April '90

COMBINATORIAL-GEOMETRIC ASPECTS OF POLYCATEGORY THEORY: PASTING SCHEMES AND HIGHER BRUHAT ORDERS (LIST OF RESULTS)

M.M.Kapranov, V.A.Voevodsky

This talk is devoted to detailed study of free n-categories and their relations with more "classical" geometric objects .Among these objects we list convex polytopes, their triangulations,configurations of hyperplanes,oriented matroids,and so-called "higher Bruhat orders",introduced by Y.I.Manin and V.V.Schechtman.

The base for our study is the notion of a pasting diagram for n-categories introduced by M.Johnson [J]. Though we feel that much of Johnson's theory can be substantially simplified, even in its present state it yields a lot of combinatorial objects,some of which are known,and the others-new and unexpected.

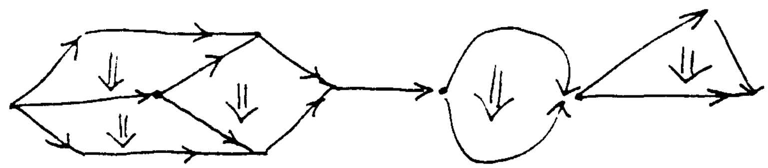
§1.Pasting schemes.

The notion of a polycategory we use is the "globular" one ([S],[J],[MS],comp.also [Gr]),in contrast with the cubical version [B]. Sometimes polycategories in our sense are called n-categories,and the cubical ones n-tuple categories.We shall use the one-sorted point of view on an n-category C,identifying it with a set MorC, equipped with mappings s_i, t_i:MorC---->MorC,

$i=0,1,\ldots,n-1$ and the partial compositions $(a,b)\dashrightarrow a*_i b$ defined when $s_i a=t_i b$. We call i-morphisms elements $a \in \mathrm{Mor}C$ such that $s_{i-1}a=t_{i-1}a=a$. Objects are just 0-morphisms.

For any two objects x,y of a n-category C, a (n-1)-category $\mathrm{Hom}_C(x,y)$ is defined ,whose objects are 1-morphisms in C from x to y.

Intuitively a pasting scheme is an "algebraic expression with indeterminate elements" which can be evaluated in an arbitrary n-category as soon as we have associated ,in a compatible way,to the indeterminates in the expression concrete polymorphisms.For example,



is a pasting scheme. In [J],a combinatorial theory of pasting schemes was developed.We shall recall some points of the theory of [J].

A pasting scheme is a collection $A=(A_i)_{i\geq 0}$ of finite sets such that $A_i=0$ for $i>>0$, equipped with binary relations $B_i,E_i\subseteq A_{i+1}\times A_i$ .They must satisfy certain conditions,the most important of which is the following. Let $\mathbb{Z}[A_i]$ be the free abelian group generated by A_i.Define the differential $\partial: A_{i+1}\text{-----}>A_i$ by the formula

$$\partial(a) = \sum_{b:(a,b)\in B_i} b \quad -- \sum_{b:(a,b)\in E_i} b \quad .$$

Then $\partial\partial$ must be equal to zero,that is,

$$\ldots\text{------}>\mathbb{Z}[A_2]\xrightarrow{\partial}\mathbb{Z}[A_1]\xrightarrow{\partial}\mathbb{Z}[A_0]$$

must be a chain complex. This complex determines A as a pasting scheme. Therefore, we can say that a pasting scheme is a based chain complex of a particular kind. We shall use this description in the sequel. We shall note dim A, the dimension of A the maximum of i such that A_i is non-empty. If $a \in A_{i+1}$, then we set $B_i(a)=\{b \in A_i : (a,b) \in B_i\}$. Similarly for E_i.

If A is a pasting scheme and $a \in A_m$, then we denote by R(a) the set of all $b \in A_i, i \le m$ such that there exists a sequence $a=a_1, a_2, \ldots, a_{m-i}=b$, in which for each j the pair (a_j, a_{j+1}) lies either in E_{m-j+1} or in B_{m-j+1}. Geometrically R(a) is to be thought of as the set of cells lying in thhe closure of a (cf. §2 below).

The really important notion is the notion of a composable pasting scheme, that is a scheme which is, in Johnson's terminology, loop free and well-formed. These conditions eliminate the following types of behavior:

For a composable pasting scheme A of dimension n, a n-category Cat(A) is defined [J]. Its polymorphisms are composable subpasting schemes (in a natural sense) in A, and the compositions are given by the union. In particular, for any composable pasting scheme A of dimension n we have composable subpasting schemes $s_iA, t_iA \subset A$ of dimension i. A realisation of a composable pasting scheme A in an n-category C is an n-functor

Cat(A)---->C. The "resulting polymorphism" of such a realisation is the value of this functor on $A \in$ Mor Cat(A). As shown in [J], Cat(A) is freely generated (in the sense of Street [S])by polymorphisms of the type R(a), $a \in A_i$, $i \geq 0$,which, in this case are composable subpasting subschemes.

§2.Geometric realisations of pasting schemes.Structures of pasting schemes on convex polytopes.

Most of pasting schemes arising in practice come from some geometric objects,e.g.polytopes. This induces an idea to consider the geometric realisation of a pasting scheme as a cellular complex.

Let A be a pasting scheme. The set $A = \cup A_i$ is partially ordered by the relation R.

<u>Definition</u> <u>2.1</u>. The geometric realisation $|A|$ of a pasting scheme A is te nerve of the cathegory associated to the poset (A,R).

Therefore, $|A|$ is a simplicial complex, whose p-dimensional simplices correspond to chains $x_0 R x_1 R \dots R x_p$, where xRy means that $y \in R(x)$. To any m and $a \in A_m$ we associate a closed subcomplex $[a] \subset |A|$ whose p-simplices correspond to chains $x_0 R x_1 R \dots R x_p$, where $x_p R a$.

<u>Theorem</u> <u>2.2</u>. If A is a composable pasting scheme of dimension n, then:

a) $|A| - |s_{n-1}A| - |t_{n-1}A|$ is homeomorphic to a disjoint union of several open n-balls.

b) For each m and $a\in A_m$ the subcomplex $[a]$ is homeomorphic to a closed m-ball. Therefore the subcomplexes of the form $[a]$, $a\in A_m$, $m\geq 0$, form a cellular decomposition of $|A|$.

In general, it is very difficult to decide, whether a given CW-complex is homeomorphic to a ball, or is a topological manifold, because this amounts to recognising asphere among other manifolds. For a 3-sphere this is the classical Poincare conjecture. The success in our situation comes from considering additional structure on the complex: the grouping of the cells lying on the boundary of a given cell, to "beginning" and "end".

Let $M\subset\mathbb{R}^n$ be a bounded convex polytope of dimension n, and

$$p=\{\mathbb{R}^n \xrightarrow{p_{n,n-1}} \mathbb{R}^{n-1} \longrightarrow \dots \longrightarrow \mathbb{R}^2 \xrightarrow{p_{2,1}} \mathbb{R}\}$$

be a system of affine projections such that any k-dimensional facet of M projects injectively to $\mathbb{R}^k$ (we shall call a system of projections with this property admissible). We shall suppose that all $\mathbb{R}^i$ are equipped with their standard orientations. Then the fibers of $p_{k,k-1}$ become oriented lines. Denote the composite projection $\mathbb{R}^n \longrightarrow \mathbb{R}^k$ by p_k.

Let $A_k(M)$ be the set of k-dimensional facets of M. Define on $A(M)=\cup A_k(M)$ a structure of a pasting scheme. Let $\Gamma\in A_k(M)$, $\Delta\in A_{k-1}(M)$, $\Delta\subset\Gamma$. Consider the image $p_k(\Gamma)\subset\mathbb{R}^k$. Let $H:\mathbb{R}^k\longrightarrow\mathbb{R}$ be an affine-linear function such that $H|_{p_k(\Delta)}=0$, $H|_{p_k(\Gamma)}\geq 0$. Say that $\Delta\in B_k(\Gamma)$ (resp. $\Delta\in E_k(\Gamma)$) if $H(t)\longrightarrow+\infty$ (resp. $H(t)\longrightarrow(-\infty)$) when t tends to the infinity along a fiber of $p_{k,k-1}:\mathbb{R}^k\longrightarrow\mathbb{R}^{k-1}$ in positive direction:

We shall call this pasting scheme A(M,p)

Theorem 2.3. If $M\subset\mathbb{R}^n$ is a bounded n-dimensional polytope and $p=\{\mathbb{R}^n \xrightarrow{p_{n,n-1}} \mathbb{R}^{n-1} \longrightarrow \ldots \longrightarrow \mathbb{R}^2 \xrightarrow{p_{2,1}} \mathbb{R}\}$ is an admissible system of projections ,Then A(M,p) is a composable pasting scheme.

Example 2.4. Let $M=\Delta^n$ be an n-dimensional simplex,and $\partial_j:A_k(\Delta^n)\to A_{k-1}(\Delta^n)$ be the standard simplicial operators,$j=0,1,\ldots,k$. Namely,denote vertices of Δ^n by $(0),(1),\ldots,(n)$. Then each facet is determined by a subset $\sigma\subset\{0,\ldots,n\}$,which we write in the increasing order:$\sigma=\{\sigma_0<\ldots<\sigma_k\}$. Then $\partial_j\sigma=\{\sigma_0<\ldots<\hat{\sigma}_j<\ldots<\sigma_k\}$. Te standard structure of pasting scheme on Δ^n,considered in [S],starts from the usual differential $\partial=\Sigma(-1)^i\partial_i$ in the chain complex of Δ^n. Therefore, for $\sigma\in A_k(\Delta^n)$, we have$B_{k-1}(\sigma)=\{\partial_j\sigma, j \text{ is even}\}$,$E_{k-1}(\sigma)=\{\partial_j\sigma, j \text{ is odd}\}$. The corresponding n-category $Cat(\Delta^n)$ was called by Street the n-th oriental.

Let us give an interpretation of this structure of pasting scheme by means of projections. Fix n+1 real numbers $t_0>\ldots>t_n\in\mathbb{R}$. Define n+1 points $v_j=(t_j,t_j^2,\ldots,t_j^n)\in\mathbb{R}^n$, $j=0,\ldots,n$. These points are in general position since the determinant of the corresponding matrix is the classical Vandermonde determinant.Therefore the convex hull of v_j is a n-simplex which we consider with the given numeration of vertices. Consider the projections $p_{i,i-1}:\mathbb{R}^i\to\mathbb{R}^{i-1}$ which forgets the last coordinate.

Theorem 2.5 The structure of pasting scheme on Δ^n given by the above projections coincides with the combinatorially defined structure used by Street.

Example 2.6. Let $I^n = \{(x_1,\dots,x_n)\in\mathbb{R}^n; 0\le x_i\le 1\}$ be the n-dimensional cube. and $\partial_i^p : A_k(I^n) \to A_{k-1}(I^n)$, $i=1,2,\dots,k$, $p=0,1$ be the standard cubic boundary operators, see [K]. Explicitly, facets of I^n have the form $F(X,Y)=\{x\in I^n : x_i=0$ for $i\in X$, $x_i=1$ for $i\in Y\}$ for $X,Y\subset\{1,\dots,n\}$, $X\cap Y=\varnothing$. Let $k=\dim F(X,Y)=n-\mathrm{Card}(X)-\mathrm{Card}(Y)$ and $a_1<\dots<a_k$ be all elements of $\{1,\dots,n\}-X-Y$. Then

$\partial_i^0 F(X,Y)=F(X\cup\{a_i\},Y)$, and $\partial_i^1 F(X,Y)=F(X,Y\cup\{a_i\})$. The differential in the chain complex of I^n is given by $\partial=\sum(-1)^{i+p}\partial_i^p$. Therefore, we introduce the relations $B_k, E_k \subset A_{k+1}(I^n)\times A_k(I^n)$ by setting

$B_k(F)=\{\partial_i^p F, i+p \text{ is even}\}$, $E_k(F)=\{\partial_i^p F, i+p \text{ is odd}\}$.

Theorem 2.7. The relations B_k, E_k define on $A(I^n)$ the structure of a composable pasting scheme.

We can deduce this theorem from the other description of this structure of pasting scheme. Fix real numbers $t_1 >> t_2 >> \dots >> t_n$, where $>>$ means "sufficiently greater than". Define vectors $v_j=(t_j, t_j^2,\dots,t_j^n)\in\mathbb{R}^n$ as above and realize the cube as the parallelotope with vertices $\sum_{i\in J} v_i$, where J runs over all subsets of $\{1,\dots,n\}$. Define projections $p_{i,i-1}:\mathbb{R}^i \to \mathbb{R}^{i-1}$ as above, by forgetting the last coordinate. This defines on I^n some structure of pasting scheme.

Theorem 2.8. The two described structures of pasting scheme on I^n coincide.

§3. Higher orders associated to a composable pasting scheme.

Categories of the form Cat(A), A being a composable pasting scheme, posess remarkable properties of order, which we shall now describe.

Definition 3.1. a) An 1-category C with a finite number of morphisms is called ordered, if the relation $Hom(x,y)\neq\emptyset$ on the set Ob C is a partial order, and Hom(x,x) is always a singleton. A category is called strictly ordered, if it is ordered and Ob C has unique maximal amd minimal elements.

b) Suppose that for $k<n$ the notion of a (strictly) ordered k-category is defined. Say that an n-category C is (strictly) ordered if:

the relation $Hom(x,y)\neq\emptyset$ on Ob C is a partial order (with unique maximal and minimal elements) ;

for any $x,y \in$ Ob C the (n-1)-category $Hom_C(x,y)$ is (strictly) ordered, and Hom(x,x) is a singleton n-category.

If C is a strictly ordered n-category, then we can define a strictly ordered (n-1) category $\Omega C=Hom_C(x_{min},x_{max})$, where $x_{min},x_{max}\in$ Ob C are maximal and minimal elements. So we can form $\Omega^2C=\Omega\Omega C$ etc.

Theorem 3.2 Let A be an n-dimensional composable pasting scheme. Then Cat(A) is a strictly ordered n-category.

So, to each composable pasting scheme A we associate a hierarchy of posets $X_k=$Ob Ω^kCat(A). There are natural surjective maps {maximal chains in X_k ---->X_{k+1} .

<u>Definition 3.3.</u> We call the higher Stasheff order $S(n,k)$ the poset Ob $\Omega^k Cat(\Delta^n)$.

<u>3.4.Examples.</u> a) $S(n,1)$ is the set of all subsets of an n-element set,(i.e. of vertices of an n-cube) partially ordered by inclusion.It has 2^n elements.

b)Elements of $S(n,2)$ are identified with triangulations of a planar convex (n+1)-gon which we shall denote $M(n+1,2)$. Namely,number the vertices by 0,1,...,n in circular order. Let T be a triangulation of $M(n+1,2)$.Lift each triangle of T with vertices i,j,k, to the corresponding triangle in Δ^n. It is clear that we thus obtain all films from $\Omega^2 Cat(\Delta^n)$,cf. [S].

It is well-known that the triangulations of $M(n+1,k)$ are in bijection with bracketings of n factors.Their number is the Catalan number $c_n=(2n-2)!/(n-1)!(n-1)!(n-1)$. These bracketings are vertices of an (n-3)-dimensional polytope constructed by J.Sfasheff [Sta],what explains our terminology.

Denote by $M(n+1,k)=p_k(\Delta^n)$ the image of the simplex under the projection to $\mathbb{R}^k$ defined in the example 2.4. In other words,$M(n+1,k)$ is the convex hull of n+1 points lying on the Veronese curve in $\mathbb{R}^k$ given by $\{(t,t^2,...,t^k),t\in\mathbb{R}\}$. It is classically called the cyclic polytope and is of importance in general theory of convex polytopes,since its face numbers posess some extremal properties , see [Gru] and references therein.

<u>Theorem 3.4.</u>Elements of the poset $\Omega^k Cat(\Delta^n)$ are in bijection with triangulations of the cyclic polytope $M(n+1,k)$ which do not add new vertices.

Remark 3.5. It would be interesting to construct a natural polytope with the set of vertices $S(n,k)$, thus generalizing the Stasheff polytope. In fact, in [GZK] for any convex polytope $Q\subset\mathbb{R}^k$ and any set $A\subset\mathbb{R}^k$ containing all vertices of Q, a new convex polytope $P(Q,A)\subset\mathbb{R}^A$ was defined, whose vertices are in bijection with those triangulations of Q with vertices in A, which are regular, i.e. admit a strictly convex piecewise-linear function. Unfortunately, we do not know, whether all triangulations of $M(n+1,k)$ are regular. It seems that the answer is negative.

Remark 3.6. It is very interesting bo calculate the number of all triangulations of $M(n+1,k)$, i.e. the higher analogue of the Catalan numbers.

§4. Free n-category generated by a n-cube and higher Bruhat orders.

In the course of study of higher-dimensional generalisations of the Yang-Baxter equation, Yu.I.Manin and V.V.Schechtman introduced in [MS 1-3] posets $B(n,k)$ called the higher Bruhat orders. The set $B(n,1)$ is the symmetric group S_n with its weak Bruhat order, and $B(n,k+1)$ is a certain quotient of the set of maximal chains in $B(n,k)$. In [MS 1-3] various connections of $B(n,k)$ with geometry were indicated. Among them are the connectiob with configurations of hyperplanes in $\mathbb{R}^k$ in general position and the structure of the convex closure of a generic orbit of S_n in $\mathbb{R}^n$. We shall not recall here the original definition of $B(n,k)$ but instead formulate our interpretation. Consider the cube I^n

with the structure of pasting scheme introduced in §2.

Theorem 4.1. There is an isomorphism of posets $B(n,k)\cong Ob\Omega^k Cat(I^n)$.

By using mutations of elements of higher Bruhat orders (analogs of multiplications of permutations by transpositions), in [MS3] a (n-1)-category $\mathbb{S}_n$ was defined, whose set of objects is the symmetric group S_n.

Theorem 4.2. There is an isomorphism of (n-1)-categories $\mathbb{S}_n\cong\Omega Cat(I^n)$.

From this theorem we easily deduce the conjecture of [MS3]. It claims that the set of indecomposable p-morphisms in $\mathbb{S}_n$ is in bijection with the set of indecomposable p-dimensional faces of a certain (n-1)dimensional polytope P_n called permutoedre [Gru]. By defininition, P_n is the convex hull of the orbit of a point $(x_1>\ldots>x_n)\in\mathbb{R}^n$ under the natural action of the group S_n. Each face of P_n is isomorphic to a product of several permutoedra of smaller dimension, and some are single permutohedra. These correspond, according to Manin-Schectman conjecture, proved ny us, to indecomposable polymorphisms of $\mathbb{S}_n$.

Consider the projection $p_k: I^n \dashrightarrow \mathbb{R}^k$ introduced in §2. Denote Z(n,k) its image. It is natural to call this polytope the cyclic zonotope.

Theorem 4.3. Elements of B(n,k) is in bijection with subcomplexes (i.e. closed subsets which are unions of facets) $\Sigma\subset I^n$ such that $p_k:\Sigma \dashrightarrow Z(n,k)$ is a bijection.

For such Σ the images of facets of Σ form a cubillage of Z(n,k) (analogue of triangulation). Consider the cell decomposition of $Z(n,k)$,dual to this cubillage. If we look at its $(k-1)$-dimensional squeleton,we obtain a configuration of n polyedral hypersurfaces in $Z(n,k)$. These hypersurfaces intersect each other as af they were hyperplanes in general position.In other words,they define an oriented matroid [FL]. Let us recall necessary definitions.

Definition 4.4. An oriented matroid is a system $M=(E,\mathcal{C},*)$,where E is a finite set,$*:E\to E$,$x\to x^*$ is a fixed point-free involution ,$\mathcal{C}\subset 2^E$ is a family of subsets (called positive cycles) satisfying the conditions:

(i) If $S\in\mathcal{C}$ and $T\subset S$,then $T=S$.

(ii)If $S\in\mathcal{C}$ and $S^*=\{x^*,x\in S\}$,then $S^*\in\mathcal{C}$ and $S\cap S^*=\emptyset$.

(iii)If $S,T\in\mathcal{C}$,$x\in S\cap T^*$,$S\neq T^*$,then there is $C\in\mathcal{C}$ such that $C\subset(S\cup T)-\{x,x^*\}$.

A basic example is given by the set E of non-zero vectors in a real vector space such that for $x\in E$ we have $(-x)\in E$. Define $x^*=-x$ for $x\in E$. Define $\mathcal{C}$ to consist of subsets $C\subset E$ minimal such that:

a) $C\cap C^*=\emptyset$

b) There are $\alpha_s\in\mathbb{R}_+$,$s\in C$ such that $\sum_{s\in S}\alpha_s s=0$.

Not any oriented matroid is realisable,i.e. comes from a system of vectors as above.

From the "dual" point of view elements of an oriented matroid wwould represent half-spaces in $\mathbb{R}^n$ arising as complements to hyperplanes of an (imaginary,non-exestent in general) configuration. Instead of half-spaces containing 0,one can imagine hemispheres in S^{n-1},the unit sphere.

One of the main result of [FL] is that oriented matroids correspond to configurations formed by not necessary genuine hemispheres, but by so-called pseudo-hemispheres,that is,by subcomplexes in S^{n-1} homeomorphic to discs and invariant under the involution. Fhis is achieved by "geometric realisation" similar to our construction in§2. Such a configuration may,however,be not stretchable.

<u>Definition</u> <u>4.5.</u> An oriented matroid $M=(E,\mathcal{C},*)$ is said to have type $F(n,k)$,if $card(E)=2n$, for each $s\in\mathcal{C}$ we have $card(C)=k+1$,and for each subset $X\subset E$, $card(X)=k+1$ there is a decomposition $X=Y\cup Z, Y\cap Z=\varnothing$ such that $Y\cup Z^{*}\in\mathcal{C}$.

Intuitively,such a matroid should represent a configuration of n hyperplanes in $\mathbb{R}^{k}$ in general position. Not every matroid of type $F(n,k)$ is realisable. In the paper of Ringel [R] there is an example of a non-realisable oriented matroid of type $F(9,3)$.

<u>Definition</u> <u>4.6.</u> The cyclic oriented matroid of rank k on n directions is the oriented matroid $C(n,k)$ in which E consists of symbols $\delta_1,\delta_1^*,\dots,\delta_n,\delta_n^*$, the involution $*$ interchanges δ_i and δ_i^* and $\mathcal{C}$ is formed by subsets $Z_I=\{\delta_i, i\in I, i \text{ is even}, \delta_i^*, i\in I, i \text{ is odd}\}$ and $Z_I^*$for all $(k+1)$-element subsets $I\subset\{1,\dots,n\}$.

The oriented matroid $C(n,k)$ is realisable by configuration of hyperplanes dual to the vertices of the cyclic polytope (see §2).

A cell of an oriented matroid M is,by definition,a cell of the cell decomposition of the sphere induced by the configuration of pseudo-hemispheres corresponding to M. Cells can be defined in a purely combinatorial ay,see [FL].

Definition 4.7 A marking of an oriented matroid M is complete flag $Z=(Z_0\subset\ldots\subset Z_{r-1})$ of cells of M. (Here $r-1$ is the dimension of the sphere, i.e. r is the rank of M)

Example. Define a marking of the cyclic oriented matroid $C(n,k)$ which e shall call the standard one. Let us view elements δ_i,δ_i^* as hemispheres in S^{k-1}. Then set

$$Z_i^{st}=\bigcap_{j=0}^{k-2+i}(\delta_j\cap\delta_j^*)\cap(\bigcap_{j=k-1+i}^{n-1}\delta_j),$$ where we set $\delta_0=\delta_n$.

If B is a cell of an oriented matroid M, then denote by $S(B)$ the unique pseudosphere in S^{r-1} of dimension $\dim(B)$ which is the intersection of some pseudohemisperes of the configuration. By $M|_{S(B)}$ we denote the oriented matroid of rank $\dim(B)+1$ defined by the configuration of pseudohemispheres induced on $S(B)$.

Theorem 4.8. The set $B(n,k)$ is in bijection with the set of marked oriented matroids (M,Z) of rank $k+1$ such that

(i) M as the type $F(n+1,k+1)$

(ii) The restriction $M|_{S(Z_{k-1})}$ is isomorphic to the cyclic oriented matroid $C(n,k)$.

(iii) The marking of $C(n,k)=M|_{S(Z_{k-1})}$ is te standard one.

For $k=3$ any oriented matroid of type $F(n+1,3)$ admits a marking satisfying (ii)-(iii).his corresponds to the numeration of affine (pseudo-)lines in $\mathbb{R}^2$ by the increasing of the slopes. For $k>3$ such a marking is not always possible.

Using this theorem we can easily disprove the conjecture from [MS2] that $B(n,2)$ classifies the combinatorial types of allangements of n lines in general position in $\mathbb{R}^2$, none of which is parallel to the fixed line. To do this, we can take the

Ringel's example [R] of non-stretchable configuration of 9 pseudo-lines in P^2, thus obtaining an element of B(9,2) which cannot be represented by a configuration of lines.

In general,one can construct from an arbitrary marked oriented matroid a composable pasting scheme.

REFERENCES

[BH] Brown R., Higgins P.The equivalence of ω-groopoids and crossed complexes.// Cah. Top. Geom. Diff., 1981,v.22,N.4,p.371-386.

[FL] Folkman J.,Lawrence J.Oriented Matroids.//J.Comb.Theory,ser.B,1978,v.25,N.2,p.199-236.

[Gr] Grothendieck A. Pursuing stacks.-Manuscript,1983.

[Gru] Grunbaum B. Convex polytopes.-Interscience,1968.

[GZK] Gelfand I.M., Zelevinsky A.V., Kapranov M.M.// Newton polytopes of principal A-determinants.// DAN SSSR,1989,v.308,N.1,p.20-23.(In Russian.)

[J]Johnson M. Pasting diagrams in n-categories.-Research report 88/16,Department of math.,Univ. of Sydney,1988.

[K] Kan D.M.Abstract homotopy theory.//Proc.Nat.Acad.Sci.USA,1955,v.41,N12,p.1092-1096.

[MS1] Manin Yu.I.,Schechtman V.V.On the higher Bruhat orders related to the symmetric group.//Func.Analiz i ego pril.,1986,v.20,N2,p.74-75.(In Russian).

[MS2]---------.Arrangements of real hyperplanes and Zamolodchhikov's equations,In: Group-theoretical methods in physics, Moscow, Nauka,1986,v.1,p.316-324.(In Russian).

[MS3]---------.Arragements of hyperplanes,Higher braid groups and higher Bruhat orders.//Advanced studies in pure matematics,1989,v.17,p.289-308.

[R] Ringel G.Teilungen der Ebene durch Geraden oder Pseudogeraden.//Math.Z.,1956,v.64,p.79-102

[S]Street R. The algebra of oriented simplexes//J.Pure Appl. Alg.,1987,v.49,N3,p.283-335.

[Sta] Stasheff J.Homotopy associativity of H-spaces.//Trans.Amer.Math.Soc,1963,

Steklov Mathematical Institute,ul.Vavilova,42,117333,Moscow,USSR.