Chapter IV. Complexity of Polytope Volume Computation

Leonid Khachiyan*

Abstract. We survey some recent results on the complexity of computing the volume of convex n-dimensional polytopes.

1. Jumps of the Derivatives

Let P be a bounded full-dimensional polyhedron in $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$ be a fixed nonzero vector. Consider the "moving halfspace" $H(t) = \{x \in \mathbb{R}^n \mid < a, x > \leq t\}$, sweeping P over the time interval $t \in (-\infty, +\infty)$. Our goal is to describe the behavior of the function $V(t) = \text{vol}_n[P \cap H(t)]$.

Let u be a vertex of P. We say that at the moment $\tau =< a, u >$ the halfspace $H(t)$ crosses u and call τ a *critical moment* for $V(t)$. Denote by $\tau_0 < \tau_1 < \cdots < \tau_N$ the critical set of $V(t)$, i.e., the set of all the instances at which $H(t)$ crosses at least one of the vertices of P. From the theory of mixed volumes it is well known (see, for example [16], Theorem 15.4) that in any time interval $t \in [\tau_k, \tau_{k+1}]$ containing no critical moments as interior points, $V(t)$ is a polynomial function of time $V(t) = p_{kn}t^n + \cdots + p_{k0}, \quad t \in [\tau_k, \tau_{k+1}]$, whose degree does not exceed n. Intuitively, we expect that in general position $V(t)$ is not analytic at any critical moment τ_k. To begin with consider the most simple case where P is an n-dimensional simplex $S_n = \text{conv.hull}\{u_0, u_1, \cdots, u_n\}$ defined by $n+1$ affine independent vertices. Suppose that $H(t)$ and $P = S_n$ are in general position i.e.,

$$P \text{ has no edges parallel to the boundary of } H(t). \tag{1.1}$$

In this case $V(t) = v_n(t) = \text{vol}_n[H(t) \cap S_n]$ has exactly $n+1$ critical moments $\tau_0 < \tau_1 < \cdots < \tau_n$. Let us show that *under the assumption* (1.1) *the first* $n-1$ *derivatives of* $v_n(t)$ *are continuous, and the jumps*

$$J_n(\tau_k) = \frac{d^n v_n(\tau_k + 0)}{dt^n} - \frac{d^n v_n(\tau_k - 0)}{dt^n}$$

* On leave from Computing Center of the USSR Academy of Sciences

of the n-th derivative have alternating signs:

$$(-1)^k J_n(\tau_k) > 0\ ,\quad k = 0, 1, \cdots, n. \tag{1.2}$$

Indeed, assume without loss of generality that $H(t)$ crosses the k-th vertex of S_n at the k-th critical moment, and write for $t > \tau_0$ the n-volume $v_n(t)$ of $H(t) \cap S_n$ as

$$v_n(t) = \frac{1}{n} \sum h(u_0, F_t)\ \mathrm{vol}_{n-1}[F_t]. \tag{1.3}$$

Here the sum is taken over the facets F_t of $H(t) \cap S_n$ and $h(u_0, F_t)$ is the height of the vertex u_0 with respect to aff.hullF_t. Clearly, only two heights in this sum are nonzero. The first one is a positive constant $h_n = n\ \mathrm{vol}_n[S_n]\ /\ \mathrm{vol}_{n-1}[S_{n-1}]$ and corresponds to the facet $H(t) \cap S_{n-1}$, where $S_{n-1} = \mathrm{conv.hull}\ \{v_1, \cdots, v_n\}$. By induction we may assume that the $(n-1)$-volume of this facet $v_{n-1}(t) = \mathrm{vol}_{n-1}[H(t) \cap S_{n-1}]$ has $n-2$ continuous derivatives, and the jump

$$J_{n-1}(\tau_k) = \frac{d^{n-1} v_{n-1}(\tau_k + 0)}{dt^{n-1}} - \frac{d^{n-1} v_{n-1}(\tau_k - 0)}{dt^{n-1}}$$

of the $(n-1)$-th derivative of $v_{n-1}(t)$ is positive at the first critical moment τ_1, negative at the second τ_2, and so on:

$$(-1)^k J_{n-1}(\tau_k) < 0,\quad k = 1, 2, \cdots, n. \tag{1.4}$$

Next, the second nonzero height in (1.3) is equal to $(t - \tau_0)/\|a\|$ and corresponds to the facet $S_n \cap \{x \in \mathbb{R}^n \mid \ < a, x > = t\ \}$, the section of S_n by the boundary of the moving halfspace $H(t)$. Clearly, the $(n-1)$-volume of that facet is equal to $\|a\| dv_n(t)/dt$. Therefore (1.3) can be written as

$$n v_n(t) = h_n v_{n-1}(t) + (t - \tau_0) \frac{dv_n(t)}{dt}.$$

The above recurrence implies that the first $n-1$ derivatives of $v_n(t)$ are continuous, and

$$J_n(\tau_k) = -\frac{h_n}{\tau_k - \tau_0} J_{n-1}(\tau_k),\quad k = 1, 2, \cdots, n.$$

Obviously $J_n(\tau_0) > 0$ and from (1.4) we obtain (1.2).

Thus, *if P is an n-dimensional simplex in general position with $H(t)$, and u is a vertex of P, then the jump $J(u)$ of the n-th derivative of $V(t)$ at the critical moment $\tau = < a, u >$ has the sign $(-1)^{e(u)}$, where $e(u)$ is the number of edges uv of P such that $< a, v > < < a, u >$.*

In fact, *one can also obtain the following explicit expression* [15] *for the jump of the n-th derivative of $V(t)$ at a vertex u :*

$$J(u) = \frac{(-1)^{e(u)}}{|\gamma_{i_1} \cdots \gamma_{i_n}\ \det[a_{i_1}, \cdots, a_{i_n}]|}. \tag{1.5}$$

Here $a_{i_1}, \cdots, a_{i_n} \in \mathbb{R}^n$ are the normal vectors of the active facets at u and $\gamma_{i_1}, \cdots, \gamma_{i_n} \in \mathbb{R}$ are the (unique) coefficients in the representation $a = \gamma_{i_1} a_{i_1} + \cdots + \gamma_{i_n} a_{i_n}$.

Observe that (1.5) is obvious for the critical moment τ_0. For $\tau_1, \cdots, \tau_n$ (1.5) follows from the fact that $|J(u)| =$ const for any of the 2^n "corners" obtained by cutting $\mathbb{R}^n$ by n hyperplanes through u.

Now let P be an arbitrary simple polyhedron, and let $H(t)$ be in general position with respect to P, see (1.1). Suppose that at a given instant τ the halfspace $H(t)$ crosses r vertices $u^1, \cdots, u^r$ of P. Since exactly n facets intersect at each vertex of P, we can cut out from the polyhedron r small simplices $S^1, \cdots, S^r$ with vertices at the points $u^1, \cdots, u^r$ and represent P as the disjoint union of the above r simplices and some polyhedron $P' = P \setminus (S^1 \cup \cdots \cup S^r)$ containing no vertices in the boundary of $H(\tau)$. This implies that *the jump of the n-th derivative of the function $V(t) = \mathrm{vol}_n[H(t) \cap P]$ is equal to the sum of the corresponding jumps over the critical vertices:*

$$\frac{d^n V(\tau + 0)}{dt^n} - \frac{d^n V(\tau - 0)}{dt^n} = \sum_{< a, u > = \tau} \frac{(-1)^{e(u)}}{|\gamma_{i_1} \cdots \gamma_{i_n}\ \det[a_{i_1}, \cdots, a_{i_n}]|}. \tag{1.6}$$

Example 1. Let $P = C_3$ be the unit 3-dimensional cube and $a = (1, 1, -2)$. At the moment $\tau = 0$ the halfspace $H(t) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - 2x_3 \le t\}$ crosses two vertices $u = (0, 0, 0)$ and $v = (1, 1, 1)$ of the cube. However, $J(u) = -J(v) = -1/2$ and $V(t)$ is analytic at the critical moment $\tau = 0$.

Example 2. Let C_n be the unit n-cube and suppose that all the coordinates of the vector $a \in \mathbb{R}^n$ are positive. Then for any vertex $u \in \{0, 1\}^n$ of C_n one has $e(u) = |u| =$ the number of 1's in u.

Since in general position the first $n-1$ derivatives of $V(t)$ are continuous, integrating (1.6) *we get the following formula* [15] *for the volume of the intersection of a simple polyhedron P with a halfspace $H(t)$*

$$V(t) = \frac{1}{n!} \sum (-1)^{e(u)} \frac{(\max(0, t - < a, u >))^n}{|\gamma_{i_1} \cdots \gamma_{i_n}\ \det[a_{i_1}, \cdots, a_{i_n}]|}, \tag{1.7}$$

where the sum is taken over all the vertices u of P.

In particular, *the polynomial*

$$\frac{1}{n!} \sum (-1)^{e(u)} \frac{(t - < a, u >)^n}{|\gamma_{i_1} \cdots \gamma_{i_n}\ \det[a_{i_1}, \cdots, a_{i_n}]|}$$

does not depend on t and is equal to the volume of P.

2. Exact Volume Computation is Hard

Consider the following well-known knapsack problem: given $(a,\tau) \in Z_+^{n+1}$ determine the solvability of the equation $< a,u >= \tau$ in Boolean variables $u \in \{0,1\}^n$. This problem can be reformulated as follows: does the moving halfspace $H(t) = \{x \in \mathbb{R}^n \mid < a,x > \le t\}$ cross a vertex of the unit n-cube C_n at the moment $t = \tau$?

Since the knapsack problem remains NP-complete under the additional assumption

$$u \in \{0,1\}^n \;\&\; < a,u >= \tau \;\Rightarrow\; |u| = \text{const}, \tag{2.1}$$

from (1.6) and *Example 2* it is easy to see that ***computing the volume of the intersection of the unit n-dimensional cube with a rational halfspace is NP-hard*** [12]. Indeed, if it were possible to compute the function $V(t) = \text{vol}_n[C_n \cap \{x \mid < a,x > \le t\}]$ for rational t in polynomial time, then by means of interpolation one could verify the condition $d^nV(\tau+0)/dt^n - d^nV(\tau-0)/dt^n \neq 0$ in polynomial time as well. The latter condition, however, is equivalent to the solvability of the knapsack problem with the property (2.1).

In fact, it can be shown [3] that *the problem of computing the volume of the intersection of the unit n-cube with a rational halfspace is #P-hard*, also see [13].

It is essential for the validity of the last statement that the coefficients $(a,\tau) \in Z_+^{n+1}$ of the halfspace $H(\tau)$ be "large", since it is known [14] that the volume of the intersection of the unit cube with an arbitrary fixed number of rational halfspaces can be computed in pseudopolynomial time. However, it was observed in [12] that if $Q = \{1,\cdots,n;\prec\}$ is a partially ordered set and

$$P(Q) = \{x \in \mathbb{R}^n \mid 0 \le x_i \le 1,\;\; i = 1,\cdots,n;\;\; x_i \le x_j \;\text{ if } i \prec j \text{ in } Q\}$$

is the order polyhedron of Q, then it is NP-hard to determine the volume of the intersection of $P(Q)$ with a rational halfspace defined by "small" (polynomial in n) coefficients $(a,\tau) \in Z_+^{n+1}$. Recently a much stronger result has been obtained in [2] as a direct corollary of the following important theorem : *the problem of computing the number of linear extensions of a given poset is #P -complete* (this theorem was conjectured in [17]). Since it is well-known, see, for example [22], that the number of linear extensions of a poset Q is equal to $n!\,\text{vol}_n[P(Q)]$, the latter result implies that determining the volume of order polyhedra $P(Q)$ is #P-hard. Thus,

computing the volume of rational polyhedra is strongly #P − hard.

The following question was posed in [3]: can the volume of a rational polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be always written as a reduced fraction whose denominator (hence, numerator) has the binary length bounded by a polynomial in the binary length of A and b? In other words, can the volume of a rational polyhedron be written in polynomial space? The answer to this question is negative [15]. This is shown by the example $P = T_a(C_n)$, where C_n is the unit n-cube, $a = (2^{-1}, 2^{-2}, \cdots, 2^{-n})$, and T_a is the projective mapping $x \to x/(1+ < a,x >)$. Clearly, for a positive a the image $T_a(C_n)$ of the unit cube is defined by the $2n$ inequalities

$$x_i \ge 0,\;\; x_i + < a,x > \le 1,\;\;\; i = 1,\cdots,n,$$

and it is easy to see that the jump (1.5) of the n-th derivative of the function

$$V(t) = \text{vol}_n\,[T_a(C_n) \cap \{x \mid < a,x > \le t\}]$$

at a vertex $T_a(u)$, $u \in \{0,1\}^n$, is given by

$$J(T_a(u)) = (-1)^{|u|}\,\frac{(1+ < a,u >)^{n-1}}{a_1 \cdots a_n}.$$

Now it follows from (1.7) that

$$V(t) = \frac{1}{n!\,a_1\cdots a_n}\sum(-1)^{|u|}(1+ < a,u >)^{n-1}(\max(0, t - \frac{< a,u >}{1+ < a,u >}))^n,$$

where the sum is taken over the vertices $u \in \{0,1\}^n$ of C_n. In particular, the volume of $T_a(C_n)$ can be written as

$$V(1) = \frac{1}{n!\,a_1\cdots a_n}\sum\frac{(-1)^{|u|}}{1+ < a,u >}.$$

Substituting $a = (2^{-1}, 2^{-2}, \cdots, 2^{-n})$ we get

$$V(1) = \frac{2^{(n^2+3)/2}}{n!}\sum_{N=2^n}^{2^{n+1}-1}\frac{(-1)^{e(N)-1}}{N},$$

where $e(N)$ is the number of 1's in the binary expansion of N. Obviously the binary length of the denominator of the latter expression is not polynomial in n.

We now turn to the complexity of computing the volume of a polytope $P \subset \mathbb{R}^n$, given as the convex hull of a set of integer points $P = \text{conv.hull}\,\{u_1,\cdots,u_m\}$, $u_1,\cdots,u_m \in Z^n$. Can the volume of P be computed in polynomial time? [18, 8]. The following negative result [3], [13] considers the problem which is "polar" to determining the volume of the intersection of a cube and a halfspace:

Let $e_1,\cdots,e_n$ be the standard basis vectors in $\mathbb{R}^n$, and let $a \in Z^n$ be a given integer vector. Computing the volume of $O(a) = \text{conv.hull}\,\{+e_1, -e_1, \cdots, +e_n, -e_n, a\}$ is #P-hard.

Indeed, $O(a) = \text{conv.hull}\,\{O, a\}$, where $O = \{x \in \mathbb{R}^n \mid |x_1| + \cdots + |x_n| \le 1\}$ is the unit n-octahedron. Hence the n-volume of $O(a)$ can be written as

$$\text{vol}\,O(a) = \text{vol}\,O + \sum_S \text{vol conv.hull}\,\{S, a\} = \frac{2^n}{n!} + \frac{1}{n!}\sum_\delta \max(0, < a,\delta > -1),$$

where the first sum is taken over the facets S of O that are visible from a, and the second sum is taken over the vectors $\delta \in \{-1,1\}^n$. Therefore

$$n!\{\text{vol}O(a+e_1)+\text{vol }O(a-e_1)-2\text{ vol }O(a)\} = \#\{\delta \in \{-1,1\}^n| < a,\delta >= 1\}.$$

But the problem of determining the number of solutions $\delta \in \{-1,1\}^n$ to the equation $< a,\delta >= 1$ is well-known to be $\#P$-complete.

It would be interesting to show that the problem of computing $\text{vol}_n \text{conv.hull}\{u_1,\cdots,u_m\}$ is strongly $\#P$-hard. Another interesting problem is to prove the $\#P$-completeness for linear extension count for posets of height 2 [2]. One more question is: is it hard to compute the volume of a zonotope? [7]

Clearly, the problem of determining the volume of a rational polytope in facial or vertex descriptions remains $\#P$-hard if it is required to approximate the volume to a given absolute accuracy , say $|u - \text{vol}_n\ P| \le 1$. In the next section we address the problem of approximating the volume to a given relative accuracy:

$$\left|\frac{u}{\text{vol}_n\ P} - 1\right| \le \varepsilon.$$

3. Volume Approximation

A convex body $P \subset \mathbb{R}^n$ is said to be ***ρ-rounded*** if $B \subseteq P \subseteq \rho B$, where $B = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ is the unit Euclidean ball. If $T: x \to Ax + b$ is an affine transformation such that the above inclusions hold for the image $T(P)$, we say that the transformation ρ-rounds P.

It is well-known [9] that for an arbitrary convex body $P \subset \mathbb{R}^n$ there exists an n-rounding affine transformation T and, in general, $\rho = n$ is the best possible value. Moreover, if $P \subset \mathbb{R}^n$ is a rational polytope in facial or vertex description, then for an arbitrary $\rho > n$ one can find a rational ρ-rounding affine transformation T in time polynomial in $\log(1/(\rho - n))$ and in the binary length of input data. In fact, one can "polynomially round" P in a more general case where P is given by a separation oracle (such an oracle can answer the question "$x \in P$?" for an arbitrary rational vector x and, if $x \notin P$ it gives a hyperplane that separates x from P). In this case it is still possible to find an $n^{3/2}$-rounding transformation $T = Ax + b$ in $O(n^4 \log \rho)$ arithmetic operations and in $O(n^2 \log \rho)$ calls to the oracle, provided that originally P is ρ-rounded, see [8]. Since vol P = vol $T(P)/|\det A|$, we assume henceforth that P itself is $n^{3/2}$-rounded:

$$B \subseteq P \subseteq n^{3/2}B.$$

In particular, one can easily approximate the n-volume of convex bodies to relative accuracy $\varepsilon = n^{3n/4} - 1$. If P is given by a separation oracle, this exponential bound is close to the best possible in the class of deterministic methods with polynomial informational complexity [5, 1]. Specifically, *for any deterministic method which estimates the volume of convex bodies $P \subset \mathbb{R}^n$ in polynomial in n number of calls to a separation oracle,*

$$\varepsilon \ge \left(\frac{n}{\log\ n}\right)^{n/2} - 1$$

for some P, see [1].

However, Dyer, Frieze and Kannan [4] have recently shown that in the class of ***randomized*** algorithms the volume of convex bodies can be approximated in polynomial time to an ***arbitrary*** fixed relative accuracy, say $\varepsilon = 0.01$. More precisely, their remarkable theorem states that *there exists a randomized algorithm which for any given $\varepsilon > 0$ and $\beta > 0$ finds an approximation u to the volume of a convex body $P \subset \mathbb{R}^n$ to relative accuracy ε with probability of error less than β*

$$\text{Prob}\left\{\left|\frac{u}{\text{vol}_n\ P} - 1\right| \le \varepsilon\right\} \ge 1 - \beta$$

in polynomial in n, $1/\varepsilon$, and $\log(1/\beta)$ *number of arithmetic operations and membership tests "$x \in P$?"*

We informally outline the main ideas of the algorithm. First the problem of approximating $\text{vol}_n\ P$ to the relative accuracy ε can be reduced to $k = O(n \log\ n)$ subproblems of estimating the ratios $\xi_i = \text{vol}_n P_i/\text{vol}_n\ P_{i-1}$, $i = 1,\cdots,k$, where

$$B = P_0 \subseteq \cdots \subseteq P_{i-1} \subseteq P_i \subseteq \cdots \subseteq P_k = P \subseteq n^{3/2}B$$

is the "tower of convex bodies" $P_i = (1 + 1/n)^i B \cap P$. Clearly, $\text{vol}_n P = \xi_1 \cdots \xi_k \text{vol}_n\ B$, where B is the unit Euclidean ball, and therefore it suffices to estimate each of the ξ_i's to the relative accuracy $\varepsilon_1 = O(\varepsilon/n \log\ n)$ with the probability of error $\beta_1 = O(\beta/n \log\ n)$. Since each ratio ξ_i in the tower is bounded from above

$$1 \le \xi_i = \text{vol}_n\ P_i/\text{vol}_n\ P_{i-1} \le (1 + 1/n)^n < e,$$

if it were possible to sample x uniformly from within P_i, then in $O(\varepsilon_1^{-2}\log(1/\beta_1))$ independent trials "$x \in P_{i-1}$?", we could estimate ξ_i as required. To generate $x \in P_i$ nearly uniformly, Dyer, Frieze and Kannan consider a discrete approximation CP_i to P_i made up of cubic "pixels"

$$C(m) = \{x \in \mathbb{R}^n \mid m_i\delta \le x_i \le (m_i + 1)\delta,\ i = 1,\cdots,n\}, \quad m \in Z^n$$

of a sufficiently small size δ. For the time being the reader may assume that CP_i consists of the cubes $C(m)$ that intersect *int* P_i. Since diam $C(m) = \delta n^{1/2}$ and P_i contains the unit ball, we have $CP_i \setminus P_i \subseteq (1+\delta n^{1/2})P_i \setminus (1-\delta n^{1/2})P_i$. This implies

$$|\ \xi_i - M_i/M_{i-1}\ | = O((1+\delta n^{1/2})^n - (1-\delta n^{1/2})^n) = O(\delta n^{3/2}),$$

where M_i is the number of cubes $C(m)$ in CP_i. Hence, with $\delta = O(\varepsilon_1 n^{-3/2})$ the problem of estimating ξ_i can be replaced by the problem of approximating the ratio M_i/M_{i-1} to the relative accuracy $O(\varepsilon_1)$ with the probability of error β_1. To pick randomly a cube $C(m)$ in CP_i, Dyer, Frieze and Kannan consider a simple random walk over the cubes $C(m)$ in CP_i, which converges to the uniform distribution. At the t-th step of the random walk, $t = 0, 1, \ldots$, we chose with probability $1/(2n)$ a facet of the present cube $C(m_t)$ and move to the cube $C(m')$ across the chosen facet $m_{t+1} = m'$ if we do not leave CP_i; otherwise we stay in the present cube: $m_{t+1} = m_t$. The convexity of P_i implies that the Markov chain of this random walk is connected i.e., for any two cubes $C(m)$ and $C(m')$ in CP_i the t step transition probability $p^{(t)}_{mm'}$ is positive for some t. In fact, $p^{(t)}_{mm'} > 0$ for all sufficiently large t, because some of the cubes $C(m)$ have self-loops with positive 1-step transition probabilities p_{mm}. In other words, the chain is ergodic and consequently, it has a unique stationary final distribution. Since the chain is symmetric $p_{mm'} = p_{m'm}$, the latter is easily seen to be uniform: $p^{(t)}_{mm'} \to p(\infty) = 1/M_i$, $t \to \infty$. Thus, for a sufficiently large t we can use the t-th cube in the random walk to sample nearly uniformly from CP_i. To bound t from above, Dyer, Frieze and Kannan use the following consequence of a recent result of Sinclair and Jerrum [21] on rapidly mixing Markov chain:

$$| p^{(t)}_{mm'} - p(\infty) | \le (1 - 0.5\Phi^2)^t, \tag{3.1}$$

where Φ is the *conductance* of the chain. The latter quantity is defined as $\Phi = \min \Phi(A,B)$, where the minimum is taken over all the partitions (A,B) of the states of the chain, and (for symmetric chains)

$$\Phi(A,B) = \frac{\sum p_{mm'} : m \in A, m' \in B}{\min\{|A|, |B|\}}.$$

Intuitively, Φ measures the minimum relative connection strength between subsets of the states. In our case $\Phi(A,B)$ admits a simple geometric interpretation:

$$\Phi(A,B) = \frac{\delta}{2n} \frac{\mathrm{vol}_{n-1}[\partial C(A) \cap \partial C(B)]}{\min\{\mathrm{vol}_n C(A), \mathrm{vol}_n C(B)\}}, \tag{3.2}$$

where $C(A) = \cup_{m \in A} C(m)$ and $C(B) = \cup_{m' \in B} C(m')$.

We now need the following isoperimetric inequality [19], [11]:

$$s(u,v) \text{ diam } Q > \min\{u,v\}, \tag{3.3}$$

where $s(u,v)$ is the $(n-1)$-volume of the minimal surface partitioning a convex body $Q \subset \mathbb{R}^n$ into two n-volumes u and v.

Consider the definition (3.2). Since $C(A) \cup C(B) = CP_i$ is "close" to the convex body P_i, and the latter is contained in the ball of diameter $D = 2n^{3/2}$, it would be fine if we could apply the isoperimetric inequality (3.3) to bound the conductance $\Phi(A,B)$ from below

$$\Phi(A,B) > \frac{\delta}{2n}\frac{1}{D} \ge \text{const } \delta n^{-5/2} \tag{3.4}$$

for *any* partition (A,B). In view of the inequality of Sinclair and Jerrum this would immediately prove the polynomiality of the algorithm. Unfortunately, we can easily obtain (3.4) from (3.3) only for the case where $\min\{\mathrm{vol}_n\ C(A), \mathrm{vol}_n\ C(B)\}$ sufficiently exceeds $\mathrm{vol}_n[CP_i \setminus P_i]$, or

$$\frac{\min\{|A|,|B|\}}{|A|+|B|} = \frac{\min\{|A|,|B|\}}{M_i} \ge \text{const } \delta n^{3/2}. \tag{3.5}$$

A possible way to bypass this difficulty is to replace P_i by its Euclidean α-neighborhood $P_i(\alpha)$ with $\alpha = O(\delta n^{3/2})$. Such a replacement is within the tolerance of estimating ξ_i and smoothes the boundary of P_i. At the same time it allows one to show that the conductance of the random walk on $CP_i(\alpha)$ is polynomially bounded away from zero, see [4]. Another approach was suggested by Lovasz and Simonovits [19]. They proved a generalization of the inequality of Sinclair and Jerrum, which allows to ignore very asymmetric partitions (A,B) in the definition of conductance and instead of pointwise convergence guarantees convergence for "big" subsets of states, provided that the initial distribution is sufficiently spread out. In particular, as a corollary for Markov chains with the uniform final distribution, Lovasz and Simonovits obtained the following result: *suppose that for every set of states C with at most M elements we have* $|\mathrm{Prob}\{m_t \in C\} - p(\infty)|C|\,| \le \nu$ *for* $t = 0$. *Then for any set of states D*

$$|\mathrm{Prob}\{m_t \in D\} - p(\infty)|D|\,| \le \nu + (1 - 0.5\Phi_M^2)^t [p(\infty)]^{-1/2} \quad \textit{for all } t. \tag{3.6}$$

Here Φ_M is the M-conductance of the chain, defined as the largest number such that

$$\left\{\sum \frac{p_{mm'} + p_{m'm}}{2} : m \in A, m' \in B\right\} \ge \Phi_M\ (\min\{|A|,|B|\} - M)$$

for any partition (A,B).

Since the latter definition actually ignores the partitions with $\min\{|A|,|B|\} \le M$, we know that $\Phi_M \ge \text{const } \delta n^{-5/2}$ for $M = O(\delta n^{3/2} M_i)$, see (3.4) and (3.5). Suppose that we use as a starting distribution for the random walk over CP_i, a distribution over CP_{i-1} such that

$$| \mathrm{Prob}\{m_0 \in D\} - |D|/M_{i-1} | \le \nu_{i-1} \quad \textit{for any set } D \textit{ in } CP_{i-1}. \tag{3.7}$$

Then for every set C in CP_i with at most $M = O(\delta n^{3/2} M_i)$ elements we have

$$| \mathrm{Prob}\{m_0 \in C\} - |C|/M_i | \le \nu_{i-1} + M\left(\frac{1}{M_{i-1}} - \frac{1}{M_i}\right) = \nu_{i-1} + O(\delta n^{3/2}).$$

We know that $\Phi_M \ge \text{const } \delta n^{-5/2}$ and therefore the second term in (3.6) is exponentially small for t, polynomial in n and $1/\delta$. Hence we conclude that after a polynomial number of random moves on CP_i we can start the random walk on CP_{i+1} with $\nu_i = \nu_{i-1} + O(\delta n^{3/2})$, see (3.7). Since we can easily obtain a good starting distribution on the discrete approximation CP_0 to the unit ball $P_0 = B$, and $i \le k = O(n \log n)$, the latter recurrence implies $\nu_i = O(\delta n^{5/2} \log n)$ for all i. Recalling that the ratios M_i/M_{i-1} must be estimated to the relative accuracy ε_1 and therefore it suffices to have $\nu_i = O(\varepsilon_1) = O(\varepsilon/n \log n)$, we finally find $\delta \simeq \varepsilon/n^{7/2} \log^2 n$, and obtain the following crude lower bound on the conductance: $\Phi_M \ge$ const $\varepsilon/n^6 \log^2 n$. We do not go into further analysis, since the polynomiality of the problem is already clear. We also skip another detail due to the fact that the question "$C(m) \cap \text{int } P_i \ne \emptyset$?" (or equivalently "is a given cube $C(m)$ in CP_i?") can be answered in polynomial time for explicitly given rational polytopes P, but not for convex bodies given by a membership oracle. In general, CP_i consists of the cubes $C(m)$ that *weakly* intersect P_i, see [4],[19].

Unfortunately, the complexity bound of the algorithm turns out to be very high: the algorithm requires $O(\varepsilon^{-4} n^{16} \log^6 n \log(n/\varepsilon) log(n/\beta))$ calls to a membership oracle for P, see [19]. Furthermore, it is clear that since the algorithm generates a random point in P by means of a diffusion process whose step distribution is centrally symmetric and *local*, it takes at least (diam $P/\delta)^2$ steps to achieve reasonable mixing on P. Thus, for radical improvements in the complexity we need more rapidly mixing "long-range" random walks on convex bodies. Sometimes we can also use a "method of finite elements" to simplify the problem of uniform generation for P. For example, if P is the order polyhedron of a poset $Q = \{1, \cdots, n; \preceq\}$, and Q has M linear extensions $m: m(1) \preceq m(2) \preceq \cdots \preceq m(n)$, we can decompose P into M simplices $S(m) = \{x \in \mathbb{R}^n | 0 \le x_{m(1)} \le x_{m(2)} \le \cdots \le x_{m(n)} \le 1\}$ and use the triangulation $P = \cup S(m)$ instead of cubic approximations to P. This gives us a simple random walk over the set of linear extensions of Q, which converges to the uniform distribution:| Prob$\{m_t = m\} - 1/M \mid \le (1 - 0.5\Phi^2)^t$. Since the triangulation $P = \cup S(m)$ is exact, $\text{vol}_{n-1}[\partial S(m)]/\text{vol}_n[S(m)] \simeq n^2$ and diam $P = n^{1/2}$, the isoperimetric inequality immediately gives us the bound $\Phi \ge$ const $n^{-2.5}$ on the conductance of this random walk. In fact, we believe that the latter bound can be improved, and $\Phi \ge$ const n^{-2}. The above almost uniform generator of linear extensions can be used for determining well-balanced comparisons [6, 10] in posets with encouraging computational results [11].

Note Added in Proof. Since the time of writing of this survey paper, there has been a substantial progress in reducing complexity bounds for polytope volume computation, obtained by Applegate and Kannan (1990), Dyer and Frieze (1991), and Lovasz and Simonovits (1991). The best currently known bound $O(\varepsilon^{-2} n^7 \log^2 n \log^3(\varepsilon^{-1}) \log(\beta^{-1}))$ is due to Lovasz and Simonovits [20].

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