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Subpolytopes of Cyclic Polytopes

by

Tibor Bisztriczky[1]
Department of Mathematics
University of Calgary
Calgary, Alta. T2N 1N4
Canada
tbisztri@math.ucalgary.ca

and

Gyula Károlyi[2,3]
School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540
U.S.A.
karolyi@math.ias.edu

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[2]DIMACS Postdoctoral Fellow.
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1 Introduction

Let P be a (convex) d-polytope in $\mathbb{E}^d$. The *combinatorial structure*, or *face lattice*, of P is the collection of all faces of P ordered by inclusion. We recall that the face lattice of P is completely determined by the set of facets of P, and that two polytopes are *combinatorially equivalent* if their face lattices are isomorphic. Next, a *facet system* of P is a pair $(\mathcal{F}, X)$ where X is a finite set, $\mathcal{F} \subseteq 2^X$ and there is a bijection $f : X \longrightarrow \text{vert}(P)$ such that

$$\{\text{conv}(\{f(v) \mid v \in X'\}) \mid X' \in \mathcal{F}\}$$

is the set of facets of P. A *subpolytope* of P is the convex hull of a subset of its vertex set. For other elementary properties of P, we refer to [2].

A d-polytope P is *neighbourly* if every $[d/2]$ vertices of P determine a face of P. A well known family of neighbourly d-polytopes are the cyclic d-polytopes.

Let $C(n,d)$ denote the convex hull of n points on the moment curve $x(t) = (t, t^2, \ldots, t^d)$ in $\mathbb{E}^d$, say,

$$C(n,d) = \text{conv}(\{x(t_i) \mid t_1 < t_2 < \ldots < t_n \text{ in } \mathbb{E}^1, n \geq d+1\}) \ .$$

A polytope that is combinatorally equivalent to some $C(n,d)$ is called a *cyclic* d-polytope.

Let C be a cyclic d-polytope with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. From the combinatorial equivalence to $C(n,d)$, we obtain (see [1]) that C is simplicial and that there is a total ordering of V, say, $v_1 < v_2 < \ldots < v_n$ and called a *vertex array*, that satisfies Gale's Evenness Condition (GEC): a d element subset Y of V determines a facet of C if and only if any two vertices of $V \setminus Y$ are separated in the vertex array by an even number of elements of Y. We note that if d is even (odd) then $v_i < v_{i+1} < \ldots < v_n < v_1 < \ldots < v_{i-1}$ $(v_n < v_{n-1} < \ldots < v_1)$ also satisfies GEC for any $2 \leq i \leq n$.

For the sake of simplicity, we say that C is cyclic with $v_1 < v_2 < \ldots < v_n$ if that vertex array of C satisfies GEC. Let $C' = \text{conv}(V')$, $V' \subset V$, be a subpolytope of C. We say that C' is cyclic with $v_1 < v_2 < \ldots < v_n$ if that vertex array induces one of C' that satisfies GEC. Finally, we say that v_i and v_{i+1} are *successive vertices* in $v_1 < v_2 < \ldots < v_n$.

As noted above, we wish to present a direct proof of the following result.

Theorem. *Let C be a cyclic d-polytope with the vertex array $v_1 < v_2 < \ldots < v_n$, $n \geq d+1 \geq 4$. If d is even (odd) then every subpolytope of C (that contains v_1 and v_n) is cyclic with $v_1 < v_2 < \ldots < v_n$.*

Proof. By way of contradiction, we suppose that there is a point $p \in H \cap \text{conv}(\{v_k, v_l\})$. Since W is a $(d-3)$-flat and $\text{aff}(F)$ is a $(d-1)$-flat, we conclude that $p \notin W$.

We note that $\text{aff}(Y_j) = H \cap \text{aff}(F_j)$ and that $\text{aff}(F_j)$ is a supporting hyperplane of C for $j \in \{i-1, i+1\}$. Thus, it follows that $H \cap C$ is contained in a closed quarterspace S of H bounded by $\text{aff}(Y_{i-1})$ and $\text{aff}(Y_{i+1})$. Then $p \in S$ and either the $(d-2)$-flat $\text{aff}(W \cup \{p\})$ strictly separates v_{i-1} and v_{i+1} in H or $p \in \text{aff}(Y_{i-1}) \cup \text{aff}(Y_{i+1})$. Since the former yields that $\text{aff}(F)$ strictly separates v_{i-1} and v_{i+1}, and so F is not a facet of C, it follows that $p \in \text{aff}(Y_{i-1})$, say. But then $v_{i-1} \in \text{aff}(W \cup \{p\}) \subset \text{aff}(F)$ and $v_{i-1} \in F$, a contradiction.

Returning to the Theorem, we assume first that d is even. Next, because of the cyclic nature of the vertex array, we assume without loss of generality that $i = n$. Then $Z = \cup_{j=1}^{m} Z_j \subset \{v_1, \ldots, v_{n-1}\}$ where $m \geq 2$ and each Z_j is a maximal set of successive vertices of C in Z. With $v_k < v_l$ for $v_k \in Z_k$, $v_l \in Z_l$ and $k < l$, we note that $|Z_1|$ and $|Z_m|$ are odd, and $|Z_j|$ is even for $1 < j < m$. As $v_1 \in Z_1$, $v_{n-1} \in Z_m$ and $W = Z \setminus \{v_1, v_{n-1}\}$, it is clear that $\text{conv}(W \cup \{v_k, v_{k+1}\})$ is a facet of C whenever $W \cap \{v_k, v_{k+1}\} = \emptyset$. Next, if $Z_j = \{v \in V \mid v_k < v < v_l\}$ for some $1 < j < m$ then it is also clear that $\text{conv}(W \cup \{v_k, v_l\})$ is a facet of C. A repeated application of the Lemma now yields that $\{v_1, \ldots, v_{n-1}\}$ is contained in one of the closed half-spaces of $\mathbb{E}^d$ bounded by H. Thus, H supports C_n and $\text{conv}(Z)$ is indeed a facet of C.

We argue similarly when d is odd.

Remark 1. Let $d \geq 3$ be odd and let $n \geq d+3$. Then there exists a cyclic d-polytope with vertex array $v_1 < v_2 < \ldots < v_n$ such that for every $V' \subset \{v_1, \ldots, v_n\}$, $\{v_1, v_2\} \not\subset V'$ and $|V'| \geq d+2$, $\text{conv}(V')$ is a non-cyclic d-polytope. Such polytopes are easy to construct.

Remark 2. A slightly different argument can be used to show that certain deformations of cyclic polytopes do not affect the cyclic property. In fact, our Theorem can be regarded as a 'limit version' of the following result, when u tends to v_{i+1}.

Theorem'. *Suppose that the vertex array $v_1 < v_2 < \ldots < v_n$ satisfies GEC in $\mathbb{E}^d$, $d \geq 3$. If u is an inner point of $\text{conv}(\{v_i, v_{i+1}\})$ for some $1 \leq i < n$ ($2 \leq i < n$ when d is odd) then the vertex array $v_1 < \ldots v_{i-1} < u < v_{i+1} < \ldots < v_n$ also satisfies GEC in $\mathbb{E}^d$.*

Proof (sketch). Let $V = \{v_1, \ldots, v_n\}$, $V' = V \setminus \{v_i\} \cup \{u\}$, and $C' = \text{conv}(V') \subset \text{conv}(V) = C$. Let $Z \subset V'$ be a d element set that satisfies the evenness part of GEC with $v_1 < \ldots < v_{i-1} < u < v_{i+1} < \ldots < v_n$. As in the previous proof, we need only to show that $\text{conv}(Z)$ is a facet of C'. Since this is immediate if $\text{conv}(Z)$ is a facet of C, we assume that $u \in Z$.