DECOMPOSITION OF CONVEX POLYTOPES INTO SIMPLICES

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Abstract. Every polytope whose facet normals are in general position can be written as a signed sum of simplices determined by the hyperplanes of the facets.

Introduction. Every convex ℓ-gon — except for the parallelograms — can be obtained from a triangle by deleting $\ell - 3$ smaller triangles so that the sides of the triangles lie on the same lines as those of the polygon. We wish to investigate here a possible generalization of this observation, namely, whether a convex polytope K in the d-dimensional Euclidean space can be decomposed to the signed sum of simplices having facets parallel to those of K.

We show the existence of such a decomposition for every K with faces in "general position" and also for the general case if we allow an auxiliary hyperplane with which the facets of the simplices may be parallel. We also make some observations on the number of the simplices needed for such a decomposition. The question is motivated by results of J. Beck (1988) in geometric discrepancy theory. Applications of our new results can be found in Károlyi (1992).

Another way of putting our results is the following. An arrangement of hyperplanes dissects the space $\mathbb{R}^d$ into a number of bounded and unbounded regions. Every simplex bounded by some of these hyperplanes is the "sum" of bounded regions contained in it. We show that, inversely, every bounded region occurs in the lattice generated by the simplices.

1. Polytopes. To avoid the discussion of how to handle boundary points, we introduce the following terminology.

Definition 1. Let $K, K_1, ..., K_n \subseteq \mathbb{R}^d$. We call K the *signed sum* of the K_i if there exist ± 1 signs $\varepsilon_1, ..., \varepsilon_n$ and $K', K_1', ..., K_n' \subseteq \mathbb{R}^d$ such that $K \triangle K' \subseteq \partial K$, $K_i \triangle K_i' \subseteq \partial K_i$ and $\chi_{K'} = \sum_{i=1}^{n} \varepsilon_i \chi_{K_i}$. In notation: $K = \sum_{i=1}^{n} \varepsilon_i K_i$.

(Here ∂A and χ_A denote the boundary and the characteristic function of A, respectively, $A \triangle B$ is the symmetric difference of A and B.)

For a convex d-polytope K, we will denote by S_K the set of the hyperplanes determined by the facets ($(d-1)$-faces) of K. We call the convex d-polytope L a K-polytope if all the facets of L lie in elements of S_K. We say that *the normals of a set $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$ are in general position* if the normal vectors of any d elements of $\mathcal{H}$ are linearly independent over $\mathbb{R}$. We say that $\mathcal{H}$ is *in general position* if its normals are and in addition no $d+1$ members of $\mathcal{H}$ go through the same point.

Theorem 1. *Let K be a convex d-polytope. If the normals of S_K are in general position, then K is the signed sum of K-simplices.*

Proof. For a convex d-polyhedron L, let $f_i(L)$ denote the number of i-faces of L $(0 \leq i \leq d-1)$ and let $c(L)$ be the greatest integer $c \leq f_{d-1}(L)$ for which every c facets of L have a common point.

Proposition 1. *For a convex d-polytope L we have $c(L) \leq d$. Equality holds if and only if L is a simplex.*

Proof. The first statement is obvious from Helly's theorem. To prove the second, assume that $F_1, ..., F_{d+1}$ are non-intersecting facets of L but every d of them have a point in common. Let $P_i \in \cap_{j \neq i} F_j$. If all the P_i's are in the same hyperplane, then by Radon's theorem there exists a partition $\{P_1, ..., P_{d+1}\} = X_1 \cup^* X_2$, $\text{conv}X_1 \cap \text{conv}X_2 \neq \emptyset$. Having $X_i \subseteq \cap_{P_j \notin X_i} F_j$ we obtain $\text{conv}X_1 \cap \text{conv}X_2 \subseteq \cap_{j=1}^{d+1} F_j$, a contradiction. Hence the points $P_1, ..., P_{d+1}$ are the vertices of a simplex wich must be identical with L.

Choose $c(K)+1$ facets of K wich have no common point: $F_1, ..., F_{c(K)+1}$. Deleting one of them, say F_i, the hyperplanes of the remaining facets determine a polyhedron K_i containing K. Then $K = K_i \setminus L_i$ with a polyhedron L_i determined by the same hyperplanes as K, more exactly, L_i is the intersection of the same $|S_K|$ halfspaces as K except one, supported by F_i, which we replace by its complement.

Proposition 2. *$f_{d-1}(K_i) < f_{d-1}(K)$, $f_{d-1}(L_i) \leq f_{d-1}(K)$ and if equality holds, then $c(L_i) < c(K)$.*

Proof. The first inequality is obvious. Suppose, by way of contradiction, that $f_{d-1}(L_i) = f_{d-1}(K)$ and the facets of L_i, lying in the hyperplanes $H_1, ..., H_{i-1}, H_{i+1}, ...,$ $H_{c(K)+1}$ containing $F_1, ..., F_{i-1}, F_{i+1}, ..., F_{c(K)+1}$, respectively, have a common point. Then $H_1, ..., H_{i-1}, H_{i+1}, ..., H_{c(K)+1}$ have a common point on both sides of the hyperplane H_i, hence they have a common point in H_i too, a contradiction.

We will prove that if K is not a simplex, then there exists an i for which K_i (and so L_i) is bounded, i.e., K_i and L_i are K-polytopes. Then repeating the procedure to K_i and L_i, and so on, in view of Proposition 2 we obtain a decomposition of K into a signed sum of K-simplices. In the light of Proposition 1, we can finish the proof by verifying

Proposition 3. *If the normals of S_K are in general position and $F_1, ..., F_{k+1}$ $(k < d)$ are facets of K without a common point, then there exists and $i \in \{1, ..., k+1\}$ for which K_i is bounded.*

To work with vertices rather than facets it is reasonable to consider the polar of K. Using the polar terminology is also a natural way of dealing with boundedness. Let us recall as much as we need here, for the details see any textbook, e.g. Grünbaum (1967).

For a subset A of $\mathbb{R}^d$ the *polar* of A is $A^* = \{x \in \mathbb{R}^d \mid xy \leq 1 \ \forall y \in A\}$. If K is a convex polyhedron having the origin 0 in its interior, then K^* is a convex polytope (possibly degenerate) containing the origin. K is bounded if and only if $0 \in \text{int}K^*$;

in this case K and K^* are dual polytopes. If the facets of K are $F_1, ..., F_n$ then the corresponding vertices of K^* can be denoted by $v_1, ..., v_n$. If K_i is the polyhedron obtained from K by deleting F_i, then $K_i \supset K$, $0 \in \text{int} K_i$ and K_i^* is the convex polytope having vertices $v_1, ..., v_{i-1}, v_{i+1}, ..., v_n$ and – if $0 \notin \text{conv}\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$ – the origin. The facets $F_{i_1}, ..., F_{i_k}$ of K have a common point if and only if the points $v_{i_1}, ..., v_{i_k}$ are vertices of the same facet of K^*. Finally we make the observation that the normals of S_K are in general position if and only if $0 \notin \text{aff}\{v_{i_1}, ..., v_{i_d}\}$ for every d-element subset of the vertices of K^*, here $\text{aff}\{x_1, ..., x_k\} = \{\sum_{i=1}^{k} \alpha_i x_i \mid \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1\}$ denotes the affine hull of the x_i's.

Now Proposition 3 follows immediately from the next lemma.

Lemma. *Let K be a convex d-polytope which is not a simplex, and let $p \in K$. If the vertices $p_1, ..., p_k$ do not lie on the same facet of K then one of them can be deleted so that the convex hull of the remaining vertices of K contains p.*

Proof. Let $\{p_1, ..., p_k, v_1, ..., v_\ell\}$ be the set of the vertices of K. If $\ell = 0$ then $k \geq d+2$ and the assertion follows by Carathéodory's theorem. So suppose that $\ell > 0$.

We show that $\text{aff}\{p_1, ..., p_k\} \cap \text{conv}\{v_1, ..., v_\ell\} \neq \emptyset$. This is obvious if $\text{aff}\{p_1, ..., p_k\} = \mathbb{R}^d$. Otherwise, there exists a hyperplane separating $\text{aff}\{p_1, ..., p_k\}$ from $\text{conv}\{v_1, ..., v_\ell\}$. This hyperplane must be parallel to $\text{aff}\{p_1, ..., p_k\}$, and so we can translate it so that it contains $\text{aff}\{p_1, ..., p_k\}$. Now we get a supporting hyperplane of K containing $\text{aff}\{p_1, ..., p_k\}$, which contradicts the assumption that the points p_i are not on the same facet.

Let $w \in \text{aff}\{p_1, ..., p_k\} \cap \text{conv}\{v_1, ..., v_\ell\}$. Writing p as a convex linear combination of the vertices of K, there exist $\alpha_1, ..., \alpha_k, \vartheta_1, ..., \vartheta_\ell \geq 0$, $\sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{\ell} \vartheta_j = 1$, $p = \sum_{i=1}^{k} \alpha_i p_i + \sum_{j=1}^{\ell} \vartheta_j v_j$.

If $\alpha_i = 0$ for some $1 \leq i \leq k$, then we can delete p_i and the convex hull of the remaining vertices still contains p. Suppose that $\alpha_i > 0$ for all i. Since the p_i do not lie on the same facet of K, this implies that $p \in \text{int} K$. But then we may assume that $\vartheta_j > 0$ for all j. Hence we have

$$u = \frac{\sum_{i=1}^{k} \alpha_i p_i}{\sum_{i=1}^{k} \alpha_i} \in \text{conv}\{p_1, ..., p_k\}$$

$$u' = \frac{\sum_{j=1}^{\ell} \vartheta_j v_j}{\sum_{j=1}^{\ell} \vartheta_j} \in \text{conv}\{v_1, ..., v_\ell\}.$$

The line connecting u and w lies in $\text{aff}\{p_1, ..., p_k\}$ so it intersects $\partial\text{conv}\{p_1, ..., p_k\}$ (in the relative topology) at two points s and s'. We may assume that $u \in \text{conv}\{w, s\}$. Since $k \geq 2$, there exists an i for which $s \in \text{conv}\{p_1, ..., p_{i-1}, p_{i+1}, ..., p_k\}$. Hence

$$u \in \text{conv}\{p_1, ..., p_{i-1}, p_{i+1}, ..., p_k, v_1, ..., v_\ell\}$$

and

$$p = (\sum_{i=1}^{k} \alpha_i)u + (\sum_{j=1}^{\ell} \vartheta_j)u' \in \text{conv}\{u, u'\} \subseteq \text{conv}\{p_1, ..., p_{i-1}, p_{i+1}, ..., p_k, v_1, ..., v_\ell\}$$

proving the lemma.

Remark. This lemma can be generalized to oriented matroids as follows: If E is the underlying set of an acyclic oriented matroid of rank r, having at least $r + 1$ extremal points, $q \in E$ is a non-extremal point, and $P \subseteq E$ is a set meeting every directed cocycle, then there exists a $p \in P$ such that q is a non-extremal point of $E \setminus \{p\}$. This can be proved translating our proof to oriented matroid arguments.

The condition that the normals of S_K are in general position is not the exact condition for Threorem 1 although the assertion is not true in general and the counterexamples are not only parallelopipeds. To formulate a result valid for all d-polytopes we need the following

Definition 2. A hyperplane H is *in general position with respect to a set of hyperplanes* S if the normal vector of H is linearly independent of any $d - 1$ of the normal vectors of elements of S. Of course, if H is in general position with respect to S then it is in general position with respect to every subset of S.

Let K be a d-polytope and H, a hyperplane in general position with respect to S_K. The elements of S_K determine finitely many intersection points. Choose a hyperplane H' parallel to H, disjoint from the convex hull of these points. A convex d-polyhedron is called a $K_{H'}$-polyhedron if its facets lie in elements of $S_K \cup \{H'\}$.

Theorem 2. *Let K be a convex d-polytope, and H, a hyperplane in general position with respect to S_K. Then K is the signed sum of suitable $K_{H'}$-simplices for arbitrary H' satisfying the condition above.*

Proof. Let S be a halfspace containing K and H' in its interior, then $\partial S = H_S$ is a hyperplane parallel to H'. The proof will be similar to the previous one. The main difference is that we allow unbounded polyhedra to appear and the procedure ends with a decomposition to simplicial K-cones. If all the cones are contained in S then cutting them with H' we obtain a desired decomposition to $K_{H'}$- simplices.

If $L \subset S$ is a K-polyhedron then $H' \cap L = \emptyset$ if and only if L is bounded. If not, then H' cuts from L a $K_{H'}$-polyhedron L'. We will consider only K-polyhedra $L \subset S$ with bounded L'. If all the facets of L have a common point (i.e. L is a cone) then $c(L) = f_{d-1}(L)$ and we cannot choose $c(L) + 1$ different facets of L. Hence we need to modify the definition of $c(L)$. To have simple analogues of Proposition 1 and 2, the most suitable way seems to be the following. Let $c'(L) = c(L)$ if $c(L) < f_{d-1}(L)$ and $c'(L) = c(L')$ if $c(L) = f_{d-1}(L)$. Using Helly's theorem and Proposition 1 we obtain

Proposition 1'. *For a convex K-polyhedron $L \subset S$ we have $c'(L) \leq d$ and equality holds if and only if L is a simplex or a simplicial cone.*

Now we can describe a step of the decomposition algorithm starting with K and ending with simplicial cones contained in S. If $L \subset S$ is a K-polyhedron, suppose first that $c(L) < f_{d-1}(L)$, then $c'(L) = c(L)$. Choose $c'(L) + 1$ non- intersecting facets of L. Using the notation of Theorem 1 with the convention that L_i and M_i play the role of K_i and L_i, respectively, we can prove the following analogue of Proposition 3.

Proposition 3'. *There exists an $i \in \{1, ..., c'(L) + 1\}$ for which $L_i \subset S$.*

Proof. Note first that for a K-polyhedron M, $M \subset S$ is equivalent to $M \subset \mathrm{int}S$ because of $K \subset \mathrm{int}S$. Hence, assuming $0 \in \mathrm{int}L$, S^* is a segment $0s$ with $s \in \mathrm{int}L^*$. If L is not a simplex then the Lemma can be applied to the point s in L^*. If L is a simplex then L^* is a simplex which is the union of $d + 1$ simplices based on the facets of L^* with apex 0. These are the polars of the possible L_i's, one of them contains s in its interior, because H_s is in general position respect to S_K, and hence respect to S_L, too.

If $c(L) = f_{d-1}(L)$, the situation is a bit different. First, choose $c'(L)+1 = c(L')+1$ non-intersecting facets of $L' : F'_1, ..., F'_{c'(L)+1}$. We may assume that $F'_{c'(L)+1} \subset H'$ and F'_i is contained in a corresponding facet F_i of L for $1 \leq i \leq c'(L)$.

Proposition 3". *There exists and $i \in \{1, ..., c'(L)\}$ for which $L'_i \subset S$, assuming that L is not a simplicial cone.*

Proof. We would like to apply the lemma again. Let the vertices of L'^* be denoted by $p_1, ..., p_{c'(L)+1}, v_1, ..., v_\ell$ where p_i belongs to F'_i. The problem is that $p_1, ..., p_{c'(L)}$ are on the same facet of L'^*. Consider L^*, its vertices are $p_1, ..., p_{c'(L)}, v_1, ..., v_\ell$ and 0, the first $c'(L) + \ell$ lying on a facet of L^*. Applying the Lemma to L^*, the point s and the vertices $p_1, ..., p_{c'(L)}, 0$, observe that 0 cannot be the deleted vertex, hence it is some p_i $(1 \leq i \leq c'(L))$. Deleting the vertex p_i of L'^* and adding a new vertex 0 if necessary we get $L_i'^*$, for which $L'_i = L_i^{**}$ satisfies the desired property.

Now deleting the facet F_i of L we have $L = L_i \backslash M_i$ with $L_i \supset L'_i$, $M_i \supset M'_i$ and $L_i \subset S$. L_i and M_i are K-polyhedra.

We can finish the proof of the theorem by proving

Proposition 2'. *$f_{d-1}(L_i) < f_{d-1}(L)$, $f_{d-1}(M_i) \leq f_{d-1}(L)$ and if equality holds, then $c'(M_i) < c'(L)$.*

Proof. Suppose $f_{d-1}(M_i) = f_{d-1}(L)$. The proof is identical to that of Proposition 2 if neither L nor M_i are cones. If L is not a cone but M_i is then the same proof yields $d \leq c(M_i) < c(L)$, a contradiction. Finally if L is a cone then so is M_i and $c'(M_i) = c(M'_i) < c(L') = c'(L)$.

2. The number of terms. The algorithm described in Theorems 1-2 gives an exponential upper bound $cd^{f_{d-1}(K)}$ for the number of K- resp. $K_{H'}$-simplices needed for a desired decomposition. In the 3-dimensional space this bound can be reduced to a linear one.

Theorem 3. *Let K be a convex 3-polytope. If the normals of S_K are in general position, then K is the signed sum of at most $cf_2(K)$ K-simplices. In the general case, K is the signed sum of at most $cf_2(K)$ $K_{H'}$-simplices.*

Proof. We prove the first assertion, the proof of the second is similar. It follows from Euler's relation that the average number of the sides of a face of K is less than 6. Thus more than half of the faces have at most 8 sides. Our aim is to find a face F_i with at most 8 sides so that the corresponding polyhedron K_i is bounded. Then $f_2(K_i) < f_2(K)$ and $f_2(L_i) \leq 9$. Repeating this procedure we can decompose K to the signed sum of less than $f_2(K)$ K-polytopes having at most 9 faces. By the proof of Theorem 1, there exists a constant c such that every K-polytope with at most 9 faces is the signed sum of at most c K-simplices, and the theorem follows.

It remains only to prove that if $f_2(K) > 9$, then there exists a face F_i of K with the desired properties. To see this, consider the polar of K again. By Carathéodory's theorem, the origin is contained in the interior of a tetrahedron determined by 4 suitable vertices of K^*. As K has at least 5 faces with at most 8 sides, we can delete one of the corresponding vertices of K^* so as to obtain a polytope K_i^* containing 0 in its interior, yielding a bounded polytope K_i.

If K is a d-polytope and $K = \sum_{j=1}^{n} \varepsilon_j \Delta_j$ with suitable simplices $\Delta_1, \dots, \Delta_n$, then each i-face of K must be contained in an i-face of some Δ_j, and no two of them can be contained in the same. Hence the number of the simplices

$$n \geq \max_{0 \leq i < d} \frac{f_i(K)}{\binom{d+1}{i}}.$$

In particular, if K is the dual of a cyclic d-polytope L on ℓ vertices, then every $\lfloor d/2 \rfloor$ vertices of L determine a $(\lfloor d/2 \rfloor - 1)$-face of L, hence the number of the $(d - \lfloor d/2 \rfloor)$-faces of K is $\binom{\ell}{\lfloor d/2 \rfloor}$. Therefore the number of the simplices

$$n \geq \frac{\binom{\ell}{\lfloor d/2 \rfloor}}{\binom{d+1}{\lceil d/2 \rceil}} \gg (f_{d-1}(K))^{\lfloor d/2 \rfloor}.$$

Problem. Is it true that every d-polytope K can be decomposed to the signed sum of at most $c_d \max_{0 \leq i < d} f_i(K)$ $K_{H'}$-simplices?

3. Hyperplane arrangements. Let $\mathcal{H}$ be an arrangement (i.e., set) of n hyperplanes in $\mathbb{R}^d$; let $R_1, \dots, R_N$ be the bounded connected components of $\mathbb{R}^d \setminus \cup \mathcal{H}$. Let $T_1, \dots, T_M$ denote the simplices formed by $d+1$ of the given hyperplanes.

Each simplex T_i is the union of those bounded regions R_j contained in it. Let

$$w_{ij} = \begin{cases} 1, & \text{if } R_j \subseteq T_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = (w_{i,j})_{i=1}^{M}{}_{j=1}^{N}$ and $\mathbf{w}_i = (w_{ij})_{j=1}^{N}$. If the normal vectors of the hyperplanes in $\mathcal{H}$ are in general position, then Theorem 1 (applied to each bounded region) implies that the vectors $\mathbf{w}_i$ generate the whole space. In fact, we get more:

Corollary. *Let $\mathcal{H}$ be a hyperplane arrangement with normals in general position. Then the lattice generated by the vectors w_i is the whole lattice $\mathbb{Z}^n$.*

This reformulation suggests an alternate proof of Theorem 1, which we only sketch. Assume that not only the normals are in general position but also $\mathcal{H}$ is (the case when more than d hyperplanes go through the same point can be reduced to this by a small perturbation). Choose any $H \in \mathcal{H}$, this splits the space into two parts A ("above H") and B ("below H"). We show that any $R_j \subseteq A$ is a signed sum of those simplices T_i that are contained in A and have one facet contained in H.

Let, say, $R_1, \ldots, R_m$ be those bounded regions contained in A and $T_1, \ldots, T_p$, those simplices contained in A with one facet contained in H. If the hyperplanes in $\mathcal{H}$ are in general position, then $m = p$. In fact, there is an easy bijection: every simplex T_i $(1 \le i \le p)$ contains a unique "highest" bounded region, namely, the region containing the vertex of T_i not on H. Conversely, every bounded region R_j contained in A has a unique vertex v farthest from H, and the hyperplanes of $\mathcal{H}$ containing v, together with H, define a simplex T_i in which R_j is the highest region.

We may assume that we have labelled the R_j so that the distance of their highest points from H increases with j, and the T_i so that R_j is the highest region in T_j.

Now if we consider the submatrix W' of W corresponding to the first $m = p$ rows and columns, then this has 1's in the main diagonal and 0's above the diagonal. Hence the determinant of W' is 1, which implies that W' is invertible and its inverse has integral entries. This is equivalent to the Corollary (and also to Theorem 1).

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