Hypersimplices: their Ehrhart series!

THE HILBERT SERIES OF ALGEBRAS OF VERONESE TYPE

MORDECHAI KATZMAN

1. Introduction

In this paper we describe the Hilbert series of algebras of Veronese type:

Definition 1.1. Fix a positive integer d and a sequence of integers $\mathbf{a} = (a_1, \dots, a_n)$ such that $1 \leq a_1 \leq \cdots \leq a_n \leq d$ and $\sum_{i=1}^n a_i > d$. Let $\mathcal{V}(\mathbf{a}; d)$ be the k-subalgebra of $k[x_1, \dots, x_n]$ generated by all monomials $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\sum_{i=1}^n \alpha_i = d$ and $\alpha_i \leq a_i$ for all $1 \leq i \leq n$.

We shall also denote by $\mathcal{S}$ all subsets S of $\{1, \dots, n\}$ with $\sum_{i \in S} a_i < d$; for any $S \subset \{1, \dots, n\}$ we define ΣS to be $\sum_{i \in S} a_i$.

Note that $\mathcal{V}(d, d, \dots, d; d)$ is the classical Veronese algebra while $\mathcal{V}(1, 1, \dots, 1; d)$ is the monomial algebra associated with the dth hypersimplex.

For the purpose of computing Hilbert series and a-invariants we will use a normalized grading on these algebras so that the degree of their generators equals one.

These monomial algebras have recently atracted considerable interest; E. DeNegri, T. Hibi ([3]) have recently classified those which are Gornestein and B. Sturmfels ([4]) described Gröbner bases arising from presentations of these algebras.

It is known that algebras of Veronese type are normal and in [3] the authors classified all such algebras which are Gorenstein. Also, in [2] the authors described the canonical modules and a-invariants of $\mathcal{V}(1, 1, \dots, 1; d)$. The aim of this section is to extend and complement these results by producing an explicit formula for the h-vectors of all algebras of Veronese type. Additionally, the explicit formulas provide a very efficient way for computing these Hilbert series.

1991 *Mathematics Subject Classification.* Primary 13A02 14M25; Secondary 05E.

M

Definition 2.4. For any positive integers n and d define the numbers $A_i^{n,d}$ by

$$\left(1+T+\cdots+T^{d-1}\right)^n = \sum_{i\geq 0} A_i^{n,d} T^i.$$

Theorem 2.5.

$$(1-t)^n \sum_{i=0}^{\infty} \binom{n+id-1}{n-1} t^i = \sum_{j\geq 0} A_{jd}^{n,d} t^j$$

Proof. Let Ξ be the set of (complex) dth roots of 1; we have

$$\frac{1}{d}\sum_{\xi\in\Xi} \frac{1}{(1-\xi t^{1/d})^n} =$$

$$\frac{1}{d}\sum_{\xi\in\Xi}\sum_{j=0}^{\infty} \binom{n+j-1}{n-1} (\xi t^{1/d})^j =$$

$$\sum_{j=0}^{\infty} \binom{n+jd-1}{n-1} t^j,$$

the last equality following from the fact that

$$\sum_{\xi\in\Xi} \xi^j = \begin{cases} d & d|j \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying both sides by $(1-t)^n = (1-(\xi t^{1/d})^d)^n$ we obtain

$$(1-t)^n \sum_{j=0}^{\infty} \binom{n+jd-1}{n-1} t^j =$$

$$\frac{1}{d}\sum_{\xi\in\Xi} \frac{\left(1-(\xi t^{1/d})^d\right)^n}{\left(1-\xi t^{1/d}\right)^n} =$$

$$\frac{1}{d}\sum_{\xi\in\Xi} \left(1+\xi t^{1/d}+\cdots+(\xi t^{1/d})^{d-1}\right)^n.$$

The coefficient of $(t^{1/d})^s$ in this expression is

$$\frac{1}{d}\sum_{\xi\in\Xi} \xi^s A_s^{n,d} = \begin{cases} A_s^{n,d} & d|s \\ 0 & \text{otherwise,} \end{cases}$$

thus only integer powers of t appear in the sum and the coefficient of t^s is $A_{sd}^{n,d}$. $\square$

Combining this with Theorem 2.1 we obtain an explicit expression for the h-vectors of $\mathcal{V}(\mathbf{a};d)$:

Theorem 2.8.

$$(1-t)^n \sum_{i=0}^{\infty} \sum_{S\in\mathcal{S}} (-1)^{|S|} \binom{i(d-\Sigma S)-|S|+n-1}{n-1} t^i =$$

$$\sum_{S\in\mathcal{S}} (-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j \sum_{l\geq 0} A_{l(d-\Sigma S)}^{n-j,d-\Sigma S} t^l.$$

Proof. For any $S \subset \{1,\dots,n\}$ we have

$$(1-t)^n(-1)^{|S|} \sum_{i=0}^{\infty} \binom{i(d-\Sigma S)-|S|+n-1}{n-1} t^i =$$

$$(1-t)^n(-1)^{|S|} P^n_{d-\Sigma S,|S|} =$$

$$(-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j ((1-t)^{n-j} P^{n-j}_{d-\Sigma S,0})$$

and by Theorem 2.5 this equals

$$(-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j \sum_{l\geq 0} A_{l(d-\Sigma S)}^{n-j,d-\Sigma S} t^l.$$

□

Corollary 2.9. *The Hilbert series of* $\mathcal{V}(1,1,\dots,1;d)$ *is*

$$(1-t)^{-n} \sum_{s=0}^{d-1} (-1)^s \binom{n}{s} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (1-t)^j \sum_{l\geq 0} A_{l(d-s)}^{n-j,d-s} t^l.$$

For $d = 2$ *this reduces to*

$$(1-t)^{-n} \left(\sum_{l\geq 0} \binom{n}{2l} t^l - nt \right).$$

Proof. The first statement follows easily from the previous Theorem. To prove the second statement note that $A_j^{n,2} = \binom{n}{j}$ and that $A_j^{n,1} = 0$ unless $j = 0$, in which case we have $A_0^{n,1} = 1$. □

Proof. The a-invariant is the degree of the Hilbert series of $\mathcal{V}(\mathbf{a};d)$ as a rational function. Note that the highest degree of t occurring in a summand of

$$\sum_{S\in\mathcal{S}}(-1)^{|S|}\sum_{j=0}^{|S|}(-1)^j\binom{|S|}{j}(1-t)^j\sum_{l\geq 0}A_{l(d-\Sigma S)}^{n-j,d-\Sigma S}t^l$$

is

$$\max_{S\in\mathcal{S}}\max_{0\leq j\leq|S|} j+\left\lfloor\frac{(n-j)(d-\Sigma S-1)}{d-\Sigma S}\right\rfloor=$$

$$j+n-j-\min_{S\in\mathcal{S}}\min_{0\leq j\leq|S|}\left\lceil\frac{n-j}{d-\Sigma S}\right\rceil=$$

$$n-\min_{S\in\mathcal{S}}\left\lceil\frac{n-|S|}{d-\Sigma S}\right\rceil\leq$$

$$n-\min_{S\in\mathcal{S}}\left\lceil\frac{n-|S|}{d-|S|}\right\rceil=n-\left\lceil\frac{n}{d}\right\rceil$$

the last equality holding for $n\geq d$. □

References

[1] G. E. Andrews. *The theory of partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley Publishing Co. Mass.-London-Amsterdam (1976)

[2] W. Bruns, W. Vasconcelos and R. Villarreal. *Degree bounds in monomial subrings (preprint)*, Illinois J. Math., **41**, No. 3, (1997), pp. 341-353.

[3] E. De Negri and T. Hibi. *Gorenstein algebras of Veronese type*, J. Algebra, **193** (1997) pp. 629–639

[4] B. Sturmfels. *Gröbner bases and convex polytopes, University Lecture Series,* **8**, American Mathematical Society, Providence, RI (1996)

E-mail address: katzman@totalise.co.uk