta^1 \times \Delta^k$ triangulations have a complete and simple description.

(4.3.8) Note that the symmetric group S_{k+1} acts on $\Delta^1 \times \Delta^k$ by permuting the vertices of the second factor and hence acts on the triangulations of $\Delta^1 \times \Delta^k$.

Let us describe the *standard* triangulation of $\Delta^1 \times \Delta^k$ used in combinatorial topology [16]. It depends on the numberings of the vertices of factors. To fix these numberings denote the vertices of our prism by pairs (a, b) where $0 \leq a \leq 1, 0 \leq b \leq k$. The triangulation T_{st} consists of the simplices $\Delta_i, 0 \leq i \leq k$, where Δ_i is the convex hull of $(0, j), j \leq k$ and $(1, j), j \geq k$. The characteristic function of this triangulation (i.e. the corresponding vertex of the secondary polytope, see §(0.2)) equals $\phi_{st}(i, j) = j + 1$.

(4.3.9) Proposition. *There exist exactly $(k+1)!$ regular triangulations of the prism $\Delta^1 \times \Delta^k$. All they can be obtained from the standard one by action of S_{k+1} .*

In fact, all the triangulation of the prism are regular, but we do not need this.

Proof: Let Σ be the secondary polytope of $\Delta^1 \times \Delta^k$. Its vertices are functions $\phi_T(i, j), i = 0, 1; j = 0, \dots, k$ where T runs over all the regular triangulations. Let us use the original interpretation of the secondary polytope as the Newton polytope of the principal determinant [22,23]. In our situation this means the following.

Consider a $2 \times (k+1)$ -matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0k} \\ a_{10} & a_{11} & \dots & a_{1k} \end{pmatrix}$$

with indeterminate entries. Consider the polynomial $E(A) = (\prod_{p,j} a_{pj}) \cdot \prod_{0 \leq i < j \leq k} D_{ij}(A)$ where $D_{ij}(A) = a_{0i}a_{1j} - a_{0j}a_{1i}$ is the minor of A on i -th and j -th column. Then, as shown in [23], Σ is the Newton polytope of E i.e. the convex hull in $Mat(2 \times (k+1), \mathbf{R})$ of integral points $\omega = \|\omega_{pj}\| \in Mat(2 \times (k+1), \mathbf{Z}_+)$ such that the monomial $\prod_{p,j} a_{pj}^{\omega_{pj}}$ enters $E(A)$ with non-zero coefficient. On the other hand, $E(A)$ can be found explicitly by means of the Vandermonde determinant:

$$E(A) = \left(\prod_{p,j} a_{pj} \right) \cdot \det \begin{pmatrix} a_{00}^k & a_{01}^k & \dots & a_{0k}^k \\ a_{00}^{k-1}a_{10} & a_{01}^{k-1}a_{11} & \dots & a_{0k}^{k-1}a_{1k} \\ a_{00}^{k-2}a_{10}^2 & a_{01}^{k-2}a_{11}^2 & \dots & a_{0k}^{k-2}a_{1k}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{10}^k & a_{11}^k & \dots & a_{1k}^k \end{pmatrix}$$

The exponent vector of any monomial of this polynomial is obtained from the vector ϕ_{st} described in (4.3.8), by a permutation of columns. Proposition is proven.

(4.3.10) Corollary. *The secondary polytope of $\Delta^1 \times \Delta^k$ is linearly isomorphic to the convex hull of the S_{k+1} -orbit of the point $(1, 2, \dots, k+1) \in \mathbb{Z}^{k+1}$.*

This polytope is known as the k -dimensional *permutohedron* and denoted \mathcal{P}_k . It is a particular case of so-called general hypersimplices associated to homogeneous spaces G/P by I.M. Gelfand and V.V. Serganova [21].

(4.3.11) The toric variety corresponding to \mathcal{P}_k will be called the k -dimensional *permutohedral space* and denoted Π^k . Here is one of the description of this space which generalizes to arbitrary G/P -hypersimplices.

(4.3.12) Proposition. [21] *The permutohedral space Π^k is isomorphic to the closure of a generic orbit of torus $(\mathbb{C}^*)^{k+1}$ on the space of complete flags of linear subspaces of \mathbb{C}^{k+1} .*

We will be interested in a slightly different point of view on Π^k realizing it as an explicit blow-up of a projective space P^k .

(4.3.13) Proposition. *The permutohedral space Π^k can be obtained from the projective space P^k by the following sequence of blow-ups. First blow up $k+1$ generic points (the projectivizations of basis vectors) then blow up the strict preimages of all coordinate lines joining them, then the strict preimages of coordinate planes etc.*

Proof: Let F_i be the space of $(1, 2, \dots, i)$ -flags in \mathbb{C}^{k+1} . Let $X = X_k$ be the closure of a generic orbit of $(\mathbb{C}^*)^k$ in F_k and X_i the projection of X to F_i . Then X_1 is the projective space $F_1 = P^k$. It is straightforward to see that each projection $X_i \rightarrow X_{i-1}$ realizes X_i as the blow-up of strict preimages of all $(i-1)$ -dimensional projective subspace spanned by basis vectors of \mathbb{C}^{k+1} .

(4.3.14) Remarks. a) Note that the orbit closure $\Pi^k = X \subset F_k$ can be mapped as well to the projective space of hyperplanes in \mathbb{C}^{k+1} . Considering the decomposition of this projection through spaces of $(i, i+1, \dots, k)$ -flags we find that X is represented as the blow-up of the dual projective space P^{k^\vee} similar to that of Proposition 4.3.13. The corresponding birational map from P^k to P^{k^\vee} is the standard Cremona inversion [24]. Thus the permutohedral space provides an explicit decomposition of the Cremona inversion to sigma-processes and their inverses.

b) In the correspondence between convex polytopes and toric varieties blowing up the closure of an orbit corresponds to chiseling of the face corresponding to this orbit, see [49]. Proposition 4.3.13 amounts to the following construction of permutohedron from the simplex. First cut out all vertices, then all edges etc.

(4.3.15) Let us relate the regular birational morphisms $\sigma_i : \overline{M}_{0,n} \rightarrow P_i^{n-3}$ and $f_{ij} : \overline{M}_{0,n} \rightarrow \Pi_{ij}^{n-3}$.

(4.3.16) Proposition. *There exist regular birational morphisms $\tau_{ij} : \Pi_{ij}^{n-3} \rightarrow P_i^{n-3}$ such that the composite morphisms $\overline{M}_{0,n} \xrightarrow{f_{ij}} \Pi_{ij}^{n-3} \xrightarrow{\tau_{ij}} P_i^{n-3}$ coincide with σ_i .*

Proof: The choice of coordinates identifies the tangent space to $G(2, n)$ at the fixed point $\langle e_i, e_j \rangle$ with the open Schubert cell in $G(2, n)$ consisting of all lines not intersecting the span of points $e_m, m \neq i, j$. For a generalized Lie complex Z the point $\sigma_i(Z)$ can be, by considerations of §4, be read off from the normal spaces to Z at a generic point of $P^{n-2}(e_i)$. Such a generic point can be contracted, by the action of the torus H , to the point $\langle e_i, e_j \rangle$. Therefore, our normal space in question can be recovered from the part of Z which can be contracted to this fixed point i.e. from $f_{ij}(Z)$.

(4.3.17) Proposition. *The space $\overline{M}_{0,n}$ coincides with the closure of the open stratum $M_{0,n}$ in the inverse limit of Π_{ij}^{n-3} and P_i^{n-3} .*

Proof: First let us show that the natural map of $\overline{M}_{0,n}$ into the said inverse limit is injective. This means that if two generalized Lie complexes Z, Z' induce the same algebraic cycles in the projectivizations of all the tangent spaces $T_{\langle e_i, e_j \rangle} G(2, n)$ then they coincide. This is obvious since any H -orbit has in its closure some fixed point.

Hence we have a regular morphism (denote it ψ) of $\overline{M}_{0,n}$ to the inverse limit in question which is bijective on \mathbb{C} -points. To show that ψ is in fact an isomorphism of algebraic varieties, it suffices to show that the

differential of ψ does not annihilate non-zero tangent vectors to $\overline{M_{0,n}}$. This is done similarly to the proof of Theorem 3.3.14.

(4.3.18) Proposition. *The map*

$$f_{ij} \times \pi_i : \overline{M_{0,n}} \rightarrow \Pi_{ij}^{n-3} \times \overline{M_{0,n-1}}$$

is an embedding of algebraic varieties.

Proof: We shall check only the injectivity on \mathbf{C} -points leaving the injectivity on tangent vectors to the reader. Let \mathcal{T} be a tree bounding the endpoints $1, \dots, n$ and $M(\mathcal{T})$ be the corresponding stratum of $\overline{M_{0,n}}$ or, what is the same, the corresponding Chow stratum in $G(2, n)/H$. Let Z be a generalized Lie complex from $M(\mathcal{T})$ and $C = Z_u$ be the corresponding stable n -pointed curve of genus 0.

The value of $f_{ij}(Z)$ depends only of the components of Z containing the fixed point $\langle e_i, e_j \rangle$. The components correspond to vertices of \mathcal{T} lying on the path $[ij]$ -the shortest edge path joining i -th and j -th endpoints. Denote these vertices, in natural order of movement from i to j , by v_1, \dots, v_r . Let s_ν be the number of edges meeting v_ν . To the chain of vertices v_ν corresponds a chain of irreducible components C_1, \dots, C_r of C and on each C_ν we have s_ν marked points. The projective configurations of these groups of points are precisely what is taken into account by the map f_{ij} on the curve from $M(\mathcal{T})$. Now our assertion means that the isomorphism class of the stable n -pointed curve C can be recovered from two groups of data:

- a) The isomorphism class of the stable $(n-1)$ -pointed curve $\pi_i(C)$;
- b) The isomorphism class of the stable curve $C' = C_1 \cup \dots \cup C_r \subset C$ pointed by x_i, x_j and all the marked and double points of C lying on C' .

This is obvious and Proposition 4.3.18 is proven.

(4.3.19) Let us now connect the spaces $\overline{M_{0,n-1}}$ and Π_{ij}^{n-3} . We view the latter space as the blow-up of P_i^{n-3} at all vertices, edges etc of the coordinate simplex formed by points $q_m, m \neq j$. Projecting these points from q_j gives a circuit in the space P_{ij}^{n-4} of lines in P_i^{n-3} meeting q_j . The space P_{ij}^{n-4} is in the same relation to $\overline{M_{0,n-1}} = \pi_i(\overline{M_{0,n}})$ as P_i^{n-3} was to $\overline{M_{0,n}}$. In particular, we have the regular birational morphism $\sigma_{j/i}$ from $\overline{M_{0,n-1}}$ to P_{ij}^{n-4} . On the other hand, consider the blow-up $Bl_{p_i} \Pi_{ij}^{n-3}$. It also possesses a projection to P_{ij}^{n-4} . Proposition 4.3.18 implies the following corollary:

(4.3.20) Corollary. *The space $\overline{M_{0,n}}$ coincides with the closure of $M_{0,n}$ in the fiber product of $\overline{M_{0,n-1}}$ and $Bl_{p_j} \Pi_{ij}^{n-3}$ over P_{ij}^{n-4} .*

This corollary can be reformulated as follows. Suppose we knew the way of constructing $\overline{M_{0,n-1}}$ as an iterated blow-up of the projective space P_{ij}^{n-4} whose centers are proper transforms of smooth subvarieties Y_1, \dots, Y_r . Then we have in $Bl_{p_j} P_j^{n-3}$ the varieties \tilde{Y}_ν which are blow-ups of cones over Y_ν with apex p_j . The corollary means that if we perform the sequence of blow-ups of $Bl_{p_j} \Pi_{ij}^{n-3}$ with centers in proper transforms of \tilde{Y}_ν then we obtain $\overline{M_{0,n}}$.

In other words, the problem of recovering $\overline{M_{0,n}}$ from the partial blow-up $Bl_{p_i} \Pi_{ij}^{n-3}$ is equivalent to the problem of recovering $\overline{M_{0,n-1}}$ from the projective space. This gives an inductive proof of Theorem 4.3.3.

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