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SUBPOLYTOPES OF CYCLIC POLYTOPES

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ABSTRACT

A remarkable result of I. Shemer [7] states that the combinatorial structure of a neighbourly $2m$-polytope determines the combinatorial structure of each of its subpolytopes. From this, it follows that every subpolytope of a cyclic $2m$-polytope is cyclic. In this note, we present a direct proof of this consequence that also yields that certain subpolytopes of a cyclic $(2m+1)$-polytope are cyclic.

1. Introduction. Let P be a (convex) d-polytope in $\mathbb{E}^d$. The *combinatorial structure*, or *face lattice*, of P is the collection of all faces of P ordered by inclusion. We recall that the face lattice of P is completely determined by the set of facets of P, and that two polytopes are *combinatorially equivalent* if their face lattices are isomorphic. Next, a *facet system* of P is a pair $(\mathcal{F}, X)$ where X is a finite set, $\mathcal{F} \subseteq 2^X$ and there is a bijection $f : X \longrightarrow \text{vert}(P)$ such that

$$\{\text{conv}(\{f(v) \mid v \in X'\}) \mid X' \in \mathcal{F}\}$$

is the set of facets of P. A *subpolytope* of P is the convex hull of a subset of its vertex set. For other elementary properties of P, we refer to [4].

A d-polytope P is *neighbourly* if every $[d/2]$ vertices of P determine a face of P. A well known family of neighbourly d-polytopes are the cyclic d-polytopes.

Let $C(n,d)$ denote the convex hull of n points on the moment curve $x(t) = (t, t^2, \ldots, t^d)$ in $\mathbb{E}^d$, say,

$$C(n,d) = \text{conv}(\{x(t_i) \mid t_1 < t_2 < \ldots < t_n \text{ in } \mathbb{E}^1, n \geq d+1\}) .$$

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A polytope that is combinatorally equivalent to some $C(n,d)$ is called a *cyclic d-polytope*. Cyclic polytopes, and simplicial neighbourly polytopes, in general, play an important rôle in the combinatorial theory of convex polytopes. They are the solutions to extremum problems (see for example the Upper Bound Theorem of McMullen [6]) and serve as bases for various constructions, from triangulations of size $\Omega(n^{[d/2]})$ with n vertices in $\mathbb{E}^d$ to bimatrix games with many equilibria ([11]).

Let C be a cyclic d-polytope with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. From the combinatorial equivalence to $C(n,d)$, we obtain (see [3]) that C is simplicial and that there is a total ordering of V, say, $v_1 < v_2 < \ldots < v_n$ and called a *vertex array*, that satisfies Gale's Evenness Condition (GEC): a d element subset Y of V determines a facet of C if and only if any two vertices of $V \setminus Y$ are separated in the vertex array by an even number of elements of Y. We note that if d is even (odd) then $v_i < v_{i+1} < \ldots < v_n < v_1 < \ldots < v_{i-1}$ ($v_n < v_{n-1} < \ldots < v_1$) also satisfies GEC for any $2 \leq i \leq n$.

For the sake of simplicity, we say that C is cyclic with $v_1 < v_2 < \ldots < v_n$ if that vertex array of C satisfies GEC. Let $C' = \mathrm{conv}(V')$, $V' \subset V$, be a subpolytope of C. We say that C' is cyclic with $v_1 < v_2 < \ldots < v_n$ if that vertex array induces one of C' that satisfies GEC. Finally, we say that v_i and v_{i+1} are *successive vertices* in $v_1 < v_2 < \ldots < v_n$.

As noted above, we wish to present a direct proof of the following result.

Theorem 1.1. *Let C be a cyclic d-polytope with the vertex array $v_1 < v_2 < \ldots < v_n$, $n \geq d+1 \geq 4$. If d is even (odd) then every subpolytope of C (that contains v_1 and v_n) is cyclic with $v_1 < v_2 < \ldots < v_n$.*

Note that there is a difference between even and odd dimensions. Indeed, the statement cannot be improved when the dimension is odd (see Section 3). As far as we know, the 'odd' part of the Theorem is a new result. However, it can also be derived from a result of Cordovil and Duchet (see Theorem 3.3) coupled with the 'even' part of the Theorem, as we will indicate it in the final section of this note. In fact, in their paper [2], which is unfortunately still unpublished, Cordovil and Duchet give a systematic account on an oriented matroid approach to cyclic polytopes, continued by Sturmfels in [10], and surveyed in [1], Section 9.4. Nevertheless, our aim here is to give direct and elementary proofs to Theorem 1.1 and Theorem 3.1, which concerns certain deformations of cyclic polytopes. We refer to [8] for some related results.

The following consequence of Theorem 1.1 is of particular interest in the light of a result of B. Sturmfels [9] that states that if every subpolytope of a cyclic d-polytope C is cyclic with respect to the original vertex array then there is an arc of order d in $\mathbb{E}^d$ that contains the vertices of C.

Corollary 1.2. *Let $d \geq 3$ be odd and let C be a cyclic d-polytope with vertex array $v_1 < v_2 < < \ldots < v_n$, $n \geq d+1$. If for every $1 \leq i < j \leq n$, $\mathrm{conv}(\{v_i, v_{i+1}, \ldots, v_j\})$ is cyclic with $v_1 < v_2 < \ldots < v_n$ then every subpolytope of C is cyclic with $v_1 < v_2 < \ldots < v_n$.*

2. The proof. We note first that if every subpolytope of C that has at least $d+1$ vertices (and contains v_1 and v_n when d is odd) is a cyclic d-polytope then for $0 \leq k < d$, every k-subpolytope of C (that contains v_1 and v_n when d is odd) is a k-simplex, and hence cyclic. Accordingly; we may asume that $n \geq d+2$, and we need only to verify that the d-subpolytope $C_i = \mathrm{conv}(\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\})$, with $1 < i < n$ for d odd and $v_0 = v_n$, $v_{n+1} = v_1$ for d even, is cyclic with $v_1 < v_2 < \ldots < v_n$.

We recall the following elementary fact from [5] or [12], Ex. 2.8.

Proposition 2.1. *If $(\mathcal{F}, X)$ and $(\mathcal{F}', X)$ are facet systems of convex d-polytopes then $\mathcal{F} \subseteq \mathcal{F}'$ implies $\mathcal{F} = \mathcal{F}'$.*

Let $V = \{v_1, v_2, \ldots, v_n\}$, and $Z \subset V \setminus \{v_i\}$ be a d element set that satisfies the evenness part of GEC with $v_1 < \ldots < v_{i-1} < v_{i+1} < \ldots < v_n$. In view of the Proposition, we need only to show that $\mathrm{conv}(Z)$ is a facet of C_i. Since this is immediate if $\mathrm{conv}(Z)$ is a facet of C, we assume that it is *not* a facet of C.

Since $v_i \notin Z$ and Z satisfies the evenness part of GEC with $v_1 < \ldots < v_{i-1} < v_{i+1} < \ldots < v_n$ but not with $v_1 < \ldots < v_{i-1} < v_i < v_{i+1} < \ldots < v_n$, it is easy to check that $\{v_{i-1}, v_{i+1}\} \subset Z$ and that both $(Z \setminus \{v_{i+1}\}) \cup \{v_i\}$ and $(Z \setminus \{v_{i-1}\}) \cup \{v_i\}$ satisfy the evenness part of GEC with $v_1 < \ldots < v_{i-1} < v_i < v_{i+1} < \ldots < v_n$.

Let $W = Z \setminus \{v_{i-1}, v_{i+1}\}$, $Y_{i-1} = W \cup \{v_{i-1}\}$ and $Y_{i+1} = W \cup \{v_{i+1}\}$. Since C is simplicial and $F_j = \mathrm{conv}(Y_j \cup \{v_i\})$, $j \in \{i-1, i+1\}$, is a facet of C, we obtain that $\mathrm{aff}(W)$ is a $(d-3)$-flat and $\mathrm{aff}(Y_j)$ is a $(d-2)$-flat of $\mathbb{E}^d$. Thus, $H = \mathrm{aff}(Z)$ is a hyperplane of $\mathbb{E}^d$.

The central notion of the proof is the following lemma.

Lemma 2.2. *Let $F = \mathrm{conv}(W \cup \{v_k, v_l\})$ be a facet of C such that $F \cap \{v_{i-1}, v_i, v_{i+1}\} = \emptyset$. Then $H \cap \mathrm{conv}(\{v_k, v_l\}) = \emptyset$.*

Proof. By way of contradiction, we suppose that there is a point $p \in H \cap \mathrm{conv}(\{v_k, v_l\})$. Since W is a $(d-3)$-flat and $\mathrm{aff}(F)$ is a $(d-1)$-flat, we conclude that $p \notin W$.

We note that $\mathrm{aff}(Y_j) = H \cap \mathrm{aff}(F_j)$ and that $\mathrm{aff}(F_j)$ is a supporting hyperplane of C for $j \in \{i-1, i+1\}$. Thus, it follows that $H \cap C$ is contained in a closed quarterspace S of H bounded by $\mathrm{aff}(Y_{i-1})$ and $\mathrm{aff}(Y_{i+1})$. Then $p \in S$ and either the $(d-2)$-flat $\mathrm{aff}(W \cup \{p\})$ strictly separates v_{i-1} and v_{i+1} in H or $p \in \mathrm{aff}(Y_{i-1}) \cup \mathrm{aff}(Y_{i+1})$. Since the former yields that $\mathrm{aff}(F)$ strictly separates v_{i-1} and v_{i+1}, and so F is not a facet of C, it follows that $p \in \mathrm{aff}(Y_{i-1})$, say. But then $v_{i-1} \in \mathrm{aff}(W \cup \{p\}) \subset \mathrm{aff}(F)$ and $v_{i-1} \in F$, a contradiction.

Returning to the Theorem, we assume first that d is even. Next, because of the cyclic nature of the vertex array, we assume without loss of generality that $i = n$. Then $Z = \cup_{j=1}^{m} Z_j \subset \{v_1, \ldots, v_{n-1}\}$ where $m \geq 2$ and each Z_j is a maximal set of successive vertices of C in Z. With $v_k < v_l$ for $v_k \in Z_k$, $v_l \in Z_l$ and $k < l$, we note that $|Z_1|$ and $|Z_m|$ are odd, and $|Z_j|$ is even for $1 < j < m$. As $v_1 \in Z_1$, $v_{n-1} \in Z_m$ and $W = Z \setminus \{v_1, v_{n-1}\}$, it is clear that $\mathrm{conv}(W \cup \{v_k, v_{k+1}\})$ is a facet of C whenever $W \cap \{v_k, v_{k+1}\} = \emptyset$. Next, if $Z_j = \{v \in V \mid v_k < v < v_l\}$ for some $1 < j < m$ then it is also clear that $\mathrm{conv}(W \cup \{v_k, v_l\})$ is a facet of C. A repeated application of the Lemma now yields that $\{v_1, \ldots, v_{n-1}\}$ is contained in one of the closed half-spaces of $\mathbb{E}^d$ bounded by H. Thus, H supports C_n and $\mathrm{conv}(Z)$ is indeed a facet of C.

We argue similarly when d is odd.

3. Remarks. Let $d \geq 3$ be odd and let $n \geq d+3$. Let $I \subset \{1, 2, \ldots, n\}$, $\{1, n\} \not\subset I$ and $|I| \geq d+2$. Then there exists a cyclic d-polytope with vertex array $v_1 < v_2 < \ldots < v_n$ such that for $V' = \{v_i \mid i \in I\}$, $\mathrm{conv}(V')$ is a non-cyclic d-polytope. Such polytopes are easy to construct.

A slightly different argument can be used to show that certain deformations of cyclic polytopes do not affect the cyclic property. In fact, Theorem 1.1 can be regarded as a 'limit version' of the following result, when u tends to v_{i+1}.

Theorem 3.1. *Suppose that the vertex array $v_1 < v_2 < \ldots < v_n$ satisfies GEC in $\mathbb{E}^d$, $d \geq 3$. If u is an inner point of $\mathrm{conv}(\{v_i, v_{i+1}\})$ for some $1 \leq i < n$ ($2 \leq i < n$ when d is odd) then the vertex array $v_1 < \ldots v_{i-1} < u < v_{i+1} < \ldots < v_n$ also satisfies GEC in $\mathbb{E}^d$.*

Proof (sketch). Let $V = \{v_1, \ldots, v_n\}$, $V' = V \setminus \{v_i\} \cup \{u\}$, and $C' = \mathrm{conv}(V') \subset \mathrm{conv}(V) = C$. Let $Z \subset V'$ be a d element set that satisfies the evenness part of GEC with $v_1 < \ldots < v_{i-1} < u < v_{i+1} < \ldots < v_n$. As in the previous proof, we need only to show that $\mathrm{conv}(Z)$ is a facet of C'. Since this is immediate if $\mathrm{conv}(Z)$ is a facet of C, we assume that $u \in Z$.

Let $Z' = Z \setminus \{u\} \cup \{v_i\}$. Then $F_{i-1} = \mathrm{conv}(Z')$ is a facet of C. If $v_{i+1} \in Z$ then $\mathrm{aff}(Z) = \mathrm{aff}(Z')$ and $\mathrm{conv}(Z)$ is indeed a facet of C'. Suppose therefore that $v_{i+1} \notin Z$. Then $v_{i-1} \in Z$, with $v_0 = v_n$ when d is even. To utilize the notation of the previous proof, let $W = Z \setminus \{v_{i-1}, u\}$, $Y_{i-1} = W \cup \{v_{i-1}\}$, $Y_{i+1} = W \cup \{u\}$,

$F_{i+1} = \mathrm{conv}(\{W \cup \{v_i, v_{i+1}\})$ and $H = \mathrm{aff}(Z)$. Then the affine subspaces $\mathrm{aff}(W)$, $\mathrm{aff}(Y_j)$ ($j \in \{i-1, i+1\}$) and H have, again, dimension $d-3$, $d-2$ and $d-1$, respectively.

Now we are ready to state the following counterpart of Lemma 2.2.

Lemma 3.2. *Let $F = \mathrm{conv}(W \cup \{v_k, v_l\})$ be a facet of C such that $F \cap \{v_{i-1}, v_i\} = \emptyset$. Then $H \cap \mathrm{conv}(\{v_k, v_l\}) = \emptyset$.*

The proof of Lemma 3.2 is literally the same as that of Lemma 2.2, if we substitute v_{i+1} by u throughout the proof. A difference emerges only when $p \in \mathrm{aff}(Y_{i+1})$. But then $\mathrm{aff}(W \cup \{p\}) = \mathrm{aff}(Y_{i+1}) \subseteq \mathrm{aff}(F) \cap \mathrm{aff}(F_{i+1})$ and either $\mathrm{aff}(F) = \mathrm{aff}(F_{i+1})$ or $\mathrm{aff}(F)$ strictly separates v_i and v_{i+1}, a contradiction in either case.

The remaining part of the proof follows as for Theorem 1.1, and we leave the details to the reader. We note only that the statement is not true when d is odd and $i = 1$.

In order to explore some connections, we close this paper with an alternative proof of the odd part of Theorem 1.1 via the following characterization of odd-dimensional cyclic polytopes.

Theorem 3.3 ([2]). *Let V be the vertex set of a convex d-polytope C, $|V| = n > d+1$, $d \geq 3$ odd. Then C is a cyclic polytope if and only if there exist $u_1, u_n \in V$ such that every hyperplane spanned by points of $V \setminus \{u_1, u_n\}$ separates u_1 and u_n, and the vertex figure C/u_1 is a cyclic $(d-1)$-polytope. (The vertex figure P/v of a d-polytope P at its vertex v is any $(d-1)$-polytope obtained by cutting P by a hyperplane that separates v from the rest of the vertices of P. Its combinatorial type is independent of the choice of the cutting hyperplane.)*

Suppose that $d \geq 3$ is odd and let C be as in Theorem 1.1. Then there exist $u_1, u_n \in V = \{v_1, v_2, \ldots, v_n\}$ as in Theorem 3.3. In fact, it is inherent in the proof of Theorem 3.3 that we may choose $u_1 = v_1$ and $u_n = v_n$. Let $C' = \mathrm{conv}(V')$, $\{v_1, v_n\} \subseteq V' \subseteq V$, be a subpolytope of C. It is straightforward, that every hyperplane spanned by points of $V' \setminus \{u_1, u_n\}$ separates v_1 from v_n. Moreover, the vertex figure C'/v_1 is a subpolytope of the cyclic $(d-1)$-polytope C/v_1, hence cyclic (note that $d-1$ is even). Thus, C' is cyclic by the 'if' part of Theorem 3.3. To see that C' is cyclic *with* vertex array $v_1 < v_2 < \ldots < v_n$ would, however, require additional work.

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