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Volumes of Polyhedra Inscribed in the Unit Sphere in E^3

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1. This paper is concerned with the problem of placing n points on the unit sphere in E^3 so as to maximize the volume of their convex hull. A necessary condition is obtained, and this is used to obtain complete solutions when $n = 4, \ldots, 8$. The results for $n = 7$ and 8 are new. The technique developed also shows that for each of the several polyhedral types considered there is only one relative maximum for the volume function.

Let $p_1, \ldots, p_n$ be points on the unit sphere, S. Denote by $H(p_1, \ldots, p_n)$ the polyhedron P which is the convex hull of the set $\{p_1, \ldots, p_n\}$. Let $C(P)$ be an oriented complex whose vertices are the vertices of P and whose edges are edges of P and diagonals of faces of P, the diagonals chosen such that the faces of $C(P)$ are all triangles. Thus, if every face of P is triangular, there is only one such complex up to orientation, while there is more than one such complex if P has a face which is not triangular.

Let K be a finite oriented complex all of whose faces are triangular and all of whose vertices are on S. If p, q, r are vertices of K which determine a face of K and are in an order determined by the orientation of K then the tetrahedron whose vertices are p, q, r and the origin is a *facial tetrahedron* and its volume is one-sixth of the value of the determinant $|p, q, r|$. Then the volume of K, $vol(K)$, is the sum of the volumes of the facial tetrahedra of K. If $P = H(p_1, \ldots, p_n)$ let $V(p_1, \ldots, p_n) = V(P)$ denote the volume of P. Note that for any oriented complex $C(P)$, $|\mathrm{vol}(C(P))| = V(P)$.

If P is a polyhedron, with n vertices, let the *valence* of a vertex of $C(P)$ be the number of edges of $C(P)$ incident with that vertex. By Euler's formula the average of the valences is $6 - 12/n$. If n is such that $6 - 12/n$ is an integer then $C(P)$ is *medial* if the valence of each vertex is $6 - 12/n$. If $6 - 12/n$ is not an integer then $C(P)$ is *medial* provided the valence of each vertex is either m or $m+1$ where $m < 6 - 12/n < m+1$. P is said to be *medial* provided all faces of P are triangular and $C(P)$ is medial. When considering the isoperimetric problem for polyhedra Goldberg [2] made a conjecture whose dual was formulated by Grace [3]: The polyhedron with n vertices on S whose volume is a maximum is a medial polyhedron provided a medial polyhedron exists for that n.

2. The Necessary Condition. Let $P = H(p_1, \ldots, p_n)$. If for each p_i there is an open set $U_i \subset S$, $p_i \in U_i$, such that if

[illegible] $p_n) \leq V(p_1, \ldots, p_n)$

for all $q \in U_i$ then P is said to have *property Z*. By any usual definition of relative maximum, if V has a relative maximum at $(p_1, \ldots, p_n)$ then P has property Z. If p_i and p_j are vertices of P, denote the line segment whose endpoints are p_i and p_j by s_{ij} and its length by $|s_{ij}|$. Also, let $n_{ij} = 1/6\, p_i \times p_j$ where $\times$ denotes the vector product in E^3.

Lemma 1. *Let P with vertices $p_1, \ldots, p_n$ have property Z. Let $C(P)$ be any oriented complex associated with P such that* $\mathrm{vol}(C(P)) \geq 0$. *Suppose $s_{12}, \ldots, s_{1r}$ are all the edges of $C(P)$ incident with p_1 and that $p_2, p_3, p_1; p_3, p_4, p_1; \ldots; p_r, p_2, p_1$ are orders for faces consistent with the orientation of $C(P)$.*

i) *Then $p_1 = m/|m|$ where $m = n_{23} + n_{34} + \cdots + n_{r2}$.*

ii) *Furthermore, each face of P is triangular.*

Proof. i) From the definition of V,

$$V(p_1, \ldots, p_n) = 1/6\,[\,|p_2, p_3, p_1| + \cdots + |p_r, p_2, p_1|\,] + \gamma = p_1 \cdot m + \gamma$$

where γ is the sum of the volumes of the facial tetrahedra of $C(P)$ for which p_1 is not a vertex. Let $q = m/|m|$. Suppose $p_1 \neq q$. Since P has property Z there is an open set U_1 containing p_1 as defined above. Let s be any point in U_1 such that $s \cdot q > p_1 \cdot q$. Let Q be the oriented complex with vertices $s, p_2, \ldots, p_n$ which is isomorphic to $C(P)$ where the isomorphism is such that $p_i \in Q$ corresponds to $p_i \in C(P)$, $i = 2, \ldots, n$, and the isomorphism preserves orientation. Then

$$V(p_1, \ldots, p_n) = p_1 \cdot m + \gamma < s \cdot m + \gamma = \mathrm{vol}(Q) \leqq V(s, p_2, \ldots, p_n).$$

This implies $p_1 = q$.

ii) If P has a quadrilateral face then conclusion i) applied to two different complexes associated with P gives a contradiction and so ii) holds in this special case, and the general case follows from this.

Note 1. If $p_2, \ldots, p_r$ lie in a plane π then p_1 is one of the two antipodal points where the plane tangent to the sphere is parallel to π. Then

$$|s_{12}| = |s_{13}| = \cdots = |s_{1r}|.$$

Note 2. If $r = 5$ and β is the sum of the volumes of the facial tetrahedra for which p_1 is a vertex then $\beta = 1/6\,|p_2 - p_4, p_3 - p_5, p_1|$. From this it follows that $p_1 \perp p_2 - p_4$ and $p_1 \perp p_3 - p_5$. Then p_1 lies on the great circles determined by the planes which are the perpendicular bisectors of s_{24} and s_{35}.

By a *double n-pyramid*, $n \geqq 5$, is meant a complex of n vertices with two vertices of valence $n-2$ each of which is connected by an edge to each of the remaining $n-2$ vertices, all of which have valence 4. Note that the $2(n-2)$ faces of a double n-pyramid are all triangular. A polyhedron P is a *double n-pyramid* provided each of its faces is triangular and some $C(P)$ is a double n-pyramid.

Lemma 2. *Let $P = H(p_1, \ldots, p_n)$. If P is a double n-pyramid with property Z then P is unique up to congruence and its volume is* $[(n-2)/3]\sin 2\pi/(n-2)$.

Proof. Let p_{n-1} and p_n be the valence $n-2$ vertices. If $n=6$ let p_{n-1} and p_n be any two vertices not incident with the same edge. Suppose p_i is a vertex of valence 4 connected by edges to p_n, p_{n-1}, p_j, and p_k. From Note 2 it follows that p_i is equidistant from p_n and p_{n-1} and that p_i is equidistant from p_j and p_k. Consequently, the valence 4 vertices form a regular $(n-2)$-gon lying in the plane which is the perpendicular bisector of $s_{n-1\,n}$. It now follows from Note 1 that p_n and p_{n-1} are antipodal. An elementary calculation gives the volume.

3. $n=4, 5, 6$. In this section polyhedra of maximal volume are considered when $n=4, 5, 6$.

For $n=4$ if $P=H(p_1, p_2, p_3, p_4)$ has property Z then, by Note 1, the three edges incident with a vertex all have the same length. Consequently, all six edges have the same length and P is a regular tetrahedron. An elementary calculation gives the volume as $8\sqrt{3}/27$.

For $n=5$, let $P=H(p_1, \ldots, p_5)$ have property Z. Then P is a double 5-pyramid, since the double 5-pyramid is the only complex with five vertices. Then, by Lemma 2, P is unique up to congruence and the volume is $\sqrt{3}/2$.

For $n=6$ it is easily seen that if P is medial then P is a double 6-pyramid. If $P=H(p_1, \ldots, p_6)$ has property Z and is medial, then, by Lemma 2, P is unique up to congruence, the volume of P is 4/3, and P is a regular octahedron. It is well known that if P is not a regular octahedron then its volume is less than 4/3 ([4], p. 264).

Some of these results are summarized in the following theorem.

Theorem 1. *For $n=4, 5, 6$, if $P=H(p_1, \ldots, p_n)$ is a medial polyhedron such that $V(p_1, \ldots, p_n)$ is a relative maximum then P is uniquely determined up to congruence and its volume may be found. In each case such a medial polyhedron exists and its volume is the absolute maximum for V.*

4. $n=7, 8$. For $n=7$ Bowen and Fisk [1] have shown that up to isomorphism there is only one polyhedron with triangular faces having no vertices of valence 3. From Euler's formula it follows that the average valence is 4 2/7. Hence the medial complex is the double 7-pyramid with two valence 5 vertices and five valence 4 vertices. If $P=H(p_1, \ldots, p_7)$ is a double 7-pyramid with property Z then, by Lemma 2, P is unique up to congruence and the volume of P is $5/3\sin 2\pi/5$, which is approximately 1.58510.

If $P=H(p_1, \ldots, p_7)$ is a double 7-pyramid such that $V(p_1, \ldots, p_7)$ is a relative maximum, then P is unique up to congruence.

Theorem 2. *If $P=H(p_1, \ldots, p_7)$ is such that $V(p_1, \ldots, p_7)$ is an absolute maximum then P is the double 7-pyramid with property Z described in Lemma 2.*

Proof. Let $Q=H(p_1, \ldots, p_7)$. Suppose Q gives the absolute maximum for V. Then Q is convex, has property Z, contains the origin, and has triangular faces. Since the double 7-pyramid is the only complex with 7 vertices none of which has valence 3, assume Q has a vertex of valence 3, say p_1. Then it remains to show that the volume of Q is less than $5/3\sin 2\pi/5$. Let s_{12}, s_{13}, s_{14} be the edges incident with p_1, and let T be the triangle with vertices p_2, p_3, p_4. Since Q has

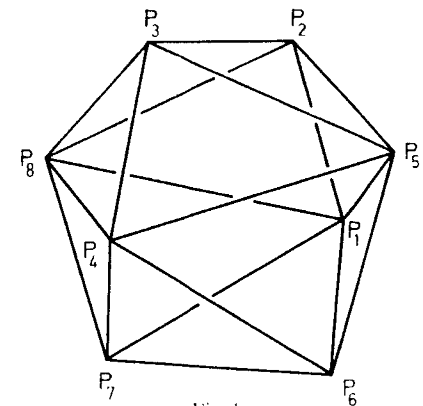

Fig. 1

property Z, the line determined by p_1 and the origin passes through the circumcenter of T. Since Q is convex, this circumcenter lies within T. Then the area of T is less than or equal to $(3\sqrt{3}/4)\,(1-1/3\tan^2(2\pi-\alpha)/6)$ ([4], p. 267), where α is the area of the central projection of T onto the sphere. Let $\theta=4\pi-\alpha$. The central projection of the seven facial tetrahedra not incident with p_1 has total area θ. The total volume of these seven tetrahedra is less than or equal to

$$7/4\tan\frac{(2\pi-\theta/7)}{6}\left[1-1/3\tan^2\frac{(2\pi-\theta/7)}{6}\right]$$
$$=7/4\tan\frac{(10\pi+\alpha)}{42}\left[1-1/3\tan^2\frac{(10\pi+\alpha)}{42}\right]$$

([4], p. 275). Then $V(Q)\leqq F(\alpha)$ where

$$F(\alpha)=\sqrt{3}/4\left[1-1/3\tan^2\frac{(2\pi-\alpha)}{6}\right]+7/4\tan\frac{(10\pi+\alpha)}{42}\left[1-1/3\tan^2\frac{(10\pi+\alpha)}{42}\right].$$

A direct calculation shows that F is concave and that its maximum is less than $5/3\sin 2\pi/5$. Hence, Q does not give the absolute maximum for V.

For $n=8$, by Euler's formula, the average valence is 4 1/2. Bowen and Fisk have shown that there exist only two non-isomorphic complexes which have no vertices of valence 3. One of these is the double 8-pyramid, which by Lemma 2 has maximal volume $\sqrt{3}$. The other has four valence 4 vertices and four valence 5 vertices, as shown in Fig. 1, and therefore it is the medial complex.

Lemma 4. *If $P=H(p_1, \ldots, p_8)$ has property Z and is medial then P is uniquely determined up to congruence and its volume is* $\left|\frac{475+29\sqrt{145}}{250}\right|^{1/2}$.

Proof. Let the vertices of P be $p_1, \ldots, p_8$ so that the labelling is consistent with that in the figure. It is assumed that the vertices are distinct. From Lemma 1 it follows that

$$p_1 = c_1(n_{25} + n_{56} + n_{67} + n_{78} + n_{82}),$$
$$p_4 = c_4(n_{38} + n_{87} + n_{76} + n_{65} + n_{53}),$$
$$p_5 = c_5(n_{61} + n_{12} + n_{23} + n_{34} + n_{46}),$$

where $c_i > 0$, $i = 1, 4, 5$. The vector product of p_1 with the above expression for p_1 gives

$$(p_1 \cdot p_5 - p_1 \cdot p_8)\, p_2 + (p_1 \cdot p_6 - p_1 \cdot p_2)\, p_5 + (p_1 \cdot p_7 - p_1 \cdot p_5)\, p_6 + (p_1 \cdot p_8 - p_1 \cdot p_6)\, p_7 + (p_1 \cdot p_2 - p_1 \cdot p_7)\, p_8 = 0. \qquad (1)$$

Similarly, from the expression for p_4, it follows that

$$(p_4 \cdot p_8 - p_4 \cdot p_5)\, p_3 + (p_3 \cdot p_4 - p_4 \cdot p_6)\, p_5 + (p_4 \cdot p_5 - p_4 \cdot p_7)\, p_6 + (p_4 \cdot p_6 - p_4 \cdot p_8)\, p_7 + (p_4 \cdot p_7 - p_3 \cdot p_4)\, p_8 = 0. \qquad (2)$$

From Note 2 it follows that $p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_4$ and $p_5 \cdot p_6 = p_6 \cdot p_7 = p_7 \cdot p_8$. The perpendicular bisector of s_{58} contains p_2 and p_3, and the perpendicular bisector of s_{14} contains p_6 and p_7. Consequently, $p_2 \cdot p_5 = p_2 \cdot p_8$, $p_3 \cdot p_5 = p_3 \cdot p_8$, $p_1 \cdot p_6 = p_4 \cdot p_6$, and $p_1 \cdot p_7 = p_4 \cdot p_7$. Adding (1) and (2) and using these equalities gives

$$(p_1 \cdot p_5 - p_1 \cdot p_8)\, p_2 + (p_4 \cdot p_8 - p_4 \cdot p_5)\, p_3 + (p_4 \cdot p_5 - p_1 \cdot p_5)\, p_6 + (p_1 \cdot p_8 - p_4 \cdot p_8)\, p_7 = 0. \qquad (3)$$

If none of the coefficients in (3) is zero then p_2, p_3, p_6, and p_7 lie in a plane, π, since the sum of the coefficients is zero. If none of p_1, p_4, p_5, p_8 is on one side of π then either s_{26} or s_{27} is an edge, a contradiction. If one of p_1, p_4, p_5, p_8, without loss of generality say p_8, is on one side of π then s_{68} is an edge, a contradiction. Since s_{58} is not an edge, p_5 and p_8 are on opposite sides of π. Similarly p_1 and p_4 are on opposite sides of π. Suppose p_1 and p_8 are on the same side of π. Then p_3 and p_6 must be opposite vertices of the quadrilateral $p_2p_3p_7p_6$ since otherwise either s_{13} or s_{68} would be an edge. Suppose p_1 and p_8 lie on the smaller of the two caps of the sphere determined by π. From this and the fact that $p_1 \cdot p_2 = p_2 \cdot p_3$ it follows that $|s_{18}| \leqq |s_{15}|$, so $p_1 \cdot p_8 \geqq p_1 \cdot p_5$. If the caps are hemispheres this same result still holds. Then the coefficient of p_2 in (3) is negative. Since s_{27} is a diagonal of the quadrilateral $p_2p_3p_7p_6$, the coefficient of p_7 must also be negative. Thus $p_4 \cdot p_8 \geqq p_1 \cdot p_8$. Since $p_6 \cdot p_7 = p_7 \cdot p_8$ it similarly follows that $|s_{18}| \leqq |s_{48}|$. But then $p_4 \cdot p_8 \leqq p_1 \cdot p_8$, a contradiction. The case where p_1 and p_8 lie on the larger of the two caps gives a similar contradiction. The case where p_1 and p_5 lie on the same side of π also gives a similar contradiction. Thus, at least one of the coefficients in (3) is zero. Suppose only one coefficient is zero, without loss of generality say $p_1 \cdot p_8 - p_4 \cdot p_8 = 0$. Since the sum of the coefficients in (3) is zero, it follows that p_2, p_3, and p_6 are collinear

which contradicts the fact that they are distinct. Suppose only two coefficients are zero. If the coefficients of p_6 and p_7, are zero then p_2 and p_3 are antipodal. Then $p_1 = p_3$, since $|s_{12}| = |s_{23}|$, a contradiction. If the coefficients of p_2 and p_7 are zero then $p_1 \cdot p_5 = p_1 \cdot p_8 = p_4 \cdot p_8$. It then follows from (3) that $p_3 = p_6$, a contradiction. Other cases are similar. If three coefficients are zero then so is the fourth. Thus,

$$p_1 \cdot p_5 = p_1 \cdot p_8 = p_4 \cdot p_8 = p_4 \cdot p_5, \quad \text{so} \quad |s_{15}| = |s_{18}| = |s_{45}| = |s_{48}|.$$

Now let

$$p_1 = (\sin 3\varphi, 0, \cos 3\varphi), \qquad p_5 = (0, -\sin 3\delta, -\cos 3\delta),$$
$$p_2 = (\sin \varphi, 0, \cos \varphi), \qquad p_6 = (0, -\sin \delta, -\cos \delta),$$
$$p_3 = (-\sin \varphi, 0, \cos \varphi), \qquad p_7 = (0, \sin \delta, -\cos \delta),$$
$$p_4 = (-\sin 3\varphi, 0, \cos 3\varphi), \qquad p_8 = (0, \sin 3\delta, -\cos 3\delta),$$

where $0 < \varphi, \delta < \pi/3$. Substituting into (1) gives

$$(\cos 2\varphi + \cos 3\varphi \cos \delta) \sin 3\delta - \cos 3\varphi \sin \delta (\cos 3\delta - \cos \delta) = 0,$$

or

$$3 \cos 3\varphi \sin 2\delta + 2 \cos 2\varphi \sin 3\delta = 0.$$

An equation analogous to (1) may be obtained from the expression for p_5 and this in turn yields

$$3 \cos 3\delta \sin 2\varphi + 2 \cos 2\delta \sin 3\varphi = 0.$$

Rewriting in terms of functions of single angles gives

$$3 \cos\varphi (4 \cos^2\varphi - 3) \cos\delta + (2 \cos^2\varphi - 1)(4 \cos^2\delta - 1) = 0,$$
$$3 \cos\delta (4 \cos^2\delta - 3) \cos\varphi + (2 \cos^2\delta - 1)(4 \cos^2\varphi - 1) = 0. \qquad (4)$$

Subtracting and factoring yields

$$(2 \cos\varphi \cos\delta + 1)(\cos^2\varphi - \cos^2\delta) = 0.$$

Then $\delta = \varphi$, so substituting into (4) and solving gives

$$\cos\varphi = [(15 + \sqrt{145})/40]^{1/2}.$$

This determines P and the volume now may be found.

Note 3. This volume is approximately 1.815716. Grace [3] describes a polyhedron he obtained with the aid of a computer which is essentially this polyhedron. He pointed out that this gives a relative maximum which may be absolute. It is indeed the case as the following theorem shows.

Theorem 3. *If* $P = H(p_1, \ldots, p_8)$ *is such that* $V(p_1, \ldots, p_8)$ *is an absolute maximum then* P *is the medial polyhedron with property* Z *described in Lemma 4.*

Proof. Let $Q = H(p_1, \ldots, p_8)$. Suppose Q gives the absolute maximum fo: Then Q is convex, has property Z, contains the origin, and has triangular fa Since the medial complex and the double 8-pyramid are the only comple with 8 vertices, none of which has valence 3, assume Q has a vertex of valence say p_1. Using an argument similar to that in the proof of Theorem 2 it can b shown that

$$V(p_1, \ldots, p_8) \leqq F(\alpha) = \frac{\sqrt{3}}{4}\left[1 - \frac{1}{3}\tan^2\frac{2\pi - \alpha}{6}\right] + \frac{9}{4}\tan\frac{14\pi + \alpha}{54}\left[1 - \frac{1}{3}\tan^2\frac{14\pi + \alpha}{54}\right]$$

A direct calculation shows that F is concave and that its maximum is less th $[(475 + 29\sqrt{145})/250]^{1/2}$.

5. Concluding Remarks. In this paper it has been shown that if P is a doub n-pyramid, a tetrahedron, or medial with 8 vertices and if P has property Z then P is uniquely determined. This raises the question: For which types o polyhedra does property Z determine a unique polyhedron. More generally for each isomorphism class of polyhedra is there one and only one polyhedron (up to congruence) which gives a relative maximum for V?

For $n = 4, \ldots, 7$ the duals of the polyhedra of maximum volume are jus those polyhedra with n faces circumscribed about the unit sphere of minimum volume, i.e., the solutions to the well known isoperimetric problem. For $n=8$ the dual of the polyhedron described in Theorem 4 is the best known solutio to the isoperimetric problem for polyhedra with 8 faces. The question naturally arises: Is this true in general?

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References

1. Bowen, R., Fisk, St.: Generation of triangulations of the sphere. Math. Comp. **21**, 250—25 (1967).
2. Goldberg, M.: The isoperimetric problem for polyhedra. Tôhoku Math. J. **40**, 226—236 (1935)
3. Grace, D. W.: Search for largest polyhedra. Math. Comp. **17**, 197—199 (1963).
4. Fejes Tóth, L.: Regular figures. New York: The Macmillan Company 1964.

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