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1980 Mathematics Subject Classification.

Primary 05B35, 52A25

Key words and phrases

Cyclic polytopes, moment curve, neighbourly simplicial polytopes, cyclic curves, geometrical types, oriented matroids, Radon types, configurations of points, matroid polytopes, alternating orientations of a free matroid, Radon partitions, combinatorial types, Gale evenness condition, Dehn-Sommerville equations.

Comment: There is an Erdős-Szekeres theorem for cyclic polytopes: Via Ramsey theory you can prove that for large enough # of points in $\mathbb{R}^d$ they contain a cyclic polytope!

CYCLIC POLYTOPES AND ORIENTED MATROIDS *

by

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ABSTRACT The combinatorial properties of cyclic polytopes C(n,d) are surveyed from the matroidal point of view. A simple characterization of the alternating oriented matroids, which constitute the geometrical type of C(n,d) is given. As a corollary, we characterize cyclic curves. The geometrical type of cyclic polytopes of even dimension is proved to be unique. For the odd dimensions, the geometrical types are characterized. The generalization of the Dehn-Sommerville equations to simplicial matroid polytopes is proved.

* *The research of the first author was partially supported by the Fundação Calouste Gulbenkian, Lisboa.*

1. INTRODUCTION

The cyclic polytope $C(n,d)$ was discovered by Caratheodory [8,9] and many times rediscovered; $C(n,d)$ is usually defined as the convex hull in $\mathbb{R}^d$ of n, $n \geq d+1$, different points $x(t_1),\dots,x(t_n)$ of the moment curve whose parametric equation is $x(t) = (t,t^2,\dots,t^d)$, $t \in \mathbb{R}$. The polytope $C(n,d)$ constitutes the simplest example of a $[\frac{d}{2}]$-neighbourly simplicial d-polytope : every subset of $[\frac{d}{2}]$ points of the vertex set of $C(n,d)$ is the set of vertices of a proper face of $C(n,d)$ and every facet is a simplex (for other examples see [24], 7.2.4). The $[\frac{d}{2}]$-neighbourly simplicial d-polytopes play a central role in the theory of polytopes since, for every k, $0 \leq k \leq d$, the number of their faces of dimension k, which depends only on k, n and d, is the maximum possible, among all d-polytopes with n vertices. This fact, conjectured by Motzkin [33] and proved by McMullen [31,32], is known as the Upper Bound theorem.

Various authors had remarked that many other curves can play the role of the moment curve for defining a "cyclic polytope". Although the problem of characterization of such cyclic curves is implicit in Grünbaum [24], it seems that nobody noticed the sufficiency of the obvious simple condition : every $d+1$ points of the curve are affinely independent (see Corollary 4.7). Of course, the classical

determination of the facial structure of $C(n,d)$, that yields the Gale's evenness condition, may be extended to all cyclic curves. But a geometrical proof, using oriented matroids, is much simpler; indeed the geometrical type (see § 2) of $C(n,d)$ is independent of a particular choice of points on the moment curve (see § 4). The consideration of geometrical types of finite sets of points in $\mathbb{R}^d$ is a recent progress in the theory of polytopes, and has been introduced in various ways : oriented matroids for Bland [3], Folkman and Lawrence [20], Las Vergnas [27] or Mandel [30]; Radon types for Eckhoff [15]; configuration of points for Goodman and Pollack [22].

The point of view or oriented matroids is the most general. The interest of this point of view is underlined by the two following facts : 1) the Upper Bound theorem is still true for matroid polytopes [30]; 2) the Dehn Sommerville equations on the number of faces are valid for simplicial matroid polytopes (see § 6).

After some preliminaries on polytopes and oriented matroids (§ 2), we present the principal properties of the cyclic polytopes with matroidal proofs. A description of the alternating oriented free matroid which constitutes the geometry of $C(n,d)$ is given in § 3. The facial structure and the geometrical type of $C(n,d)$ are investigated in § 4. The geometrical types of all polytopes combinatorially equivalent to $C(n,d)$ (i. e. polytopes with the facial structure of $C(n,d)$) are determined in § 5. We count the faces for simplicial matroid polytopes in § 6 and for cyclic polytopes in § 7.

For a short history and additional informations about cyclic polytopes and neighbourly polytopes, the reader should consult [24,32]. For a recent investigation on cyclic polytopes see [1,2].

The reader is supposed to be familiar with the general properties of matroids [40], oriented matroids [4,20] and polytopes [24]. Bland and Las Vergnas' notations [4,28] are followed, with minor changes; we briefly describe what we use in § 2. The integer part of a real x is denoted by $[x]$, the deletion of sets by $\setminus$, the ordinary convex hull of a set S in Euclidian spaces by $\mathrm{conv}(S)$.

The proof of Theorem 5.3 uses the notion of a simple graph G on a set of vertices V : it is a collection of 2-element subsets of V , called edges of G . An automorphism of G is a one-to-one correspondence $\alpha: V \to V$ such that $\{\alpha(x),\alpha(y)\}$ is an edge of G if and only if $\{x,y\}$ is an edge of G .

2. ORIENTED MATROIDS AND POLYTOPES

The notion of oriented matroids, suggested by Rockafellar [34] was independently introduced by Bland [3], Las Vergnas [27], Folkman and Lawrence [20,29]. See also Bland and Las Vergnas [4] and Mandel [30].

Let S be a finite set. A signed set in S is an ordered pair $X = (X^+, X^-)$ with $X^+ \subseteq S$, $X^- \subseteq S$ and $X^+ \cap X^-$ The set $\underline{X} = X^+ \cup X^-$ is called the support of X . We say that X contains a if $a \in \underline{X}$ and that a and b appear in X with the same sign (resp. opposite sign) if $a,b \in X^+$ or $a,b \in X^-$ (resp. $a \in X^+$ and $b \in X^-$, or $a \in X^-$ and $b \in X^+$) . The set X^+ (resp. X^-) is called the set of positive (resp. negative) elements of X . The opposite of X is the signed set $-X = (X^-, X^+)$.

The pair $M = (S, \mathcal{O})$ is an oriented matroid on S if $\mathcal{O}$ is a collection of signed sets in S , called signed circuits (or shortly circuits) satisfying :

(O1) $X \in \mathcal{O}$ implies $\underline{X} \neq \emptyset$ and $-X \in \mathcal{O}$;

$X_1, X_2 \in \mathcal{O}$ and $\underline{X}_1 \subseteq \underline{X}_2$ imply $X_1 = X_2$ or $X_1 = -X_2$;

(O2) (Signed elimination property) for all $X_1, X_2 \in \mathcal{O}$, $x \in X_1^+ \cap X_2^-$ and $y \in X_1^+ \setminus X_2^-$, there is $X_3 \in \mathcal{O}$ such that $y \in \underline{X}_3$, $x \notin \underline{X}_3$ and $X_3^+ \subseteq (X_1^+ \cup X_2^+)$, $X_3^- \subseteq (X_1^- \cup X_2^-)$.

The sign of an element $x \in \underline{X}$ in a signed circuit X will be denoted by $sg_X(x)$. By forgetting the orientation of M, we obtain an unoriented matroid $\underline{M}$ defined by its collection $\underline{\mathcal{O}} = \{\underline{X} : X \in \mathcal{O}\}$ of circuits. The cocircuits of $\underline{M}$, i. e. the circuits of the orthogonal matroid $\underline{M}^{\perp}$ can be oriented (=signed) in an unique way such that the collection $\mathcal{O}^{\perp}$ of signed cocircuits of M satisfies the orthogonality property :

(O3) For all $X \in \mathcal{O}$ and $Y \in \mathcal{O}^{\perp}$ such that $|\underline{X} \cap \underline{Y}| \geq 2$, both $(X^+ \cap Y^+) \cup (X^- \cap Y^-)$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+)$ are non-empty.

$\mathcal{O}^{\perp}$ satisfies **(O1)** and **(O2)** and defines the orthogonal oriented matroid $M^{\perp}$.

A collection $\mathcal{O}$ of signed circuits of an oriented matroid M is often called a signature of the underlying unoriented matroid $\underline{M}$ or of the collection $\underline{\mathcal{O}}$ of its circuits.

Let S be a finite set of points in $\mathbb{R}^d$. The set of affinely dependent subsets of S is the set of dependent sets of a matroid **Aff**(S) over S. The natural ordering of $\mathbb{R}$ induces a canonical orientation $\mathcal{O}$ of **Aff**(S) : observe that if C is a circuit of **Aff**(S) then a mapping $\lambda : C \to \mathbb{R} - \{0\}$ such that $\sum_{x \in C} \lambda(x).x = 0$ and $\sum_{x \in C} \lambda(x) = 0$ is unique up to multiplication by a non-zero real number. Then (C, λ) determines a signed set $X = (X^+ = \{x \in C, \lambda(x) > 0\}, X^- = \{x \in C, \lambda(x) < 0\})$.

Aff(S) = (S, $\mathcal{O}$) is called the oriented matroid of affine **dependencies of** S **over** $\mathbb{R}$ [4]. **Aff**(S) **is also** called the geometry of S. We will say that two finite

sets S and S' of $\mathbb{R}^d$ are geometrically equivalent when the oriented matroids **Aff**(S) and **Aff**(S') are **isomorphic oriented matroids : i. e. if there is a** bijection $\varphi : S \to S'$ which preserves signed circuits. Clearly, the geometrical equivalence is an equivalence relation; **its representatives are called** geometrical types. The pair $X = \{X_1, X_2\}$ is called a Radon partition in S provided it is a partition of a subset of S and $conv(X_1) \cap conv(X_2) \neq \emptyset$. (The reader is referred to [14,15,16] for a discussion of Radon partitions). X is called a primitive Radon partition in S if X is minimal, in the sense that it does not extend any other partition in S [25]. It turns out that X is a signed circuit of Aff(S) if and only if $\{X^+, X^-\}$ is a primitive Radon partition in S. Thus the Radon types of finite sets of points in $\mathbb{R}^d$ of Eckhoff [16] coincide with the geometrical type defined above.

A signed circuit X of an oriented matroid is called positive if $X^- = 0$ or if $X^+ = 0$. The oriented matroid M is said to be acyclic when it has no positive circuit. For any set A of elements of M, the oriented matroid obtained by reversing signs over A [4] is denoted by $_{\bar{A}}M$. The partitions $\{A, S \setminus A\}$ of the underlying set S of M such that $_{\bar{A}}M$ is acyclic (or equivalently $_{\overline{S \setminus A}}M$ is acyclic) are called the non-Radon partitions of M [10]. Indeed, for a finite subset S of $\mathbb{R}^d$, the matroid **Aff**(S) is acyclic and for every subset $A \subseteq S$, the matroid $_{\bar{A}}$**Aff**(S) is acyclic if and only if $\mathrm{conv}(A) \cap \mathrm{conv}(B) = \emptyset$; if both A and $S \setminus A$ are not empty, this condition is equivalent to the existence of a hyperplane that strictly separates A and $S \setminus A$. An oriented matroid is completely determined by its non-Radon partition [10], thus the geometries of finite subsets of $\mathbb{R}^d$ are also characterized by their non-Radon partitions. For a matroid theoretic point of view on the (non-)Radon partitions see [7,10,12,30,41]. Further informations are in [17,23]. Relevant is the work of Mandel [30] who takes as approach to oriented matroids a notion similar to non-Radon partitions.

A d-polytope is the ordinary convex hull of a finite subset of $\mathbb{R}^d$ whose affine dimension is d . The classical notions of Convexity Theory may be extended to acyclic oriented matroids (see Las Vergnas [28]). For this reason, acyclic oriented matroids will be called <u>matroid polytopes</u>. In general, a facet of an oriented matroid $M = (S,\mathcal{O})$ is a hyperplane H (of the unoriented matroid $\underline{M}$) such that $S \setminus H$ supports a positive cocircuit of M . A <u>face</u> is an intersection of facets, i. e. a subset F of S such that $S \setminus F$ is a union of positive cocircuits of M . An <u>extreme point</u> of M (or a <u>vertex</u>) is a face of rank 1 . A matroid polytope M has facets and any subset of elements of M is the convex hull of its extreme points (Las Vergnas [28]). A matroid polytope M is said to be <u>simplicial</u> when every facet of M is an independent subset of $\underline{M}$. The faces of M ordered by inclusion form a lattice [28], called the <u>facial structure</u> of M and denot[ed] by $\mathcal{F}(M)$. It should be remarked that a subset F of elements of an oriented matroid $M = (S,\mathcal{O})$ is a face of M if and only if the contracted matroid M/F is acycli[c].

Let V_1 (resp. V_2) be the set of vertices of a polytope P_1 (resp. P_2) . The polytopes P_1 and P_2 are <u>combinatorially equivalent</u> (resp. <u>geometrically equivalent</u>) when the lattices $\mathcal{A}\!ff(V_1)$ and $\mathcal{A}\!ff(V_2)$ are isomorphic (resp. V_1 and V_2 are geometrically equivalent). The combinatorial (resp. geometrical) equivalence of polytopes is an equivalence relation; its representative are called, respectively, <u>combinatorial types</u> and <u>geometrical types</u>.

With the language of primitive Radon partitions, Breen [5] remarked :

<u>PROPOSITION 2.1</u> [5]. <u>The combinatorial type of a polytope is determined by its geometrical type</u>. □

Even for points in general position, combinatorial equivalence of polytopes does not imply geometrical equivalences.

The next results extend to matroid polytopes simila[r] theorems of [5].

PROPOSITION 2.2 . Let $M = (S, \mathcal{O})$ be a matroid polytope and let $F \subseteq S$. Then F is a face of M if and only if, for every circuit $X = (X^+, X^-)$ of M , $X^+ \subseteq F$ implies $X^- \subseteq F$.

Proof :

F is a face of M if and only if $S \setminus F$ is a union of positive cocircuits.

The intersection of a circuit and a cocircuit of $\underline{M}$ is either empty or contains at least two elements. Then if $X = (X^+, X^-)$ is a circuit of M and $X^+ \cap (S \setminus F) = \emptyset$, by the orthogonality property (O3) we also have $X^- \cap (S \setminus F) = \emptyset$. □

COROLLARY 2.3 . Let $M = (S, \mathcal{O})$ be a simplicial matroid polytope. Then $F \subseteq S$ is a face of M if and only if $\{A, S \setminus A\}$ is a non-Radon partition of M for every $A \subseteq F$.

Proof :

The "only if" part results trivially of Proposition 2.2. Indeed if $X = (X^+, X^-)$ is a circuit of M and F is a face of M , we have $\underline{X} \not\subseteq F$ because M is simplicial. Then, by Proposition 2.2 , we have also $(S \setminus F) \cap X^+ \neq \emptyset$ and $(S \setminus F) \cap X^- \neq \emptyset$ (considering the circuit $-X$). But in this case $(F, S \setminus F)$ is orthogonal to every circuit X of M , and then is a non-Radon partition of M .

Conversely suppose that ${}_{\bar{A}}M$ is acyclic for every $A \subseteq F$. Then no circuit (X^+, X^-) can verify $X^+ \subseteq F$. Hence F is a face of M by Proposition 2.2. □

A central problem in the theory of polytopes is the characterization of its combinatorial types. The remarks above allow to think that the characterization of the combinatorial types of matroid polytopes should be easier and perhaps more interesting. But certainly the characterization of the geometrical types of polytopes is a more fundamental question, raised by Eckhoff [16] under the form : characterize the Radon types of finite sets of points in $\mathbb{R}^d$.

We propose a related problem, that seems easier :

PROBLEM 2.4 . Characterize the different classes of orientations of the free matroid of rank r on a set of n elements.

We recall that the free matroid of rank r on a set S has all $(r+1)$-element subsets of S as circuits. Two oriented matroids M and M' on the same set S , are in the same class of orientation when $M' = {}_{\bar{A}}M$ for some $A \subseteq S$.

3 . ALTERNATING ORIENTATIONS OF FREE MATROIDS

Let S be a n-element set, and suppose that $1 \le r \le n-1$. The free matroid of rank r on S , denoted $F_r(S)$, has as its bases all r-elements subsets of S . Bland and Las Vergnas [4] have pointed out that an alternating circuit signature $\mathcal{O}$ of $F_r(S)$ can be associated to every linear order $S_< = s_1 < \ldots < s_n$ of S : for every signed circuit $X \in \mathcal{O}$, with $\underline{X} = \{s_{i_1} < \ldots < s_{i_{r+1}}\}$, $sg_X(s_{i_{j+1}}) = -sg_X(s_{i_j})$, $j = 1,\ldots,s-1$, where $sg_X(s_j)$ denotes the sign of s_j in X . The oriented matroid $(S,\mathcal{O})$ is called here, shortly, the alternating free matroid of rank r on $S_<$ and is denoted by $\mathbb{F}_r(S_<)$; or by $\mathbb{F}_r(s_1 < s_2 < \ldots < s_n)$.

THEOREM 3.1 . Let $S = \{s_1,\ldots,s_n\}$ be an n-element set. Suppose $\mathcal{O}$ is a circuit signature of $F_r(S)$ such that, for every signed circuit X of $\mathcal{O}$ (resp. signed cocircuit Y of $\mathcal{O}^\perp$) and for all $s_i, s_{i+1} \in \underline{X}$ (resp. $s_i, s_{i+1} \in \underline{Y}$) , we have $sg_X(s_{i+1}) = -sg_X(s_i)$ (resp. $sg_Y(s_{i+1}) = sg_Y(s_i)$). Then $\mathcal{O}$ is the alternating circuit signature of $F_r(S)$ with respect to the order $s_1 < \ldots < s_n$.

Proof :

By the orthogonality property (O3), the assertions relative to circuits and to cocircuits are equivalent. We establish the proposition for circuits, proving that, for every signed circuit X , with $\underline{X} = \{s_{i_1} < \ldots < s_{i_{r+1}}\}$, we have

(3.1.2) $sg_X(s_{i_p}) = (-1)^{q-p} sg_X(s_{i_p})$.

We use induction on $i_q - i_p$. If $i_q = i_p + 1$, (3.1
the hypothesis. Suppose $i_q - i_p > 1$ and that (3.1.2) is t
for all integers $i_{p'}$ and $i_{q'}$ such that $1 \le i_{q'} - i_{p'} <$
If $p+1 < q$, we have $s_{i_{p+1}} \in \underline{X}$ and $s_{i_p} < s_{i_{p+1}} < s_{i_q}$;
then the result follows by the induction hypothesis. No
suppose $q = p+1$. Let x_j be some element of $S \setminus \underline{X}$ with $i_p < j < i_{p+1}$. Let X' (resp. X'') be the signed circuit of $\mathbb{F}_r(S_<)$ of support $(\underline{X} \setminus \{s_{i_p}\}) \cup \{s_j\}$ (resp. $(\underline{X} - \{s_{i_{p+1}}\}) \cup \{s_j\}$) . By induction hypothesis $sg_{X'}(s_j) = -sg_{X'}(s_{i_{p+1}})$ and $sg_{X''}(s_{i_p}) = -sg_{X''}(s_j)$. Hen
by the signed elimination property (O2), $sg_X(s_{i_p}) =$

$= -sg_X(s_{i_{p+1}})$. □

PROPOSITION 3.2 [4]. Let $\mathcal{O}$ be the alternating circuit signature of $\mathbb{F}_r(s_1 < \ldots < s_n)$ and $E = \{s_i$; i e
$1 \le i \le n\}$. Then $_{\bar{E}}\mathcal{O}^\perp$ is the alternating circuit signatu
of $\mathbb{F}_{n-r}(s_1 < \ldots < s_n)$.

Proof :

Let Y be a signed cocircuit of $\mathbb{F}_r(s_1 < \ldots < s_n)$ with support $\underline{Y} = \{s_{i_1} < \ldots < s_{i_{n-r+1}}\}$. Applying (3.1.2) and the definition of orthogonal orientation we obtain (cf. Prop. 3.9 in [4]) :

(3.2.1) $sg_Y(s_{i_q}) = (-1)^{p-q+i_q-i_p} sg_Y(s_{i_p})$. The value of

$(-1)^{p-q+i_q-i_p}$ equals $-(-1)^{p-q}$ if and only if exactly one of the integers i_p, i_q is even. Hence, for every signed circuit Y' of $_{\bar{E}}\mathcal{O}^{\perp}$ we have $sg_{Y'}(s_{i_q}) = (-1)^{q-p} sg_{Y'}(s_{i_p})$. The proposition follows, by Prop. 3.1. □

<u>PROPOSITION 3.3</u> . <u>Let</u> $M = \mathbb{F}_r(s_1<\ldots<s_n)$ <u>with</u> $1 \leq r < m$; <u>for</u> $1 \leq i < m$ <u>put</u> $S_i = \{s_j \; ; \; i < j \leq n\}$. <u>Then</u> <u>we have</u> :

$$M \setminus s_i = \mathbb{F}_r(s_1 < s_2 < \ldots < s_{i-1} < s_{i+1} < \ldots < s_n) \quad \underline{\text{and}}$$

$$M / s_i = {}_{\bar{S_i}}\mathbb{F}_{r-1}(s_1 < s_2 < \ldots < s_{i-1} < s_{i+1} < \ldots < s_n) \; .$$

<u>Proof</u> :

The statement relative to $M \setminus s_i$ is clear. Put $E = \{s_j \; ; \; j \text{ even} \; , \; 1 \leq j \leq n\}$ and $E' = \{s_j \; ; \; j \text{ even} \; , \; 1 \leq j < i\} \cup \{s_j : j \text{ odd} \; , \; i < j \leq n\}$. By a known property of the oriented matroids (see [4], Prop. 4.1), $(M/s_i)^{\perp} = M^{\perp} \setminus s_i$. Thus by Prop. 3.2 , $(M/s_i)^{\perp} = {}_{\bar{E}}\mathbb{F}_{n-r}(s_1 < \ldots < s_{i-1} < s_{i+1} < \ldots < s_n)$. Applying Prop. 3.2 again, we obtain $M/s_i = {}_{\bar{E}}({}_{\bar{E'}}\mathbb{F}_{r-1}(s_1 < \ldots < s_{i-1} < s_{i+1} < \ldots < s_n))$, which is the required conclusion. □

4 . <u>CYCLIC POLYTOPES : AN INTRODUCTION</u>

The cyclic polytopes C(n,d) , defined as the convex hull in $\mathbb{R}^d$ of m different points of the moment curve, are combinatorially equivalent. This point is an immediate consequence of the Gale's evenness condition (Theorem 4.2 below) that characterizes the facets (hence the lattice of faces) of C(n,d). Thus we may speak about the combinatorial type C(n,d) . In fact, the polytopes C(n,d) are geometrically equivalent (Theorem 4.1 below). This property was discovered by M. Breen [6], by the way of primitive Radon partitions. The proof uses only very few of the properties of the moment curve. As Grünbaum [24] noticed, it is not surprising that many other curves can take the place of the moment curve for developping the theory of cyclic polytopes : examples can be found in [13, 19, 33,36]. A complete characterization of these curves is possible, via a characterization of the geometrical type of polytopes C(n,d) (Theorem 4.6 and Corollary 4.7).

More surprising is the phenomenon that happens in even dimension ; <u>every polytope which is combinatorially equivalent to</u> C(n,2k) <u>is geometrically equivalent to</u> C(n,2k) . In other words, an alternating free matroid of odd rank (i. e. the geometry of n points on the moment curve of even dimension) is uniquely determined by its lattice of faces. This result and the characterization of the geometries combinatorially equivalent to C(n,2k+1) will be settled in Section 5.

THEOREM 4.1 ([6] and [4] Corollary 3.9.1). Let $x_i = x(t_i)$, $1 \le i \le n$, be n points on the moment curve $x(t) = t, t^2, \dots, t^d)$ in $\mathbb{R}^d$, with $t_1 < t_2 < \dots < t_n$. Then the oriented matroid of affine dependencies of $\{x_1, \dots, x_n\}$ over $\mathbb{R}$, is the alternating free matroid $\mathbb{F}_{d+1}(x_1 < \dots < x_n)$.

Proof :

Breen's proof [6] uses Gale's evenness condition. In fact, as suggested in [4], an effective simple calculus using Vandermonde's determinants suffices : it is clear that $\mathbf{Aff}(\{x_1, \dots, x_n\})$ is a free matroid of rank $d+1$. Let $\{x_{i_1}, \dots, x_{i_{d+2}}\}$ be a circuit of $\mathbf{Aff}(x_1, \dots, x_n)$, $i_1 < i_2 < \dots < i_{d+2}$. The calculus of the coefficients $\lambda_1, \dots, \lambda_{d+2}$ of an affine combination, $\sum_j \lambda_j x_{i_j} = 0$ and $\sum_j \lambda_j = 0$, shows that λ_{i+1} and λ_i have opposite signs for $i = 1, \dots, d+1$. Hence, by definition of $\mathbf{Aff}(x_1, \dots, x_n)$ and of $\mathbb{F}_{d+1}(x_1 < \dots < x_n)$, the conclusion follows. □

Let V be the set of the vertices of a polytope P. We emphasize the fact that the determination of faces of P only depends of the geometry of V by presenting a matroidal version of Gale's evenness condition :

THEOREM 4.2 (Gale's evenness condition [24] for alternating free matroids). Let $V = \{v_1, \dots, v_n\}$ be a set with n elements and let $V_d \subseteq V$ be a d-element subset of V. Then V_d is a facet of $\mathbb{F}_{d+1}(v_1 < \dots < v_n)$ if and only if every two points of $V \setminus V_d$ are separated (for the order $v_1 < \dots < v_n$) by an even number of points of V_d.

Proof :

Put $\underline{Y} = V - V_d = \{v_{i_1}, v_{i_2}, \dots, v_{i_{n-d}}\}$, $i_1 < i_2 < \dots < i_{n-d}$. The number of elements of V_d between v_{i_p} and v_{i_q} is $i_q - i_p + p - q$. V_d is a facet of $F_{d+1}(v_1 < \dots < v_n)$ if and only if $\underline{Y}$ supports a positive cocircuit Y. By (3.2.1) v_{i_p} and v_{i_q} have the same sign in Y if and only if $p - q + i_q - i_p$ is even. Hence the theorem follows. □

Every polytope combinatorially equivalent to $C(n,d)$ will be called a cyclic d-polytope (or shortly a cyclic polytope). Note that the set V of the vertices of a cyclic polytope P can be ordered in such a way that the Gale's evenness condition holds for the facets of P. Every order of V that satisfies the Gale's criterion will be called an admissible order for P. The characterization of all admissible orders for a cyclic polytope will be given in Section 5 (Theorem 5.3).

Shephard [37] gives an extension of Gale's criterion to faces of any dimension of cyclic polytopes. Let $V = \{v_1 < \dots < v_n\}$ be an ordered set and let $W \subset V$. A subset $X \subseteq W$ will be called a contiguous subset of W if for some $1 < i \le j < n$, $X = \{v_i, v_{i+1}, \dots, v_j\}$, $v_{i-1} \notin W$ and $v_{j+1} \notin W$. X is said to be even (resp. odd) when $|X|$ is even (resp. odd).

COROLLARY 4.3 [37]. Let P be a cyclic d-polytope with vertex set V and an admissible order $v_1<\ldots<v_n$. Let $W \subset V$. Then conv W is a face of dimension k of P if and only if W has $k+1$ elements and admits at most $d-k-1$ odd contiguous subsets (with respect to the admissible order)

Proof :

The corollary is a straightforward consequence of Gale's evenness condition and of the following simple remark that results from definitions :

REMARK 4.4 . Let W be a k-element subset of an ordered set $V = \{v_1<\ldots<v_n\}$. Let m be the number of odd contiguous subsets of W; then there is a $(k+m)$-element subset F of V containing W and such that every contiguous subset of F is even.

The remaining of the proof is left to the reader. □

A polytope P is said to be k-neighbourly if every subset of k points of the vertex-set V of P is the set of vertices of a proper face of P [24]. An immediate consequence of Corollary 4.3 is :

COROLLARY 4.5 (Motzkin [33]) A cyclic d-polytope is a simplicial $[\frac{d}{2}]$-neighbourly polytope. □

Remark that Corollary 4.5 is also a simple consequence of Corollary 2.3 and Theorem 4.1.

We may formulate the characterization of the geometrical type of polytopes $C(n,d)$ as follows :

THEOREM 4.6 . Let V, $V \subseteq \mathbb{R}^d$, $d \geq 2$, be a n-element set. Then $P = \text{conv}(V)$ is geometrically equivalent to $C(n,d)$ if and only if V satisfies both the conditions :

(4.6.1) the points of V are in general position in $\mathbb{R}^d$;

(4.6.2) there is an order $v_1<\ldots<v_n$ of V such that no hyperplane determined by d points of V separates strictly v_k from v_{k+1}, for every $k=1,\ldots,n-1$.

Proof :

The polytope conv(V) is geometrically equivalent to $C(n,d)$ if and only if $\mathcal{A}ff(V)$ is the alternating free matroid $\mathbb{F}_{d+1}(v_1<\ldots<v_n)$ for some order $v_1<\ldots<v_n$ of V (Theorem 4.1). If, for every $k=1,\ldots,n-1$, no hyperplane determined by d points of V separates strictly v_k from v_{k+1}, then for every signed cocircuit Y of $\mathcal{A}ff(V)$, such that $v_k, v_{k+1} \in \underline{Y}$ the elements v_k and v_{k+1} appear in Y with the same sign. Then Theorem 4.6 follows from Theorem 3.1. □

A curve (i. e. a continuous mapping $x : \mathbb{R} \to \mathbb{R}^d$) is said a cyclic d-curve when $\text{conv}(x(t_1),\ldots,x(t_n))$ is combinatorially equivalent to $C(n,d)$ for any different reals $t_1,\ldots,t_n$. A straightforward consequence of the previous theorem yields :

COROLLARY 4.7 . A curve $x : \mathbb{R} \to \mathbb{R}^d$ is a cyclic d-curve if and only if $x(t_1),\dots,x(t_{d+1})$ are in general position for any different reals $t_1,\dots,t_{d+1}$. □

Let v be a vertex of a polytope P and let H be a hyperplane that separates strictly v from the other vertices of P . Then the combinatorial type of $H \cap P$ does not depend on the choice of H , and is called the vertex-figure of P at v [24]; it should be noted that the vertex-figure of a polytope does not depend on the geometrical type, but only on its combinatorial type, since the lattice of faces of the vertex-figure of P at v is isomorphic to the interval $[v,P]$ in the lattice of faces of P .

PROPOSITION 4.8 . Let P be a cyclic d-polytope, $d \geq 3$, with vertex set V and admissible order $v_1<\dots<v_n$. Then if d is odd (resp. even) the vertex-figure of P at v_1 or v_n (resp. at v_i , for $1 \leq i \leq n$) is combinatorially equivalent to $C(n-1,d-1)$.

Proof :

It suffices to prove the proposition when P is $C(n,d)$. It is not difficult to see that the lattice of faces of the vertex-figure of a polytope $P = \mathrm{conv}(V)$ at $v \in V$ is isomorphic to the lattice of faces of the matroid polytope $\mathcal{A}ff(V)/v$. Thus the proposition is a consequence of Proposition 3.3 and Theorem 4.1, remarking that the matroid $\bar{S}_i \mathbb{F}_d(\{v_1 < v_2 < \dots < v_{i-1} < v_{i+1} < \dots < v_n\})$, with $S_i = \{v_j \ ; \ i < j \leq n\}$, coincides, when d is even, with $\mathbb{F}_d(v_{i+1}<\dots<v_n<v_1<\dots<v_{i-1})$.

Proposition 3.8 shows the strong regularity of the cyclic polytopes of even dimensions. The special role of two vertices in the case of odd dimension will be illustrated by the construction of cyclic polytopes of odd dimension proposed in the next section.

5. CYCLIC POLYTOPES : NEW RESULTS

THEOREM 5.1 . Let V be a n-element subset of $\mathbb{R}^{2k}$, where $k \geq 1$ and $n \geq 2k+1$. Then conv(V) is a cyclic 2k-polytope with n vertices if and only if $\mathcal{A}ff(V)$ is an alternating free matroid of rank $2k+1$.

Proof :

We only prove the non trivial part of the theorem : We suppose $P = \text{conv}(V)$ is a cyclic polytope and we prove that $\mathcal{A}ff(V)$ is an alternating free matroid. We put $\mathcal{F} = \{F : F \subset V, \text{conv}(F) \text{ is a face of } P\}$ and let $v_1 < \ldots < v_n$ be an admissible order for P. The Gale's criterion can be stated as follows (see Corollary 4.3) :

$F \in \mathcal{F}$ if and only if every contiguous subset X of F is even.

As P is a simplicial polytope, no member of a primitive Radon partition of V can be included in one of its facets (see Corollary 2.3). Since P is a k-neighbourly polytope (see Corollary 4.5), every member of a primitive Radon partition of V has at least $k+1$ elements (hence has exactly $k+1$ elements) and $\mathcal{A}ff(V)$ is a free matroid. Let $X = (X^+, X^-)$ be a signed circuit of $\mathcal{A}ff(V)$. Suppose there are two consecutive vertices v_i and v_{i+1} (with respect to the admissible order of V) contained in X^+. Then the number of odd contiguous subsets of X^+ is at most $|X^+|-2 = k-1$ and, by Remark 4.4, X^+ must be contained in a member of $\mathcal{F}$

which is absurd. We conclude that for every signed circuit X of $\mathcal{A}ff(V)$ and for every consecutive vertices v_i and v_{i+1}, $i = 1,\ldots,n-1$, $sg_X(v_i) = -sg_X(v_{i+1})$: i.e. $\mathcal{A}ff(V)$ is the alternating free matroid $\mathbb{F}_{k+1}(v_1 < \ldots < v_n)$ by Proposition 3.1. □

For odd dimensions, a similar result does not hold, even if the points are supposed in general position :

THEOREM 5.2 . Let V be a n-element subset on R^{2k+1}, $n \geq 2k+2$. Suppose conv(V) is a 2k+1-polytope with n vertices. Then conv(V) is a cyclic polytope if and only if there are two vertices v_1, v_n of V with the following properties :

(5.2.1) Every hyperplane spanned by points of $V - \{v_1, v_n\}$ separates strictly v_1 from v_n ;

(5.2.2) the vertex-figure of P at v_1 is a cyclic 2k-polytope.

Proof :

We begin with the "only if" part : we suppose $P = \text{conv}(V)$ is a cyclic (2k+1)-polytope and $v_1 < v_2 < \ldots < v_n$ is an admissible order for P. Then $\mathcal{A}ff(V)/v_1$ is a cyclic 2k-polytope with n-1 vertices (see Proposition 4.8); hence 5.2.2 follows . To see 5.2.1, note that if $X = (X^+, X^-)$ is a signed circuit of $\mathcal{A}ff(V)$ and $v_1 \in X^+$, $v_n \in X^-$, then we have $|X^+| > k+1$ and $|X^-| > k+1$ which is absurd. Indeed if we had $|X^+| \leq k+1$ (resp. $|X^-| \leq k+1$)

then $X^+ - \{v_1\}$ (resp. $X^- - \{v_n\}$) would be a face of $\mathcal{A}ff(V)/v_1$ (resp. $\mathcal{A}ff(V)/v_n$) , by Corollaries 4.5 and 4.8; consequently X^+ (resp. X^-) would also be a face of the simplicial polytope $\mathcal{A}ff(V)$, a contradiction with Corollary 2.3.

To prove the "if" part of the theorem, suppose $P = $ onv(V) is a (2k+1)-polytope with n vertices and the vertices v_1, v_n satisfying both the conditions 5.2.1 and 5.2.2. Then for every facet F of $\mathcal{A}ff(V)$, $F \cap \{v_1, v_n\} \neq \emptyset$ and $|F| = d$, by condition 5.2.1. Thus P is a simplicial polytope. Since every face of dimension 2k-1 is contained into two facets, if $v_1 \in F$ and $v_n \notin F$ (respectively $v_n \in F$ and $v_1 \notin F$) then $(F\setminus\{v_1\}) \cup \{v_n\}$ (resp. $(F\setminus\{v_n\}) \cup \{v_1\}$) is also a facet of $Aff(V)$. We denote by $\bar{v}_i$, $i = 2,\dots,n$, the vertex of $Aff(V)/v_1$ corresponding to the line $\overline{v_1 v_i}$ joining v_1 and v_i . Then there is an admissible order $\bar{v}_{i_1} < \dots < \bar{v}_{i_{n-1}}$ of the vertices of $\mathcal{A}ff(V)/v_1$ such that $\bar{v}_{i_{n-1}} = \bar{v}_n$. But in this case it is easy to see that the facets of $\mathcal{A}ff(V)$ verify the Gale's criterion relatively to the order $v_1 < v_{i_1} < \dots < v_{i_{n-2}} < v_n$ and the theorem follows. □

We now indicate a simple construction of all geometrical types of cyclic polytopes P of odd dimension d . Assuming $n \geq d+2$, we choose a cyclic polytope P_o of type $C(n-2,d-1)$ in $\mathbb{R}^{d-1}$, with vertices $v_2,\dots,v_{n-1}$

and admissible order $v_2 < \dots < v_{n-1}$. By an affine transformation, we may choose the origine O not in P_o but very close to a facet F_o of P_o , so that the hyperplane spanned by F_o is the only hyperplane spanned by a facet of P_o that separates strictly O from the interior of P_o . Considering the immersion $I : \mathbb{R}^{d-1} \to \mathbb{R}^d$, $I(x) = (x,0)$, we choose $v_1 = (0,\dots,0,-1)$ and $v_n = (0,\dots,0,1)$. It is not difficult to see that $P_1 = conv(v_1,\dots,v_n)$ is a cyclic d-polytope with admissible order $v_1 < \dots < v_n$. Since P_1 is simplicial, some positive real exists such that if n points $w_1,\dots,w_n$ are choosen respectively in the balls of radius ε and centers $v_1,\dots,v_n$, then $P = conv(w_1,\dots,w_n)$ is combinatorially equivalent to P_1 . The reader may verify that every geometrical type of the cyclic d-polytopes with n vertices can be represented by some polytope P obtained by this way. Theorems 5.1 and 5.2 explain that a complete description of admissible orders of cyclic polytopes is possible :

THEOREM 5.3 : *Let P be a cyclic d-polytope with vertex set V and admissible order $v_1 < \dots < v_n$.*

(5.3.1) *If d is odd and $n > d+2$, P admits exactly four admissible orders:* $v_1 < v_2 < \dots < v_n$, $v_1 < v_{n-1} < v_{n-2} < \dots < v_2 < v_n$ *and their reversals.*

(5.3.2) *If $d = 2m+1$ and $n = d+2$, P admits exactly $m!(m+1)!$ admissible orders which are the orders of the form* $v_{\sigma(1)} < v_{\tau(2)} < v_{\sigma(3)} < \dots < v_{\tau(d+1)} < v_{\sigma(d+2)}$ *where σ (resp. τ) is any permutation of the* odd (resp. even) numbers of $\{1,2,\dots,d+2\}$.

(5.3.3) If d is even and $m > d+2$, P admits exactly $2n$ admissible orders which are orders $v_i < v_{i+1} < \dots < v_n < v_1 < v_2 < \dots < v_{i-1}$ and their reversals.

(5.3.4) If $d = 2m$ and $n = d+2$, P admits exactly $(m!)^2$ admissible orders of the form

$$v_{\sigma(1)} < v_{\tau(2)} < v_{\sigma(3)} < v_{\tau(4)} < \dots < v_{\sigma(d+1)} < v_{\tau(d+2)}$$

where σ (resp. τ) is any permutation of the odd (resp. even) numbers of $\{1,2,\dots,d+2\}$.

(5.3.5) If $n = d+1$, P is a simplex and any total order of V is an admissible order.

Proof :

The proof relies upon the fact that a cyclic polytope has a special type of facets, whose vertices constitute a circular interval of the circular order associated to an admissible order. (The circular order determined by the order $i_1 < i_2 < \dots < i_n$ is $\cdots i_1 \prec i_2 \prec \dots \prec i_n \prec i_1 \cdots$).

Let $\mathcal{F}$ be the set of all d-element subsets of V that determine a facet of P (i. e. $\mathcal{F}$ is the set of facets of $\mathrm{Aff}(V)$). For every $F \in \mathcal{F}$ and $x \in F$, there is a unique element $y \in V-F$ such that $(F \setminus \{x\}) \cup \{y\} \in \mathcal{F}$ (because P is simplicial and $F-\{x\}$ determines a face of dimension $d-1$ of P). A vertex $y \in V-F$ such that $(F-\{x\}) \cup \{y\} \in \mathcal{F}$ for some $x \in F$ will be said adjacent to F . A member F of $\mathcal{F}$ will be called a special set when exactly two vertices of $V-F$ are adjacent to F .

Assume $n > d+2$. For every admissible order ξ , the Gale's criterion for P implies : an element F of $\mathcal{F}$ is a special set if and only if one of the following situations arises :

(i) F is a circular interval of the circular order determined by ξ .

(ii) d is odd and $F = \{\xi(1),\xi(n-d+1),\xi(n-d+2),\dots,\xi(n-1)\}$ of $F = \{\xi(2),\xi(3),\dots\xi(d),\xi(n)\}$ where $\xi(k)$ denotes the k-th element of V , with respect to ξ .

Two vertices a and b of P will be said contiguous when $\{a,b\}$ is the intersection of all special sets containing them. The contiguity relation defines a simple graph G on V that depends only on the facial structure of P . But using (i) and (ii) we may determine G by the way of an admissible order ξ . When d is even, G is a cycle of n elements; two vertices are contiguous if and only if they are consecutive in the circular order determined by ξ . When d is odd, G is a cycle with edges $\{\xi(1),\xi(2)\},\dots,\{\xi(k),\xi(k+1)\},\dots,\{\xi(n-1),\xi(n)\}$ augmented with the two edges $\{\xi(2),\xi(n)\}$ and $\{\xi(1),\xi(n-1)\}$. Every admissible order must produce the same graph G . Hence to every admissible order corresponds an automorphism of G . It is not difficult to see that conversely every automorphism of G can be associated to an admissible order. Thus (5.3.1) and (5.3.3) follow.

Whenever $n = d+2$, V supports a signed circuit of $\mathrm{Aff}(V)$; thus (5.3.2) and (5.3.4) follow (the details are left to the reader). Finally (5.3.5) is trivial. □

6. DEHN-SOMMERVILLE EQUATIONS

The Dehn-Sommerville equations for simplicial polytopes [24, 26,32, 38] relate the numbers of faces of given dimension. They constitute a good way of obtaining the number of faces of dimension k, $k \leq d$, for every $[\frac{d}{2}]$-neighbourly simplicial d-polytope, hence for cyclic polytopes (see Section 7).

In [11] it is stated, without proof, that a suggestion of Stanley in [39] can be used to prove that Dehn-Sommerville equations are valid for simplicial matroid polytopes. For the sake of completeness we give here a complete proof of this result.

Let M be a matroid polytope of rand r. $f_k(M)$ or simply f_k will denote the number of faces of rank k of M. Let C_k be the number of chains $0 = F_0 < F_1 < \ldots < F_k = M$ in the lattice L of the faces of M. The zeta polynomial $Z(n)$ of L is defined by :

$$Z(n) = \sum_{k=0}^{r} \binom{n}{k} C_k .$$

The zeta polynomial was first explicitely defined by Stanley [39] in an equivalent form. (For more details and recent results of the theory of zeta polynomial see [18]). We remark that the zeta polynomial is usually defined by the identity $Z(n) = \zeta^n(0,1)$, where ζ^n denotes the nth power of the zeta function on the incidence algebra of L.

From the definition of multiplication it is easy to deduce that $\zeta^k(F,G)$ = the number of multichains $F = F_0 \leq F_1 \leq \ldots \leq$ Observe that $\mu^n(F,G) = \zeta^{-n}(F,G)$ where μ^n denotes the nth power of the Möbius function. From the Euler relat for the matroid polytope M we have $\mu(F,G) = (-1)^{\text{rank } F\text{-ra}}$ (see [11], Corollary 3.2). Thus we have proved the followin theorem :

THEOREM 6.1 ([11], Corollary 3.4) . Let M be a matroid of rank r. Let $Z(n)$ denote the number of multich $0 = F_0 \leq F_1 \leq \ldots \leq F_n$ of faces of M between 0 and M. Then $Z(n)$ is a polynomial in n, of degree r, satisfyin $Z(-n) = (-1)^r Z(n)$. □

As Stanley has pointed out in the case of simplicial polytopes ([39], Proposition 3.3) Theorem 6.1 may be viewed as a generalization of the Dehn-Sommerville equations :

COROLLARY 6.2 (Dehn-Sommerville equations) Let M be a simplicial matroid polytope of rank $r \geq 2$. Then for every k, $0 \leq k \leq r-2$, we have :

$$(-1)^{r+1} f_k = \sum_{i=k}^{i=r-1} (-1)^i \binom{i}{k} f_i .$$

Proof :

Let F be a face of M of rank $i = |F|$. Hence the number of multichains

$$0 = F_0 \leq \ldots \leq F_j = F < F_{j+1} = F_{j+2} = \ldots F_n = M$$

of faces of M is $\sum_{j=0}^{n-1} j^i$. (Note that the multichain

$0 = F_0 \leq \ldots \leq F_j = F$ determines uniquely an application $\varphi : F \to \{1,2,\ldots,j\}$, where $\varphi^{-1}(i) = F_i \setminus \bigcup_{i'=1}^{i-1} F_{i'}$). From this observation we derive

$$Z(n) = \sum_{i=0}^{r-1} f_i (0^i + 1^i + \ldots + (n-1)^i) .$$

In this case

$Z(n+1) - Z(n) = \sum_{i=0}^{r-1} f_i n^i$ and also

$Z(-n-1) - Z(-n) = -\sum_{i=0}^{r-1} f_i (-n-1)^i$. By Theorem 6.1 we have

$$(6.2.1) \quad \sum_{i=0}^{r-1} f_i (-n-1)^i = (-1)^{r+1} \sum_{i=0}^{r-1} f_i n^i .$$

As for every k, $0 \leq k \leq r-2$, the coefficient of n^k in the first member of (6.2.1) is $\sum_{j=0}^{r-1-k} (-1)^{k+j} \binom{k+j}{k} f_{k+j}$, the corollary follows . □

7. NUMBER OF FACES OF CYCLIC POLYTOPES

In the sequel $f_k(n,d)$ will denote the number of faces of dimension k of $C(n,d)$. It should be noted that the Dehn-Sommerville equations may be used to determine all the f_k's from $f_0, f_1, \ldots, f_{m-1}$ where $m = [\frac{d}{2}]$. Explicit formula may be found in [24] or [32] (McDonald formulas). Hence the number of faces of dimension k is the same for every simplicial $[\frac{d}{2}]$-neighbourly d-polytope with n vertices. Another way of obtaining $f_K(n,d)$ is by using the generalized Gale's evenness condition (Corollary 4.3) introduced by Shephards [32, 37].

Here we will determine $f_{d-1}(n,d)$ by a simple matroidal argument. The calculus of other $f_K(n,d)$ becomes simpler when we know $f_{d-1}(n,d)$, by a direct use of Dehn-Sommerville equations (for details on this point see Grünbaum [24]).

LEMMA 7.1

(7.1.1) $f_{2m}(n,2m+1) = f_{2m-1}(n-1,2m) + \frac{1}{2} f_{2m}(n-1,2m+1)$, <u>for</u>, $n \geqslant 2m+2$.

(7.1.2) $(k+1) f_k(n,2m) = n f_{k-1}(n-1,2m-1)$, <u>for</u> $n \geqslant 2m+1$.

<u>Proof</u> :

We prove (7.1.1). Let $v_1 < v_2 < \ldots < v_n$ be an admissible order of the vertex set V of $C(n,2m+1)$. Then $\bar{v}_n / \mathbf{Aff}(V)$ equals $\mathbb{F}_{2m+2}(v_n < v_1 < v_2 < \ldots < v_{n-1})$. Thus the number f

of facets F of $C(n,2m+1)$ such that v_n is not a vertex of F is half of the number of the facets of $\mathrm{Aff}(V) \setminus v_n$. On the other hand, by the definitions, the number of facets of $C(n,2m+1)$ such that v_n is a vertex of F, equals the number of facets of the vertex figure of $C(m,2n+1)$ at v_n. By Proposition 4.8, this number is $f_{2m-1}(n-1,2m)$.

(7.1.2) is an immediate consequence of Proposition 4.8 since :

$$nf_{k-1}(n-1,2m-1) = \sum_{v \in V} |\{F : F \text{ is a } k\text{-face of } C(n,2m) \text{ containing } v\}| =$$

$$= (k+1) f_k(n,2m) .$$

PROPOSITION 7.2 (Motzkin [33])

(7.2.1) $f_{2m-1}(n,2m) = \frac{n}{n-m} \binom{n-m}{m}$;

(7.2.2) $f_{2m}(n,2m+1) = 2\binom{n-m-1}{m}$.

Proof :

By Lemma 7.1 we have :

$f_{2m}(n,2m+1) = \frac{n-1}{2m} f_{2m-2}(n-2,2m-1) + \frac{1}{2} f_{2m}(n-1,2m+1)$. This yields (7.2.2), by an easy induction. Then (7.2.1) results of (7.2.2) and (7.1.2). □

The result of the calculus of $f_K(n,d)$ using (7.2.1.), (7.2.2) and Dehn-Sommerville equations is :

THEOREM 7.3 (Motzkin [33], [24])

(7.3.1) $f_k(n,2m) = \sum_{j=1}^{n} \frac{n}{j} \binom{n-j-1}{j-1} \binom{j}{k-j+1}$;

(7.3.2) $f_k(n,2m+1) = \sum_{j=0}^{m} \frac{k+2}{j+1} \binom{n-j-1}{j} \binom{j+1}{k-j+1}$. □

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