The number and average length of decreasing paths in the Klee-Minty cube

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1 Introduction

In [1], Gärtner, Henk and Ziegler studied several randomized simplex algorithms on the n-dimensional Klee-Minty cube. They proved that the expected length of a path chosen by any of these algorithms is polynomially bounded, whereas the *average* length Φ_n of a path going from the top vertex to the bottom vertex of the cube is exponential in n. More precisely, they gave the following bounds for Φ_n:

$$\left(1+1/\sqrt{5}\right)^{n-1} < \Phi_n < \left(1+1/\sqrt{5}\right) 2^{n-1}.$$

In this note, we improve their lower bound, proving that there exists a positive constant C such that

$$\Phi_n \geq C\ 2^n. \tag{1}$$

Our starting point will be the following result [1].

Proposition 1.1 *The average length of decreasing paths in the n-dimensional Klee-Minty cube is*

$$\Phi_n = \frac{\sum_{k=1}^{2^{n-1}} (2k-1)\phi(k,n)}{\sum_{k=1}^{2^{n-1}} \phi(k,n)}, \tag{2}$$

where $\phi(1,1)=1$, $\phi(k,n)=0$ if $k \leq 0$ or $k > 2^{n-1}$, and for $n \geq 2$:

$$\phi(k,n) = \sum_j \binom{k}{2j} \phi(k-j, n-1). \tag{3}$$

Actually, $\phi_{k,n}$ is the number of decreasing paths of length $2k-1$, which justifies (2).

As suggested by Günter Ziegler in his talk at the University of Minnesota (June 1996), generating functions are a convenient way to handle recurrence relations like (3). Thus for $n \geq 1$, let us define the polynomial $F_n(s)$ by

$$F_n(s) = \sum_k s^k \phi(k,n).$$

We note that $F_n(1)$ is the number of decreasing paths in the n-dimensional Klee-Minty cube. The average length of these paths is $\Phi_n = -1 + 2F_n'(1)/F_n(1)$.

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2 The number of decreasing paths

In this section we study the asymptotics of $F_n(s) = \sum s^k \phi(k,n)$. Here is our main result.

Proposition 2.1 *For $s > 0$, there exists a constant $\nu(s) > 1$ such that*

$$\lim_{n\to\infty} (F_n(s))^{1/2^n} = \nu(s).$$

Moreover, $\sqrt{s} \le \nu(s) \le \sqrt{s+1}$. In particular, the radius of convergence of the series $\sum_n F_n(s)t^n$ is zero.

We first need to translate the recurrence relation (3) in terms of the polynomials $F_n(s)$.

Lemma 2.2 *Let s be a positive real number. The sequence of polynomials $(F_n)_n$ satisfies $F_1(s) = s$ and for $n \ge 2$,*

$$F_n(s) = \frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)F_{n-1}(s\alpha(s)) + \frac{1}{2}\left(1-\sqrt{\frac{s}{4+s}}\right)F_{n-1}\left(s/\alpha(s)\right),$$

where

$$\alpha(s) = \frac{s+2+\sqrt{s^2+4s}}{2}.$$

Proof. Using (3) gives $F_n(s)$ in terms of the $\phi(i, n-1)$:

$$F_n(s) = \sum_i \phi(i,n-1)A_i s^i \tag{4}$$

where for $i \ge 0$, A_i is the following polynomial in s:

$$A_i = \sum_{\ell=0}^{i} \binom{i+\ell}{i-\ell} s^\ell.$$

Let us define similarly B_i, for $i \ge 0$, by

$$B_i = \sum_{\ell=0}^{i-1} \binom{i+\ell}{i-\ell-1} s^\ell.$$

One checks easily that

$$A_i = A_{i-1} + sB_i \quad \text{and} \quad B_i = B_{i-1} + A_{i-1}.$$

This gives

$$A_i = (2+s)A_{i-1} - A_{i-2},$$

and the solution of this recurrence relation is

$$A_i = \frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)(\alpha(s))^i + \frac{1}{2}\left(1-\sqrt{\frac{s}{4+s}}\right)(\alpha(s))^{-i}.$$

Substituting this identity in (4) concludes the proof of the lemma.

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Note that $\alpha(s) > 1$ if $s > 0$. In particular, $s/\alpha(s) < s < s\alpha(s)$, and Lemma 2.2 implies that for $s > 0$

$$\frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)F_{n-1}(s\alpha(s)) \le F_n(s) \le F_{n-1}(s\alpha(s)),$$

or, with $\tau(s) = s\alpha(s)$,

$$\frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)F_{n-1}(\tau(s)) \le F_n(s) \le F_{n-1}(\tau(s)). \tag{5}$$

We want to iterate this inequality. Let us define the functions τ_i, $i \ge 0$ by

$$\tau_0(s) = s, \quad \tau_1(s) = \tau(s) \quad \text{and} \quad \tau_i(s) = \tau_{i-1}(\tau(s)) \ \text{for } i \ge 2.$$

The asymptotic behaviour of these functions is described by the following lemma.

Lemma 2.3 *For $s > 0$, the sequence $(\tau_n(s))^{1/2^n}$ is increasing. Moreover, its limit $\mu(s)$ is finite and satisfies*

$$1 < \mu(s) \quad \text{and} \quad s \leq \mu(s) \leq s+1.$$

Proof. Let us first prove that $\tau_n(s)$ increases to $+\infty$. We first note that $\tau_n(s)$ is positive for $s > 0$ and $n \geq 0$. Then, as $\alpha(u) > 1$ if $u > 0$, we have

$$\tau_n(s) = \tau(\tau_{n-1}(s)) = \tau_{n-1}(s)\alpha(\tau_{n-1}(s)) > \tau_{n-1}(s) > 0.$$

Thus the sequence $\tau_n(s)$ is increasing, and hence has a limit in $\mathbb{R}^* \cup \{+\infty\}$. If this limit $\ell(s)$ were finite, it would satisfy the equation

$$\ell(s) = \tau(\ell(s)),$$

which has no positive solution. Therefore $\tau_n(s)$ increases to $+\infty$.

Now, we also have $\alpha(u) > u$ for $u > 0$. This implies that

$$\tau_n(s) = \tau(\tau_{n-1}(s)) = \tau_{n-1}(s)\alpha(\tau_{n-1}(s)) > (\tau_{n-1}(s))^2,$$

and hence the sequence $(\tau_n(s))^{1/2^n}$ is increasing; as $\tau_n(s) \to \infty$, $(\tau_n(s))^{1/2^n}$ is larger than 1 for a large enough n. Thus the sequence $(\tau_n(s))^{1/2^n}$ has a limit $\mu(s)$ in $]1, +\infty]$. Now, using the fact that $\tau_n(s) = \tau(\tau_{n-1}(s))$ and $1 < \alpha(u) < u+2$, we obtain by induction on n

$$s^{2^n} < \tau_n(s) < s\prod_{i=0}^{n-1}\left(1 + (s+1)^{2^i}\right), \tag{6}$$

which proves that $\mu(s) \in [s, s+1]$.

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Let us now iterate (5). Using the fact that $F_1(s) = s$, we obtain

$$\prod_{i=0}^{n-2}\left[\frac{1}{2}\left(1 + \sqrt{\frac{\tau_i(s)}{4+\tau_i(s)}}\right)\right] \leq \frac{F_n(s)}{\tau_{n-1}(s)} \leq 1.$$

The sequence

$$C_n(s) = \prod_{i=0}^{n-2}\left[\frac{1}{2}\left(1 + \sqrt{\frac{\tau_i(s)}{4+\tau_i(s)}}\right)\right] \tag{7}$$

is decreasing and positive, so it converges towards its limit

$$C(s) = \prod_{i\geq 0}\left[\frac{1}{2}\left(1 + \sqrt{\frac{\tau_i(s)}{4+\tau_i(s)}}\right)\right]. \tag{8}$$

We want to prove is that $C(s)$ is positive. As

$$\frac{1}{2}\left(1 + \sqrt{\frac{u}{4+u}}\right) > 1 - \frac{1}{u},$$

it suffices to prove that the series $\sum_i 1/\tau_i(s)$ is finite. But for all $s > 0$, there exists an n_0 such that $\tau_{n_0}(s) > 1$. Then for $n > n_0$, we have, according to (6),

$$(\tau_{n_0}(s))^{2^{n-n_0}} < \tau_n(s),$$

which proves that the series $\sum_i 1/\tau_i(s)$ is finite. Finally, we have obtained the inequality

$$0 < C(s) = \prod_{i\geq 0}\left[\frac{1}{2}\left(1 + \sqrt{\frac{\tau_i(s)}{4+\tau_i(s)}}\right)\right] \leq \frac{F_n(s)}{\tau_{n-1}(s)} \leq 1.$$

Combining it with Lemma 2.3 proves Proposition 2.1. The constant $\nu(s)$ is actually $\sqrt{\mu(s)}$. The functions μ and ν both satisfy the functional equation (in f):

$$f(s) = \sqrt{f(\tau(s))}.$$

3 The average length of decreasing paths

We study in this section the asymptotics of $G_n(s) = sF_n'(s)/F_n(s)$. Here is our central result.

Proposition 3.1 *For $s > 0$, there exists a positive constant $D(s)$ such that*

$$G_n(s) = \frac{sF_n'(s)}{F_n(s)} \geq D(s)\ 2^n.$$

As the average length of decreasing paths in the Klee-Minty cube is $\Phi_n = -1 + 2G_n(1)$, this implies (1).

Proof. Let us differentiate the identity of Lemma 2.2 with respect to s. Denoting $H_n(s) = sF_n'(s)$, we obtain:

$$H_n(s) = \frac{s}{(4+s)\sqrt{s^2+4s}}\left[F_{n-1}(s\alpha(s)) - F_{n-1}(s/\alpha(s))\right]$$

$$+\frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)^2 H_{n-1}(s\alpha(s)) + \frac{1}{2}\left(1-\sqrt{\frac{s}{4+s}}\right)^2 H_{n-1}(s/\alpha(s)),$$

and hence

$$H_n(s) \geq \frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)^2 H_{n-1}(s\alpha(s)).$$

Moreover, we have seen that $F_n(s) \leq F_{n-1}(s\alpha(s))$. Hence

$$G_n(s) = \frac{H_n(s)}{F_n(s)} \geq \frac{1}{2}\left(1+\sqrt{\frac{s}{4+s}}\right)^2 G_{n-1}(\tau(s)).$$

Let us now iterate this identity, and use the fact that $G_1(s) = 1$. We obtain

$$\frac{G_n(s)}{2^n} \geq \frac{1}{2}(C_n(s))^2 > \frac{1}{2}(C(s))^2 > 0,$$

where $C_n(s)$ and its positive limit $C(s)$ are respectively defined by (7) and (8).

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References

[1] B. Gärtner, M. Henk and G. M. Ziegler, Randomized simplex algorithms on Klee-Minty cubes, Technishe Universität Berlin, 1996.