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Counterexamples to the Strong d-Step Conjecture for $d \geq 5$*

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Abstract. A *Dantzig figure* is a triple (P, x, y) in which P is a simple d-polytope with precisely $2d$ facets, x and y are vertices of P, and each facet is incident to x or y but not both. The famous *d-step conjecture* of linear programming is equivalent to the claim that always $\#^d P(x, y) \geq 1$, where $\#^d P(x, y)$ denotes the number of paths that connect x to y by using precisely d edges of P. The recently formulated *strong d-step conjecture* makes a still stronger claim—namely, that always $\#^d P(x, y) \geq 2^{d-1}$. It is shown here that the strong d-step conjecture holds for $d \leq 4$, but fails for $d \geq 5$.

Introduction

A path formed from k edges of a graph is here called a *k-path*. When x and y are vertices of a polytope P, $\delta_P(x, y)$ denotes the *distance* from x to y in P's graph; thus $\delta_P(x, y)$ is the smallest k such that x and y are joined by a k-path. The maximum of $\delta_P(x, y)$, as x and y range over all vertices of P, is called the *diameter* of P and is denoted by $\delta(P)$. For each $n > d$ $\Delta(d, n)$ denotes the maximum of $\delta(P)$ as P ranges over all convex d-polytopes that have precisely n *facets* ($(d-1)$-faces). In the geometric form reported by Dantzig [D1], [D2], the *d-step conjecture* of linear programming (first formulated by W. M. Hirsch) asserts that $\Delta(d, 2d) = d$, and the formally stronger *Hirsch conjecture* asserts that $\Delta(d, n) \leq n - d$ for all d and all $n > d$.

A d-polytope is called *simple* if each of its vertices is incident to precisely d edges, or, equivalently, to precisely d facets. We use the term *(d, n)-polytope* to refer to a simple d-polytope that has precisely n facets. Two vertices of a polytope will be called *estranged* if they do not share a facet. In the course of showing that the d-step conjecture and the Hirsch conjecture are equivalent (though not necessarily on a dimension-for-dimension

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basis), Klee and Walkup [KW] introduced the notion of a d-dimensional *Dantzig figure*, this being a triple (P, x, y) such that P is a $(d, 2d)$-polytope and x and y are estranged vertices of P.

When x and y are vertices of a polytope P, we use $\#^k P(x, y)$ to denote the number of k-paths from x to y in P. As was shown in [KW], the d-step conjecture is equivalent to the claim that $\#^d P(x, y) \geq 1$ for each d-dimensional Dantzig figure (P, x, y). Using this equivalence, the d-step conjecture was proved in [KW] for $d \leq 5$, but it is still open for all $d \geq 6$. In [LPR], Lagarias *et al.* observed that for each d-dimensional Dantzig figure (P, x, y), $\#^d P(x, y) \leq d!$, and they formulated what they called the *strong d-step conjecture*, asserting that $\#^d P(x, y) \geq 2^{d-1}$. They verified this conjecture for $d \leq 3$ and they produced extensive numerical evidence in its favor for $4 \leq d \leq 15$. Subsequently, Lagarias and Prabhu [LP] showed for each d, that if r is either the minimum $(d^2 - d + 2)$ or the maximum number of vertices that a $(d, 2d)$-polytope can have, then there exists a d-dimensional Dantzig figure (P, x, y) such that $\#^d P(x, y) = 2^{d-1}$ and P has precisely r vertices.

This paper shows that the strong d-step conjecture is correct when $d = 4$ but fails for all $d \geq 5$. The proof for $d = 4$ is a routine computation based on the Grünbaum–Sreedharan catalog [GS] of the 37 combinatorial types of simple 4-polytopes with 8 facets. The disproof for $d \geq 5$ starts with a $(4, 9)$ dual-neighborly polytope of diameter 5 that was first constructed in [KW], and then applies the wedging operation of [KW] to show that for each $d \geq 5$ there exists a d-dimensional Dantzig figure (P, x, y) for which $\#^d P(x, y) = 3 \cdot 2^{d-3} < 2^{d-1}$. (In the constructed examples, the number of vertices is $d^2 + 9d - 28$.)

As general references on the combinatorial structure of polytopes, the books by Grünbaum [G] and Ziegler [Z] are recommended. Both discuss the d-step conjecture.

1. Computational Procedure

The following procedure finds, for each estranged pair of vertices of a simple d-polytope P, the number of d-paths that join the two vertices.

(0) (Input.) For a simple d-polytope P with n facets and m vertices, let M denote the $n \times m$ facet-versus-vertex incidence matrix of P. The ith row of M tells which vertices are incident to facet i. The jth column of M tells which facets are incident to vertex j.

(1) $S := M^T M$. (S is an $m \times m$ matrix (s_{ij}) in which s_{ij} is the number of facets shared by vertex i and vertex j.)

(2) $B := (s_{ij} \stackrel{?}{=} 0)$, an $m \times m$, 0–1 matrix (b_{ij}) in which the 1 entries correspond to pairs of vertices that are estranged. If $B = 0$, there are no estranged pairs and the computation halts.

(3) $A := (s_{ij} \stackrel{?}{=} d - 1)$, the $m \times m$ adjacency matrix of the graph formed by P's vertices and edges.

(4) (Output.) $N := A^d \circ B$, in which $\circ$ denotes the *Hadamard* (entry-by-entry) product. The (i, j) entry of A^d is the number of walks of length d from vertex i to vertex j. However, when two vertices x and y of a simple d-polytope P are estranged, they cannot be connected by a walk of length less than d, and hence each walk of length d from x to y must, in fact, be a d-path. Thus the matrix N tells, for each estranged pair of vertices (x, y), the number $\#^d P(x, y)$ of d-paths that connect the two vertices.

2. Proof for $d \leq 4$

2.1. Theorem. *The strong d-step conjecture is correct for $d \leq 4$.*

Proof. The strong d-step conjecture is obvious for $d = 2$, and [LPR] noted that it also holds for $d = 3$. Verification for $d = 3$ is almost immediate, because there are only two different combinatorial types of $(3, 6)$ polytopes. The first is the 3-cube I^3, for which $\#^3 I^3(x, y) = 6$. (In general, I^d has 2^{d-1} estranged pairs (x, y), and $\#^d I^d(x, y) = d!$ for each such pair.) The second $(3, 6)$-polytope Q is combinatorially equivalent both to a triangular prism truncated at one vertex and to the wedge over a pentagon with an edge as foot. In Q there are two estranged pairs (x, y), and $\#^3 Q(x, y) = 4$ for each of them.

To verify the strong d-step conjecture for $d = 4$, we use the complete catalog of simplicial 4-polytopes with eight vertices that was published in 1967 by Grünbaum and Sreedharan [GS], correcting a 1909 list of Brückner [Br]. With the aid of the usual polarity, this may also be regarded as a catalog of simple 4-polytopes with eight facets. There are 37 different combinatorial types. In terms of the indexing of [GS], the procedure described in Section 1 yields the information that is listed below concerning the numbers of d-paths connecting estranged pairs of vertices.

The indices in parentheses are the identification numbers used in [GS]. An "na" indicates that the polytope in question has no estranged pairs. Polytope number (34) is the 4-cube, in which there are eight estranged pairs and each pair is connected by twenty-four 4-paths. In polytope number (25) there are four estranged pairs, with one such pair connected by eight 4-paths, another pair connected by ten 4-paths, and two pairs for each of which there are eleven 4-paths. The other data are interpreted similarly.

(1) na; (2) na; (3) na; (4) na; (5) 8_2; (6) 8_2; (7) 12_2; (8) $8_1, 10_1$; (9) 10_2; (10) na; (11) 8_2; (12) 8_2; (13) na; (14) 8_2; (15) 8_4; (16) 12_2; (17) 16_4; (18) 10_4; (19) 13_2; (20) 10_2; (21) 13_4; (22) $12_2, 14_2$; (23) 11_2; (24) 10_2; (25) $8_1, 10_1, 11_2$; (26) 18_6; (27) $14_2, 15_2$; (28) $12_2, 13_2$; (29) $8_1, 12_4, 14_1$; (30) $10_2, 12_2$; (31) $8_1, 10_1, 11_1, 12_1$; (32) 12_2; (33) $8_4, 11_2$; (34) 24_8; (35) $8_4, 12_2$; (36) $8_1, 9_2, 12_1$; (37) $8_2, 9_2$.

Note that for each of the 37 polytopes, each estranged pair is connected by at least eight 4-paths. This proves the strong d-step conjecture for $d = 4$. □

3. Wedging and Truncation

Suppose that P is a d-polytope in $\mathbb{R}^d$, and that F is a face of P. In the terminology of [KW], a *wedge* over P with *foot* F is a $(d+1)$-polytope $\omega_F(P)$ that is formed by intersecting the "cylinder" $C = P \times [0, \infty[$ with a closed half-space J in $\mathbb{R}^{d+1}$ such that the intersection $J \cap C$ is bounded and has nonempty interior, and the bounding hyperplane H of J is such that $H \cap (\mathbb{R}^d \times \{0\}) = F \times \{0\}$. The boundary complex of $\omega_F(P)$ is combinatorially equivalent to the complex formed from the boundary complex of the prism $P \times [0,1]$ by identifying $\{p\} \times [0,1]$ with $(p,0)$ for each point p of F. Henceforth, we specialize to the case in which F is a facet of P. Then, in effect, the identification process replaces the facet (d-face) $F \times [0,1]$ of the prism by a ridge ($(d-1)$-face) R that is a copy of F. In the wedge $\omega_F(P)$, there are two facets that contain the ridge R, and each of these facets is combinatorially equivalent to P. We shall denote these facets by B $(= P \times \{0\})$ and T $(= P \times \{1\})$ and call them the *base* and the *top* of the wedge; thus $R = B \cap T$. Since each vertex of $\omega_F(P)$ is incident to T or B, it corresponds naturally to a vertex in P. Each vertex $v \in F$ has a unique natural image in the ridge R in $\omega_F(P)$. Each vertex $v \in P \backslash F$ has a natural image in the base B and a second natural image in the top T; we denote these images by v_b $(= v \times \{0\})$ and v^t $(= v \times \{1\})$, respectively. If P is a (d,n)-polytope and F is a facet of P, then the wedge $\omega_F(P)$ is a $(d+1, n+1)$-polytope.

To derive the incidence matrix for $\omega_F(P)$ from the incidence matrix $M(P)$ of P, we first determine the index of F: $f_i = F$. Recall that the rows of M correspond to facets and the columns to vertices. Let C_i be the submatrix of $M(P)$ consisting of the columns that correspond to vertices not incident to f_i, and let E_i be a matrix of the same dimensions as C_i $(n \times (f_0(P) - f_0(F)))$ in which all entries are zero, except those in the ith row which are all ones. Then

$$M(\omega_{f_i}(P)) = \begin{pmatrix} C_i + E_i : M(P) \\ \langle 0 \rangle \quad : \quad \langle 1 \rangle \end{pmatrix}.$$

With $M(\omega_F(P))$ so constructed, we have the base $B = f_i$, and the new row is the top $T = f_{n+1}$. The vertices of the foot are indicated precisely by the columns that have 1's in both of these rows.

When F is any face of a d-polytope P, and x and y are vertices of P, we denote by $\#P(x,y)$ the number of shortest paths from x to y in P, and by $\#P(x,F,y)$ the number of shortest paths from x to y that visit F. Note that this differs from the practice of [LPR] and [LP], who use $\#P(x,y)$ to denote the number of d-paths from x to y in a d-dimensional Dantzig figure (P,x,y). (For that specialized purpose, we have used the notation $\#^d P(x,y)$.)

Let $W = \omega_F(P)$. Since the facets B and T are combinatorially equivalent to P, each vertex v of P has two natural images in W, and we denote these by v_b and v^t; if v is incident to F, then these two images coincide: $v_b = v^t = v$. Since a vertex w of W is incident to at least one of B or T, w has a natural image in P, which we denote by $\overline{w}$. Thus $\overline{v_b} = \overline{v^t} = v$ for each vertex v of P.

From these maps of vertices, we obtain for each path in W a unique natural image in P. Let $[w_0, w_1, \ldots, w_m]$ be a path in W. For each i, $[w_i, w_{i+1}]$ is an edge in W, so either $[\overline{w_i}, \overline{w_{i+1}}]$ is an edge of P or $\overline{w_i} = \overline{w_{i+1}}$ (i.e., $\{w_i, w_{i+1}\} = \{v_b, v^t\}$ for some vertex v of P). In the latter case, we say that $[w_i, w_{i+1}]$ is a *vertical* edge. The natural image of a vertical edge in W is a vertex in P. The natural image of $[w_0, w_1, \ldots, w_m]$ is $[\overline{w_0}, \overline{w_1}, \ldots, \overline{w_m}]$, to which sequence of vertices we apply the contraction that replaces v, v by v. In effect, we eliminate the vertical edges and map the remaining edges to their natural images in P

The natural image of an m path in W is a k-path in P with $k = m - e$, e the number of vertical edges in the m-path. For a path ρ in P and fixed images w_0 and w_m of its endpoints in W, we define the *tight natural images* of ρ from w_0 to w_m to be those paths of minimal length among all the paths from w_0 to w_m in W whose natural image is ρ. For shortest paths, we have the following result.

3.1. Wedging Lemmas. *Suppose x and y are vertices and F is a facet of the (d,n)-polytope P. Then the wedge $W = \omega_F(P)$ is a $(d+1, n+1)$-polytope.*

(1) Case (i). *If no shortest path from x to y visits F, then*

$$\delta_W(x_b, y^t) = \delta_P(x,y) + 1,$$

and each shortest path from x to y in P corresponds naturally to $\delta_P(x,y)+1$ shortest paths from x_b to y^t in W. Further,

$$\#W(x_b, x^t, y^t) = \#P(x,y),$$

and for each neighbor v of x in P

$$\#W(x_b, v_b, y^t) = \delta_P(x,y) \cdot \#P(x,v,y) + \sum_\rho 2^{r_\rho - 1},$$

the sum being taken over all $(\delta_P(x,y)+1)$-paths ρ from x to y via v which visit F r_ρ (>0) times.

(2) Case (ii). *If some shortest path from x to y visits F, then*

$$\delta_W(x_b, y^t) = \delta_P(x,y),$$

and each shortest path in P from x to y that visits F r times corresponds naturally to 2^{r-1} shortest paths from x_b to y^t in W.

If every shortest path in P from x to y that visits F does so only once, then the shortest paths from x to y are in natural one-to-one correspondence with the shortest paths in W from x_b to y^t. Under this nonrevisiting assumption,

$$\#W(x_b, y^t) = \#P(x,F,y).$$

If v is a neighbor of x in P, then

$$\#W(x_b, v_b, y^t) = \#P(v,F,y),$$

and

$$\#W(x_b, x^t, y^t) = 0.$$

Proof. Let $[x = v_0, v_1, \ldots, v_m = y]$ be an m-path from x to y in P which does not visit F. Then $[x_b = v_{0b}, \ldots, v_{ib}, v_i^t, \ldots, v_m^t = y^t]$ is an $(m+1)$-path from x_b to y^t in W, for each $0 \leq i \leq m$. These $m+1$ distinct paths are the shortest paths in W for which the natural image in P is the given path. Moving from the base to the top requires the addition of a vertical edge somewhere in the path.

Now suppose that in the m-path $[x = v_0, v_1, \ldots, v_m = y]$, v_i is incident to F. Then $[x_b = v_{0b}, \ldots, v_{(i-1)b}, v_i, v_{(i+1)}^t, \ldots, v_m^t = y^t]$ is an m-path from x_b to y^t in W. Moving from the base to the top requires no additional edge.

For an m-path from x to y in P which visits F, its tight natural images from x_b to y' in W necessarily enter the first visit to F from the base and leave the last visit to F on the top. After visiting F the first time and before visiting F the last time, any choice of base or top between visits to F yields a tight natural image from x_b to y'. There are 2^{r-1} ways of choosing whether the natural image in W of each of the $r-1$ sequences of vertices between visits to F is in the base or top. Thus a path in P which visits F r times has 2^{r-1} distinct tight natural images from x_b to y' in W.

Now let $m = \delta_P(x, y)$, and consider the set of shortest paths from x to y in P. Those which do not visit F have $m+1$ tight natural images from x_b to y' in W, each of length $m+1$. Those which visit F r times ($r > 0$) have 2^{r-1} tight natural images from x_b to y', each of length m.

In the case that none of the shortest paths from x to y in P visits F, we have established all the claims except the specific counts of shortest paths from x_b to y' incident to given neighbors. Let v be a neighbor of x in P. Any shortest path from x to y via v consists of the edge $[x, v]$ prepended to a shortest path from v to y. Necessarily, $\delta_P(v, y) = \delta_P(x, y) - 1$, and each of the $\delta_P(x, y)$ tight natural images of a shortest path from v_b to y' can be prepended to a shortest path from x_b to y'. We have accounted for all the shortest paths from x_b to y' via v_b which do not visit F. However, an $(m+1)$-path from x to y via v and visiting F r times has 2^{r-1} tight natural images from x_b to y' in W, each of length $m+1$; hence each of these images will be a shortest path from x_b to y'. We summarize this accounting in

$$\#W(x_b, v_b, y') = m \cdot \#P(x, v, y) + \sum_{\rho} 2^{r_\rho - 1}.$$

An $(m+1)$-path from x_b to y' via x' consists of the initial edge $[x_b, x']$ followed by an m-path ρ from x' to y'. Since none of the m-paths from x to y in P visits F, ρ must lie entirely in T, and so $\overline{\rho}$ is an m-path in P from x to y. On the other hand, for every m-path β from x to y in P, the tight natural image $[x_b, \beta']$ is an $(m+1)$-path from x_b to y' in W. From this natural one-to-one correspondence, we have

$$\#W(x_b, x', y') = \#P(x, y).$$

We now address case (ii), in which some shortest m-path from x to y visits F. No path from x_b to y' can have length less than m, but the tight natural images of an m-path which visits F has length m; hence $\delta_W(x_b, y') = \delta_P(x, y)$, and as observed above, an m-path in P which visits F r times has 2^{r-1} tight natural images in W, each of length m. For any path from x to y in P which does not visit F, the tight natural images from x_b to y' are of length $m+1$ and so are not shortest paths. Summing over all shortest paths ρ from x to y in P which visit F r_ρ times, we have

$$\#W(x_b, y') = \sum_{\rho} 2^{r_\rho - 1}.$$

We now assume further that the shortest paths from x to y which visit F do so only once ($r = 1$). Under this assumption, each shortest path from x to y which visits F has a unique tight natural image from x_b to y' in W. Hence, for each neighbor v of x in P,

$$\#W(x_b, v_b, y') = \#P(v, F, y).$$

and we can rewrite the above sum

$$\#W(x_b, y') = \#P(x, F, y).$$

To see finally that $\#W(x_b, x', y') = 0$, we can either observe that no shortest paths from x_b to y' are left uncounted, or we could observe that an m-path from x_b to y' via x' would have as its natural image in P a path from x to y of length less than m. □

When a simple d-polytope P and two vertices x and y of P are fixed, we define a function γ_x on the neighbors of x in P by setting $\gamma_x(v) = \#P(x, v, y)$ for each neighbor v. We can list γ_x as a d-vector since P is simple:

$$\gamma_x = (\#P(x, v_1, y), \ldots, \#P(x, v_d, y)).$$

The conclusion of the second case in the above lemma can now be written succinctly:

$$\gamma_{x_b} = (\gamma_x, 0),$$

by which we mean $\gamma_{x_b}(v_b) = \gamma_x(v)$ for neighbors v of x in P, and $\gamma_{x_b}(x') = 0$.

In the construction of counterexamples, we also employ the operation of truncating a (d, n)-polytope P at a vertex v. To perform the truncation geometrically, we form the intersection $\tau_v(P)$ of P with any closed half-space that misses v and whose bounding hyperplane passes strictly between v and the remaining vertices of P. Again note that since P is simple, $\tau_v(P)$ is a $(d, n+1)$-polytope with new facet $\tau(v)$ and $d-1$ additional vertices.

Combinatorially, the vertex v is replaced by a $(d-1)$-simplex $\Sigma(v)$ with one of its vertices on each edge incident to v. For example, if u is a neighbor of v in P, then in $\tau_v(P)$, $\sigma(u)$ is a vertex in $\Sigma(v)$ whose neighbors are the $d-1$ other vertices in $\Sigma(v)$ and u.

We form the incidence matrix for the truncated polytope $\tau_v(P)$ from that of P thus:

$$M(\tau_v(P)) = \begin{pmatrix} M(P\backslash v) & : & M(\Sigma(v)\backslash\tau(v)) \\ (0) & : & (1) \end{pmatrix}.$$

We start with a copy of $M(P)$ and remove the column corresponding to v; this is the upper-left block $M(P\backslash v)$. We take d copies of the column for v, and in each copy replace one of the d 1's by a 0 so that no two of these columns are the same; this is the upper-right block $M(\Sigma(v)\backslash\tau(v))$. Finally, we append a new row with 1's under these rightmost d columns and 0's under $M(P\backslash v)$; this new row corresponds to the facet $\tau(v)$.

We note some natural correspondences between paths on P and paths on $Q = \tau_v(P)$. Paths in Q have unique natural images in P, obtained by replacing each occurrence of a vertex in $\Sigma(v)$ with v and then applying the contraction that replaces v, v by v. For a fixed path ρ in P, we define a *tight natural image* of ρ in Q to be a path of minimal length in Q whose natural image in P is ρ. Every path in P has a unique tight natural image in Q. In particular, for distinct neighbors u and w of v in P, the paths $[u, v]$ and $[u, v, w]$ correspond, respectively, to the paths $[u, \sigma(u)]$ and $[u, \sigma(u), \sigma(w), w]$ in Q. Note that the tight natural images in Q of m-paths in P which do not visit v, except possibly as a terminal vertex, are also of length m; if an m-path in P does not terminate at v but visits v r times, then its tight natural image is an $(m+r)$-path in Q.

3.2. Truncation Lemmas. *Suppose x and v are distinct vertices in the (d, n)-polytope P, and u and w are distinct neighbors of v in P. Then $Q = \tau_v(P)$ is a $(d, n+1)$-polytope.*

(1) Case (i). *If* $\delta_P(x, w) = \delta_P(x, v)$, *then* $\delta_Q(x, \sigma(w)) = \delta_P(x, v) + 1$,

$$\#Q(x, w, \sigma(w)) = \#P(x, w),$$

and

$$\#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v).$$

(2) Case (ii). *If* $\delta_P(x, w) = \delta_P(x, v) - 1$, *then* $\delta_Q(x, \sigma(w)) = \delta_P(x, v)$,

$$\#Q(x, w, \sigma(w)) = \#P(x, w, v),$$

and

$$\#Q(x, \sigma(u), \sigma(w)) = 0.$$

(3) Case (iii). *If* $\delta_P(x, w) = \delta_P(x, v) + 1$, *then* $\delta_Q(x, \sigma(w)) = \delta_P(x, v) + 1$,

$$\#Q(x, w, \sigma(w)) = 0,$$

and

$$\#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v).$$

Proof. Let w be a neighbor of v in P. Since w is a neighbor of v, their distances from x differ by at most 1. For case (i) let $m = \delta_P(x, w) = \delta_P(x, v)$. Necessarily, $\#P(x, w, v) = \#P(x, v, w) = 0$. The tight natural image of any m-path in P from x to v via a neighbor $u \neq w$ is an m-path in Q from x to $\sigma(u)$, which extends to an $(m+1)$-path from x to $\sigma(w)$. Each m-path from x to w in P can be identified with its tight natural image in Q and then extended to an $(m+1)$-path from x to $\sigma(w)$. Thus, $\delta_Q(x, \sigma(w)) = m + 1$; moreover, we have the specific counts $\#Q(x, u, \sigma(w)) = \#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v)$, and $\#Q(x, w, \sigma(w)) = \#P(x, w)$.

In case (ii) we let $m = \delta_P(x, v) = \delta_P(x, w) + 1$. So $\#P(x, w, v) = \#P(x, w)$, and $\#P(x, v, w) = 0$. The tight natural image of any $(m-1)$-path in P from x to w can be extended in Q to an m-path from x to $\sigma(w)$. On the other hand, for any other neighbor u of v, a path in Q from x to $\sigma(w)$ via $\sigma(u)$ has length at least $m + 1$. We conclude, in this case, that $\delta_Q(x, \sigma(w)) = m$ with $\#Q(x, w, \sigma(w)) = \#P(x, w) = \#P(x, w, v)$ and $\#Q(x, \sigma(u), \sigma(w)) = 0$.

For case (iii) we let $m = \delta_P(x, w) = \delta_P(x, v) + 1$. In this case, $\#P(x, w, v) = 0$ and $\#P(x, v, w) = \#P(x, v)$. Any m-path in P from x to w can be identified with its tight natural image in Q and then extended to an $(m+1)$-path from x to $\sigma(w)$ via w. On the other hand, an $(m-1)$-path from x to v in P must arrive at v via a neighbor $u \neq w$, and so its tight natural image is an $(m-1)$-path in Q from x to $\sigma(u)$, which can be extended to an m-path from x to $\sigma(w)$. Thus $\delta_Q(x, \sigma(w)) = m$ with $\#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v)$ and $\#Q(x, w, \sigma(w)) = 0$. □

4. Disproof for $d = 5$

4.1. Theorem. *There is a five-dimensional Dantzig figure (P, x, y) for which $\#P(x, y) = 12$. Hence the strong d-step conjecture fails for $d = 5$.*

Proof. We produce the counterexample for $d = 5$ as the wedge over a certain $(4, 9)$-polytope Q_4 which was first constructed in [KW]. The polytope Q_4 has 9 facets and 27 vertices, and is the only $(4, 9)$-polytope of diameter 5. The combinatorial structure of Q_4 is described explicitly on p. 741 of [KK′]. With a convenient numbering of facets and vertices, Q_4's incidence matrix is as follows. The estranged vertices $x(= v_1)$ and $y(= v_{15})$ of Q_4 have $\delta_{Q_4}(x, y) = 5$, and the facet $F(= f_9)$ misses both x and y. The facet F has 12 vertices.

$$M(Q_4) = \begin{pmatrix} 110111000000100111111000000 \\ 101000111000010110000111100 \\ 111000111111000101000010000 \\ 111111000111000010100100000 \\ 001100100100101000110101011 \\ 010010010010011001001010111 \\ 000011011011111000011000010 \\ 000101101101111000000001101 \\ 000000000000000111111111111 \end{pmatrix}.$$

Let P_5 denote the wedge over Q_4 with foot F. Then F becomes a ridge in P_5, and each vertex v of $Q_4 \backslash F$ has two images in P_5: an image v_b in the base B and an image v^t in the top T, connected by an edge. There are 15 such pairs, and with the 12 vertices in F this yields a total of 42 vertices in P_5.

Following the method in Section 3, we produce the incidence matrix $M(P_5)$ from $M(Q_4)$.

$$M(P_5) = \begin{pmatrix} 110111000000100111111000000110111000000100 \\ 101000111000010110000111100101000111000010 \\ 111000111111000101000010000111000111111000 \\ 111111000111000010100100000111111000111000 \\ 001100100100101000110101011001100100100101 \\ 010010010010011001001010111010010010010011 \\ 000011011011111000011000010000011011011111 \\ 000101101101111000000001101000101101101111 \\ 111111111111111111111111111000000000000000 \\ 000000000000000111111111111111111111111111 \end{pmatrix}.$$

In this incidence matrix we have the base $B = f_9$, the top $T = f_{10}$, and the vertices $x_b = v_1$, $y_b = v_{15}$, $x^t = v_{28}$, and $y^t = v_{42}$.

When applied to $M(P_5)$, the procedure of Section 1 yields as output a 42×42 matrix $N(P_5)$ whose only nonzero entries are

$n_{1,42} = 12$, $n_{4,35} = 36$, $n_{5,34} = 36$, [illegible] $= 36$, [illegible] $= 36$, $n_{15,28} = 12$,

$n_{42,1} = 12$, $n_{35,4} = 36$, $n_{34,5} = 36$, $n_{32,7} = 36$, $n_{31,8} = 36$, $n_{28,15} = 12$.

Using the same notation as in Section 2, the summary statistic for P_5 is $12_2, 36_4$. That is:

- P_5 has six estranged pairs in all, each of distance 5.
- There are thirty-six shortest paths for each of four estranged pairs.
- For two of the estranged pairs, (x_b, y^t) and (x^t, y_b), there are only twelve shortest paths.

In Q_4 there are sixteen 5-paths from x to y, but only twelve of those paths visit F. From the Wedging Lemmas, as confirmed by the computational procedure, we have $\#^5 P_5(x_b, y^t) = 12$. Since (P_5, x_b, y^t) is a five-dimensional Dantzig figure, and $\#^5 P_5(x_b, y^t) < 16$, this is a counterexample to the strong 5-step conjecture. □

5. Disproof for $d \geq 6$

With $M(P_5)$ as in Section 4, truncate P_5 at v_{42} to produce $\tau(P_5)$. Then

$$M(\tau(P_5)) = \begin{pmatrix} 11011100000001001111111000000110111000000001000000 \\ 10100011100000101100001111001010001110000100000 \\ 11100011111110001010000100001110001111110000000 \\ 11111100011100001010010000001111110001110000000 \\ 0011001001001010001101010110011001001001001111 \\ 0100100100100110010010101110100100100100110111 \\ 0000110110111110000110000100000110110111111011 \\ 000101101101111100000000011010001011011011111101 \\ 111111111111111111111111111110000000000000000000 \\ 00000000000000011111111111111111111111111111110 \\ 00000000000000000000000000000000000000000011111 \end{pmatrix}.$$

Let P_6 be the wedge over $\tau(P_5)$ with foot f_{10}. Then

$$M(P_6) = \begin{pmatrix} 1101110000000100011011100000001001111111000000110111000000001000000 \\ 1010001110000010010100011100000101100001111001010001110000100000 \\ 11100011111100001110001111110001010000100001110001111110000000 \\ 111111000111000011111100011100001010010000001111110001110000000 \\ 00110010010010110011001001001010001101010110011001001001001111 \\ 01001001001001110100100100100110010010101110100100100100110111 \\ 000011011011111110000110110111110000110000100000110110111111011 \\ 0001011011011111100010110110111110000000001101000101101101111101 \\ 11111111111111101111111111111111111111111111110000000000000000000 \\ 1111111111111110000000000000000111111111111111111111111111111110 \\ 00000000000000010000000000000000000000000000000000000000001111 \\ 0000000000000000111111111111111111111111111111111111111111111111 \end{pmatrix}.$$

Applying the procedure of Section 1 to this incidence matrix, we find that there are only two estranged pairs, (v_1, v_{62}) and (v_{17}, v_{16}), with summary statistic 24_2. Since the

strong 6-step conjecture would require this number to be at least $32 = 2^{6-1}$, P_6 is a counterexample.

In the remainder of this section we show that the process of truncating and wedging can be repeated to produce a family of counterexamples to the strong d-step conjecture for all $d > 5$.

A triple (P, x, y^t) is a *W_d-figure* iff P is a $(d, 2d)$-polytope and is also a wedge $P = \omega_F(Q)$, with vertices $x \in B \backslash F$ and $y^t \in T \backslash F$ such that $\delta_P(x, y^t) = \delta_P(x, y_b) = d$.

For a W_d-figure (P_d, x, y^t), truncation at y^t yields a $(d, 2d+1)$-polytope Q with a vertex $z = \sigma(y_b)$ that is estranged from x, and with $\delta_Q(x, z) = d + 1$. The truncated top $\tau(T)$ is the unique facet of Q not incident to either x or z. Taking the wedge over Q with foot $\tau(T)$ yields a $(d+1, 2d+2)$-polytope P_{d+1} with only two estranged pairs (x_b, z^t) and (x^t, z_b), each at distance $d+1$. Since (P_d, x, y^t) is a W_d-figure, we can obtain a stronger result.

Proposition 5.1. *If (P_d, x, y^t) is a W_d-figure with $\#P_d(x, y^t) = k$, and*

$$P_{d+1} = \omega_{\tau(T)} \tau_{y^t}(P_d),$$

then (P_{d+1}, x_b, z^t) is a W_{d+1}-figure with

$$\#P_{d+1}(x_b, z^t) = 2k,$$

$$\gamma_{x_b} = (2\gamma_x, 0),$$

and

$$\gamma_{z^t} = (\gamma_{y^t}, k).$$

Proof. Since (P_d, x, y^t) is a W_d-figure, $P_d = \omega_F(Q)$ for some $(d-1, 2d-1)$-polytope Q with facet F, and every d-path from x to y^t visits F. The polytope P_d satisfies the first case of the Truncation Lemmas, with $v = y^t$ and $w = y_b$. Let $z = \sigma(y_b)$ in $\tau_{y^t}(P_d)$. Then from the Truncation Lemmas it follows that the collection of shortest paths from x to z is in natural bijection with the union of the collection of shortest paths in P_d from x to y^t and the collection of shortest paths in P_d from x to y_b.

Once we take the wedge over $\tau_{y^t}(P_d)$ with foot $\tau(T)$, the shortest paths from x_b to z^t are in natural bijection with shortest paths from x to z that visit $\tau(T)$. This includes all those shortest paths on $\tau_{y^t}(P_d)$ which correspond to shortest paths from x to y^t on P_d; it also includes those shortest paths on $\tau_{y^t}(P_d)$ which correspond to shortest paths from x to y_b on P_d which visit F, since $F \subset T$.

By the Wedging Lemmas there is a natural bijection between shortest paths in P_d from x to y^t and those from x to y_b which visit F. In particular,

$$\#P_d(x, y^t) = \#P_d(x, F, y_b)$$

($= k$ by assumption). Thus, from these natural correspondences, we conclude not only that

$$\#P_{d+1}(x_b, z^t) = 2k,$$

but also that

$$\gamma_{x_b} = (2\gamma_x, 0)$$

and

$$\gamma_{z'} = (\gamma_{y'}, k).$$

We note also that P_{d+1} is a W_{d+1}-figure. □

Corollary 5.2. *If (P_d, x, y') is a W_d-figure and a counterexample to the strong d-step conjecture, then with $P_{d+1} = \omega_{\tau(T)}\tau_{y'}(P_d)$, (P_{d+1}, x_b, z') is a W_{d+1}-figure and a counterexample to the strong $(d+1)$-conjecture.*

Corollary 5.3. *Let Q be a $(c, 2c+1)$-polytope of diameter $c+1$ with an estranged pair (x, y) at distance $c+1$, and $\#Q(x, F, y) < 2^c$ for F the unique facet F not incident to x or y. Then $(\omega_F(Q), x_b, y')$ is a counterexample to the strong $(c+1)$-conjecture and is a W_{c+1}-figure.*

That is, any polytope Q with the prescribed properties serves as the seed for a family of counterexamples to the strong d-step conjecture for all $d > c$, simply by iterating the construction in Proposition 5.1 above. The Q_4 of Section 4 is such a polytope, and serves as the seed for the family of counterexamples constructed here.

For this first family of counterexamples, denoting by x the vertex x_b in every iterate P_d, γ_x has only four nonzero entries, an extreme case of a phenomenon already noted in [KK′]. Only four of the d edges incident to x occur in a shortest path from x to y'; for large d, most choices of pivot at x will not yield a shortest path. For example in P_5, $\gamma_x = (4, 4, 2, 2, 0)$, and in P_6, $\gamma_x = (8, 8, 4, 4, 0, 0)$. In this family,

$$\gamma_x = (2^{d-3}, 2^{d-3}, 2^{d-4}, 2^{d-4}, 0, \ldots, 0),$$

and

$$\gamma_{y'} = (0, 2, 2, 4, 4, 12, 24, \ldots, 3 \cdot 2^{d-4}).$$

Since there is only one 0 in $\gamma_{y'}$ for each iterate, the truncation-and-wedge construction is unique at y'; that is, once we have truncated at y', there is a unique choice of $z \in \Sigma(y')$ to produce a counterexample. However, many variations of this family can be constructed by applying the truncation at x in any iterate; $z \in \Sigma(x)$ can be chosen to be $\sigma(u)$ for any of the $d-4$ neighbors u of x with $\gamma_x(u) = 0$. Although many combinatorial types of counterexamples may be produced in this way, with many γ_x and $\gamma_{y'}$, and with small variations in the number of vertices, all such counterexamples P will have $\#^d P(x, y') = 3 \cdot 2^{d-3}$. In fact, except for P_5, all counterexamples constructed in these ways will have summary statistic $(3 \cdot 2^{d-3})_2$; to prove this, all we have left to show is the following.

Proposition 5.4. *If (P_d, x, y') is a W_d-figure, and $P_{d+1} = \omega_{\tau(T)}\tau_{y'}(P_d)$ with $z = \sigma(y_b)$, then there are only two estranged pairs in P_{d+1}: (x_b, z') and (y', z_b).*

Proof. Since P_{d+1} is a wedge, one vertex of any estranged pair must lie in the top, the other in the base, and neither in the foot $\tau(T)$. So suppose without loss of generality that u_b and v' are estranged vertices in P_{d+1} with u_b in the base, v' in the top. Then in $\tau_{y'}(P_d)$, u and v are estranged vertices, neither incident to $\tau(T)$. However, P_d is itself a wedge, so either $u \in B$ and $v = z$, or $u = z$ and $v \in B$. Since neither u nor v is incident to $\tau(T)$ in this $(d, 2d+1)$-polytope, there is only one vertex in B estranged from z, but x is estranged from z and so must be this vertex. Hence, either $u = x$ and $v = z$, or $u = z$ and $v = x$, and the result follows. □

6. Additional Comments

If (P, x, y) is a (simple) d-dimensional Dantzig figure, then the polar polytope Q is simplicial. The boundary complex of Q is a triangulated $(d-1)$-sphere with $2d$ vertices and the facets ($(d-1)$-simplices) F_x and F_y of Q that correspond to x and y do not share a vertex and hence may be called *estranged*. Under polarity, the paths (*edge-paths*) of length d from x to y in P correspond to *ridge-paths* of length d from F_x to F_y in Q. (See [KK′] for details.) The computational procedure of Section 1 applies without change to determine, for each estranged pair of facets of a triangulated $(d-1)$-manifold, the number of ridge-paths of length d joining the two facets.

In addition to the 37 different combinatorial types of simplicial 4-polytopes with 8 vertices, there are nonpolytopal triangulated 3-spheres with 8 vertices. The Brückner sphere, listed in [GS], does not have any estranged pair of facets. The Barnette sphere [Ba] has summary statistic 15_2.

In cataloging the triangulated 3-manifolds with 9 vertices, Altshuler and Steinberg [AS] found 1297 different combinatorial types. With the aid of Bokowski (as reported in [ABS]), these were found to consist of one nonsphere, 154 nonpolytopal spheres, and 1142 polytopes. A tape containing their catalog was (many years ago) sent by Steinberg to Klee, who found that all but one of those manifolds is of ridge-diameter ≤ 4. The sole exception was the simplicial 4-polytope that is dual to the simple 4-polytope Q_4 (with 9 facets and edge-diameter 5) that was used in Section 3 as the basis for our constructions.

Early in the study of the d-step conjecture, it was felt that the dual-cyclic polytopes and other dual-neighborly polytopes were the most natural candidates for counterexamples to the conjecture. However, the Hirsch conjecture was proved by [K1] for the dual-cyclic polytopes, and Lagarias and Prabhu [LP] have proved the strong d-step conjecture for these polytopes. Both the d-step conjecture and the strong d-step conjecture are still open for more general dual-neighborly polytopes, but Kalai [K1] established a weaker form of the d-step conjecture (and of the Hirsch conjecture), showing that $\delta(P) \leq d^2(n-d)^2 \log n$ for each dual-neighborly (d, n)-polytope.

Among the $(d, 2d)$ polytopes, the minimum possible number of vertices is $d^2 - d + 2$ and the maximum is

$$2\binom{(3d-1)/2}{d} \quad \text{or} \quad \frac{4}{3}\binom{3d/2}{d}$$

according as d is odd or even. The maximum is attained by the polars of cyclic polytopes and the minimum by the polars of stacked polytopes, and the strong d-step conjecture has been verified for both of these classes by Lagarias and Prabhu [LP]. The number of vertices is relatively small for the counterexamples to the strong d-step conjecture constructed (for $d \geq 5$) in Sections 4 and 5; the number of vertices of P_d is $d^2 + 9d - 28$.

Finally, it should be mentioned that Kalai [K2], [K3], Kalai and Kleitman [KK], and Matoušek, Sharir, and Welzl [MSW] have established subexponential upper bounds on $\Delta(d, n)$, and that Frieze and Teng [FT] have shown that computing the diameter of a polytope is an $\mathbb{NP}$-hard problem.

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The Complexity of Stratification Computation

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Abstract. This paper investigates the complexity of stratification computation for semi-algebraic sets. An upper bound for the computation of canonical stratifications is given for a wide class of stratifying conditions called here *admissible*. For such conditions, the stratifying process is at most doubly exponential in the depth of the stratification. Usual conditions of regularity like Whitney conditions (a) and (b) or Bekka condition (C) are admissible. A useful criterion and tools are given in order to prove easily other admissibilities.

Introduction

The basic fact in the theory of stratifications is that it is always possible to decompose an algebraic set or a semialgebraic set into a union of connected smooth manifolds of various dimensions, called strata. Roughly speaking, a stratification is such a decomposition.

The first idea in order to construct a stratification is to construct the singular locus of the algebraic set, which is an algebraic set of lower dimension. Then again the singular locus of the singular locus is constructed, and so on. The depth of the stratification is the number of times this construction has to be iterated. Unfortunately, this construction does not permit a good control of the topology of the set along strata, and it is necessary to introduce regularity conditions on stratifications (e.g., conditions on the limits of the tangent spaces).

In order to construct stratifications satisfying certain regularity conditions, it is possible to do as follows: for every pair of strata X and Y already computed with $\dim(X) \leq \dim(Y)$ for which the closure of Y contains X, the set of bad points of X is constructed as the set of points at which the regularity condition is not satisfied. In several important examples, the dimension of the set of bad points is always strictly smaller than the dimension of X, which means that the construction will terminate. Regularity conditions with such properties are called stratifying conditions. The depth of the stratification is the number of iterations needed by the construction to be ended.