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WHICH SPHERES ARE SHELLABLE?*

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1. Introduction

When $\boldsymbol{B}$ is a finite collection of topological d-balls belonging to a cell-complex, a *partial shelling* of $\boldsymbol{B}$ is a sequence $(B_1, \ldots, B_k)$ of distinct members of $\boldsymbol{B}$ such that the intersection $B_j \cap (\bigcup_{i=1}^{j-1} B_i)$ is topologically a $(d-1)$-ball for $1 < j \leq k$ except that when $j = k = |\boldsymbol{B}|$ it may instead be a $(d-1)$-sphere. A partial shelling $(B_1, \ldots, B_k)$ is *maximal* if it is not an initial segment of another partial shelling, and is a *shelling* if $k = |\boldsymbol{B}|$. The collection $\boldsymbol{B}$ is *shellable* if it admits a shelling.

A shelling is an especially nice and useful way of assembling $\boldsymbol{B}$ from its component parts. As was observed in [28, pp. 141–142], the notion first appeared in the second half of the nineteenth century, when many of the early "proofs" of the Euler–Poincaré relation for a convex $(d+1)$-polytope were based on the then unproved assumption that the collection of all d-faces of such a polytope is shellable. The current knowledge of shellability is summarized in the present paper, where attention is confined to the case in which $\boldsymbol{B}$ is the set of all d-simplices in a d-dimensional simplicial complex. The following questions are of particular interest:

How efficiently can shellability be tested?
Are all 3-spheres shellable?
Are all 4-spheres shellable?
Are all combinatorial spheres shellable?

These questions are still unsettled. The purpose of this paper is to explain why they may be important, to suggest the algorithmic study of shellability as a subject for research, and to describe what little progress has thus far been made in that study.

The remaining sections of the paper are as follows: Definition and notation; Current knowledge of shellability and some related matters; Comments on the preceding section; An algorithm that finds all maximal extensions of a partial shelling; Computational results; Noted added in proof.

2. Definitions and notation

For the sake of simplicity, the present study of shelling is confined to the case in which the members of the collection $\boldsymbol{B}$ are all geometric simplices. For that case an

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equivalent purely combinatorial formulation is available, as described below. As the term is used here, a *complex* (often called an *abstract simplicial complex*) is a finite collection S of finite sets such that each subset of a member of S is itself a member of S. The $(d+1)$-sets that belong to S are called *d-simplices*, and S is *d-dimensional* if it includes a d-simplex but no $(d+1)$-simplex. A *geometric realization* $G(S)$ of S is defined in the usual way; the d-dimensional topological space $\bigcup G(S)$ is what is often called a *Euclidean polyhedron*.

A *polytope* is a subset P of a real vector space such that P is the convex hull of a finite set or, equivalently, is the bounded intersection of a finite number of closed halfspaces. The *faces* of P are the empty set, P itself, and the intersections of P with its various supporting hyperplanes. A $(d+1)$-dimensional polytope P is *simplicial* if its d-dimensional faces are all geometric simplices, and P's *boundary complex* is then the d-complex consisting of the vertex-sets of the various faces of P other than P itself.

When A is a subcomplex of the boundary complex S of a geometric d-simplex then $\bigcup G(A)$ is a topological $(d-1)$-ball if and only if A is generated by m $(d-1)$-simplices of S with $1 \leq m \leq d$, and $\bigcup G(A)$ is a topological $(d-1)$-sphere if and only if $A = S$. These characterizations underly the definitions in the next paragraph.

Let d be a positive integer. The role of the collection B of topological d-balls mentioned earlier is played here by a nonempty finite collection F of $(d+1)$-sets, called a *$(d+1)$-family*. The members of F are called *facets*, and a *partial shelling* of F is a sequence $(F_1, \ldots, F_k)$ of distinct members of F such that the following two conditions hold for $1 < j \leq k$ except that (b) may fail when $j = k = |F|$:

(a) for each $i < j$ there exists $h < j$ such that $|F_h \cap F_j| = d$ and $F_h \cap F_j \supset F_i \cap F_j$;

(b) there is a d-set in F_j that is not contained in F_i for any $i < j$.

The terms *maximal*, *shelling* and *shellable* are then defined as they were earlier.

For a $(d+1)$-family F, let $S(F)$ denote the d-complex consisting of all subsets of members of F, whence of course $F \subset S(F)$; and let $G(F) = \bigcup G(S(F))$. For $0 \leq m \leq d$, let $f_m(F)$ denote the number of $(m+1)$-sets (or *m-simplices*) that belong to $S(F)$. Of special interest are the parameters $v = f_0(F) = |\bigcup F|$, the number of *vertices* of F, and $f = f_d(F) = |F|$, the number of facets of F.

As the term is used here, a *d-ball* (resp. *d-sphere*) is a $(d+1)$-family F such that $G(F)$ is a topological d-ball (resp. d-sphere). A *combinatorial d-ball* (resp. *combinatorial d-sphere*) is a $(d+1)$-family F such that some simplicial subdivision of $G(S(F))$ is combinatorially equivalent to a simplicial subdivision of a geometric d-simplex (resp. of the boundary complex of a geometric $(d+1)$-simplex). A *convex d-sphere* is a $(d+1)$-family that is combinatorially equivalent to the boundary complex of a simplicial $(d+1)$-polytope.

When F is a $(d+1)$-family, a *ridge* of F is a d-set ($(d-1)$-simplex) that belongs to $S(F)$. A *d-pseudomanifold with boundary* is a $(d+1)$-family F such that each ridge of F lies in at most two facets. The *boundary* of F is then the d-family consisting of all ridges that lie in only one facet, and F is a *d-pseudomanifold* (resp. *d-manifold*) if its boundary is empty (resp. if $G(F)$ is connected and is locally homeomorphic with Euclidean d-space). In a similar manner, other terms from topology (e.g., *combinatorial d-manifold, homology d-manifold*) are used here to denote certain sorts of $(d+1)$-families. Henceforth, when no serious ambiguity results, the notations F, $S(F)$, $G(S(F))$ and $\bigcup G(S(F))$ may be used interchangeably.

The statement that *there exists an algorithm* for a decision problem means that the problem is effectively solvable in the sense of [18, pp. 41–42].

3. Current knowledge of shellability and some related matters

The present paper is motivated by the facts listed below, and an important part of its purpose is simply to assemble those facts. The statements all refer to $(d+1)$-families, but several of them actually apply to more general cell-complexes as can be seen by consulting some of the cited references. Also see the comments in the next section.

(1) If $(F_1, \ldots, F_f)$ is a shelling of a d-pseudomanifold F with boundary, then F is a combinatorial d-sphere or combinatorial d-ball according as condition (b) fails or holds when $j = f$, and according as F's boundary is empty or nonempty [36, p. 39] [16, pp. 107–108] [22, p. 444].

(2) Each convex d-sphere is shellable [20, p. 202], and in fact there always exist shellings that satisfy various strong restrictions on the order in which the facets appear [20, p. 203] [47, p. 183] [39, p. 8] [22, p. 449].

(3) There exists an algorithm that decides whether a given $(d+1)$-family F is a convex d-sphere [28, pp. 90–92], but for $d \geq 3$ no specific algorithm is known even when F is given as a combinatorial d-sphere.

(4) For a 3-family F, the following three conditions are equivalent: F is a shellable pseudomanifold; F is a 2-sphere; F is a convex 2-sphere.

(5) For a 2-pseudomanifold F with boundary, the following three conditions are equivalent: F is shellable; each partial shelling of F is an initial segment of a shelling; F is a 2-sphere or 2-ball [55, p. 1401] [50, p. 174] [60, pp. 913–914] [16, p. 107] [24].

(6) There is an algorithm of time-complexity $O(f_1(F))$ that tests the shellability of an arbitrary 2-pseudomanifold F with boundary and finds a shelling if one exists [24].

(7) There is a straightforward backtrack algorithm that tests the shellability of an arbitrary $(d+1)$-family and finds a shelling if one exists [24]. However, for each $d \geq 3$ it is unknown whether for some $p_d < \infty$ there exists an algorithm of time-complexity $O(|F|^{p_d})$ that tests the shellability of an arbitrary d-pseudomanifold (or even combinatorial d-sphere) F.

(8) For $d \geq 3$, each d-sphere with at most $d+4$ vertices is convex but there exist nonconvex d-spheres with $d+5$ vertices [45]. For $d \neq 4$, each d-manifold with at most $d+5$ vertices is a combinatorial d-sphere [4] [15].

(9) For $d \geq 3$ it is unknown whether there exists an algorithm that decides whether a given $(d+1)$-family (or even a given combinatorial d-manifold) is a d-sphere or d-ball [32, p. 149] [33, pp. 438–439]. (See Section 7.)

(10) For $d \geq 3$ there exist combinatorial d-balls that are not shellable [55, pp. 1403–1405] [26, pp. 361–364] [68] [59] [16, pp. 108–111].

(11) For $d \leq 3$ all d-spheres and d-balls are combinatorial [51]. It is unknown whether all 4-spheres and 4-balls are combinatorial. A recent announcement [25], supplemented by a personal communication from its author, implies that for $d \geq 5$ there exist d-spheres and d-balls which are not combinatorial and hence by (1) not shellable.

(12) It is unknown whether all 3-spheres are shellable. If they are (as has been conjectured by B. Grünbaum) then testing for shellability decides whether a given 3-pseudomanifold is a 3-sphere.

(13) It is unknown whether all 4-spheres are shellable. If they are then all are combinatorial and testing for shellability decides whether a given 4-pseudomanifold is a 4-sphere.

(14) For $d \geq 3$ it is unknown whether all combinatorial d-spheres are shellable. If they are then testing for shellability decides whether a given combinatorial d-manifold is a d-sphere.

4. Comments on the preceding section

The comments below are keyed to the numbered statements of the preceding section.

(2) The shellability of convex spheres, proved in [20] in response to a question of [28], was used in the study of Cohen–Macaulay rings [34, 35, 62, 64] and played a key role in the first complete proof of the upper bound result for convex polytopes [47, 48]. The latter provides sharp upper bounds for the numbers $f_k(\boldsymbol{F})$ $(1 \leq k \leq d)$ as $\boldsymbol{F}$ ranges over all convex d-spheres with a given number v of vertices; in particular,

$$(*) \qquad f_d(\boldsymbol{F}) \leq \binom{v-[d/2]}{v-d+1} + \binom{v-[(d+1)/2]}{v-d+1}.$$

The upper bound result was later extended [63] to arbitrary d-spheres by the use of heavy machinery from commutative algebra, having been proved earlier by more elementary methods [38] for all d-spheres with a sufficiently large number of vertices. Equality holds in $(*)$ precisely when the d-sphere $\boldsymbol{F}$ is *neighborly*, meaning that each set of $[(d+1)/2]$ points of $\bigcup \boldsymbol{F}$ lies in some member of $\boldsymbol{F}$ and hence is a member of $\boldsymbol{S}(\boldsymbol{F})$.

(2)(12) The cited results on shelling order are different from each other. The result of [22] is the most prescriptive, but here we are especially concerned with [20], which asserts that the first and last facets in the shelling of a convex d-sphere can be specified arbitrarily. Grünbaum has conjectured (in a private communication) the same is true of an arbitrary 3-sphere, and that has been verified for all 3-spheres with at most 9 vertices. See the final section of this paper.

(3) For each $d \geq 3$, a purely combinatorial characterization of convex d-spheres (which must exist by [28]) would be of great interest if it were not too complicated to be useful. (See (4)–(5) for $d = 2$.) Though there exist nonconvex shellable 3-spheres, it seems conceivable that the convexity of a d-sphere can be characterized in terms of the existence of a sufficiently rich collection of shellings such as described in [22], or by extendable shellability as defined below. However, such conditions are difficult to check even for particular examples. (See also [70–73].)

(3)(9) Like the inequality $(*)$ above, several other combinatorial properties first established for convex d-spheres were later extended to arbitrary d-spheres and even to arbitrary d-manifolds [12, 13]. See also [28, 29]. Of special interest is the lower bound result [28, 69, 10, 11, 12, 40], which provides sharp lower bounds for the numbers $f_k(\boldsymbol{F})$ $(1 \leq k \leq d)$ as $\boldsymbol{F}$ ranges over all convex d-spheres with a given number v of vertices; in particular,

$$(\dagger) \qquad f_d(\boldsymbol{F}) \geq (v-d-1)d + 2.$$

(See [49] for a far-reaching conjectured extension of the lower-bound result.) Equality holds in (†) precisely when the d-sphere $\boldsymbol{F}$ is *stacked*, meaning that it can be obtained from the boundary complex of a $(d+1)$-simplex by successive replacements of facets by pyramids over them (that is, replace a facet $F \in \boldsymbol{F}$ by the $d+1$ facets $R \cup \{p\}$, where p is a point not in $\bigcup \boldsymbol{F}$ and R ranges over the d-sets in F).

(4) It follows from a theorem of [65] (see [28, pp. 235–242]) that, in our special terminology, all 2-spheres are convex. In conjunction with (1) and (2), that establishes (4).

(5) Let us say that a $(d+1)$-family $\boldsymbol{F}$ is *extendably shellable* if every partial shelling of $\boldsymbol{F}$ is an initial segment of a shelling. By (5), each 2-sphere is extendably shellable. H. Tverberg has asked whether, for $d \geq 3$, each convex d-sphere is extendably shellable. (See Section 7.)

(6) In (6) and elsewhere in this paper, estimates of complexity are based on the RAM model of random access computation [3, pp. 5–14], using the uniform cost criterion.

(5)–(7) Let us refer to the sort of d-pseudomanifolds considered here, or to their topological analogues, as *simplicial* d-pseudomanifolds. As can be seen from [24], (5) and (6) remain valid (when $d = 2$) for much more general structures, which are here called *cell* d-pseudomanifolds. (The precise definition when $d = 2$ can be inferred from [24]. For general d it is a bit involved and "noncombinatorial," so we do not give it in detail but only remark that it does not require the cells (topological d-balls) to have connected intersections.) It can be verified that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv), where these statements are as follows:

(i) each cell d-sphere is extendably shellable;

(ii) each facet of a cell d-sphere is the first facet of a shelling;

(iii) each simplicial d-sphere is extendably shellable;

(iv) the shellability of a simplicial d-pseudomanifold F can be tested by an algorithm of time-complexity $O(|F|^3 d)$.

It is known that (i) is true for $d \leq 2$ and (iii) is false for $d \geq 5$ (see (11)). What happens when d is 3 or 4? (See Section 7.)

From the shellability of cell 2-balls it follows that if B is a cell d-sphere and $d = 2$, then

(v) each facet of B is the last facet of a shelling.

For an arbitrary d, (v) holds by [20] when B is the boundary complex of a $(d+1)$-polytope. A simple construction based on (10) shows that when $d \geq 3$, (v) does not apply to all cell d-spheres (in the general sense indicated above), but it is not clear what happens when d is 3 or 4 and the cell d-sphere B is a cell-complex in the sense of [2].

(8) The result of [45] is extended in [41] to nonsimplicial spheres, and in [42] the analogous results for symmetric spheres are established. [15] shows that a homology d-manifold with at most $d+5$ vertices is a combinatorial d-sphere when $d \neq 4$, and when $d = 4$ is a combinatorial d-sphere or not a homology d-sphere. It would be interesting to know, for each d, what is the minimum number of vertices and of facets for a nonorientable d-manifold, and for an orientable d-manifold that is not a sphere. For $d = 2$, a more general problem was solved by [58]. For $d = 3$ the numbers of vertices are respectively 9 and 10 [6, 7].

(7) It would not surprise us to learn that for some $d \geq 3$ the problem of deciding whether a d-pseudomanifold (or even a d-sphere) is shellable is NP-complete in the sense of [21] [37] [3, p. 373]. On the other hand, it is conceivable that there is an algorithm of polynomial time-complexity $p(d, f)$ such that, given any d-pseudomanifold F with f facets, the algorithm decides whether F is shellable. Certainly there is such an algorithm that either finds a shelling or concludes F is not extendably shellable. (See the next section of this paper.)

(7)(9)(12) Note that a d-sphere F is unshellable if and only if the d-ball $F-\{F\}$ is unshellable for each $F \in F$. Even if it turns out that unshellable 3-spheres exist, it will still be of interest to be able to test pseudomanifolds for shellability as efficiently as possible. When a pseudomanifold is suspected of being a sphere, it is reasonable to check shellability as a first step toward verifying the suspicion, especially if a good algorithm is available. (In [4] a 3-pseudomanifold with 10 vertices and 35 facets is proved to be a sphere by showing it is shellable.) Further, there may exist an algorithm A which accepts as input a combinatorial manifold F with boundary and produces as output another combinatorial manifold F_A with boundary such that $G(F_A)$ is homeomorphic with $G(F)$ and F_A is shellable if it is a sphere or ball. Application of A, followed by testing F_A for shellability, would then constitute an algorithm for deciding when F is a sphere or ball. Though no such algorithm A is known, the following facts seem to favor its existence, at least in the important case $d = 3$.

(i) If F is combinatorial d-sphere or combinatorial d-ball then F admits a simplicial subdivision E that is shellable [60, 20]. When $d = 3$ and the 3-ball F is geometrically realized in Euclidean 3-space, $|E|$ can be bounded by an exponential (though perhaps not by any polynomial?) function of $|F|$ [67].

(ii) If F is a 3-manifold with boundary there is a 3-manifold E with boundary such that (α) the boundary of E is equal to the boundary of F, (β) $G(E)$ is homeomorphic with $G(F)$, and (γ) each 3-ball in E is shellable [52]. To satisfy (γ) it suffices to take for E a minimum 3-manifold (one minimizing $|E|$) that satisfies (β) [66].

(Some of the language of the preceding paragraph, and of other parts of this paper, may seem strange to topologists among the readers. That is because we have chosen to emphasize the purely combinatorial viewpoint.)

(9) The 3-dimensional *Poincaré conjecture* [56, 16, 32] asserts that a 3-manifold F is a sphere if it is simply connected — that is, if the one-dimensional homotopy $\pi_1(F)$ is trivial. If one believes the conjecture to be false and has a candidate for a counterexample, there arises the necessity of showing it is simply connected and is not a sphere. However, not only is it unknown whether there exists a finite algorithm for deciding whether a 3-manifold is a sphere, but it is also unknown whether there exists a finite algorithm for deciding whether a 3-manifold is simply connected. [31] shows these decision problems are both equivalent to purely group-theoretic decision problems, but in view of the next paragraph, that is not in itself reassuring. See [32, 17] for partial results on the problem of deciding algorithmically when a 3-manifold is a sphere, see [54] for simple connectedness.

(9) [1, 57] proved the *isomorphism problem* for finitely presented groups is recursively unsolvable — there is no algorithm which accepts an arbitrary pair of presentations and decides whether the associated groups are isomorphic. [46] gave an algorithm for assigning, to each finite presentation P of a group G_P, a 4-manifold M_P with $\pi_1(M_P) = G_P$, the procedure being such that M_Q and M_P are homeomorphic if and only if the groups G_Q and G_P are isomorphic. That showed there is no algorithm for deciding when two 4-manifolds are homeomorphic. See [33, 18] for further information about topological decision problems.

(9)(13)(14) From [57] it follows that the *triviality problem* for finitely presented groups is recursively unsolvable — there is no algorithm which accepts an arbitrary presentation and decides whether the group is trivial. It follows from results of [18] that the isomorphism problem is unsolvable even in the case of *balanced presentations* (those having the same number of relations as generators), but it is unknown whether this is true of the triviality problem. The question is of interest because of connections with both the 3-dimensional and the 4-dimensional Poincaré conjecture, where the latter asserts that a 4-manifold F is a sphere if both $\pi_1(F)$ and $\pi_2(F)$ are trivial. In particular, E. Brown has observed (private communication) that if (a) the 4-dimensional Poincaré conjecture is true and (b) the triviality problem is unsolvable for balanced presentations, then (c) there is no finite algorithm for deciding whether a combinatorial 4-manifold is a sphere. It follows that if every combinatorial 4-sphere is shellable (or if, for combinatorial 4-manifolds there is an algorithm A of the sort described under (7)(9)(12) above), then (a) fails or (b) fails.

(10)(12) In an unpublished note, Branko Grünbaum points out that in [44], the "proof" of an interesting result tacitly assumes an affirmative answer to the following open question: Is every 3-ball with more than one facet the union of two facet-disjoint 3-balls? Grünbaum also asks: Is there a k such that every 3-sphere is the union of k shellable 3-balls, no two of which have a common facet? What about $k = 2$?

(10) If a facet F of a 3-ball $\boldsymbol{F}$ is such that $G(\boldsymbol{F} \sim \{F\})$ is not a topological 2-ball, then no shelling of $\boldsymbol{F}$ ends with F. If $G(\boldsymbol{F} \sim \{F\})$ is bad for every $F \in \boldsymbol{F}$, then $\boldsymbol{F}$ is said to be *strongly unshellable*. Most of the examples of unshellable 3-balls are strongly unshellable, and their constructions rely so heavily on the presence of $\boldsymbol{F}$'s boundary that they seemingly offer little chance of being extended to the construction of an unshellable 3-sphere. However, the second example of [16] is based on knottedness considerations that seem to offer a better chance of being extended. It would be of interest to know what is the minimum number of vertices, and of facets, for an unshellable 3-ball.

(10) Unshellable 3-balls provide good test problems for the development of efficient shelling algorithms. We do not know of any algorithm that will demonstrate the unshellability of even one such ball in a "reasonable" time. (See [23] for more detailed information.) The example of [59] has 14 vertices and 41 facets and is geometrically realized in Euclidean 3-space as a subdivision of a tetrahedron into 41 smaller tetrahedra. Representing the U_i's of [59] by 1 2, X_i's by 3 4 5 6, Y_i's by 7 8 9 10, and Z_i's by 11 12 13 14, the vertex-sets of the 41 tetrahedra are 3 4 7 11, 4 5 8 12, 5 6 9 13, 6 3 10 14, 3 4 7 12, 4 5 8 13, 5 6 9 14, 6 3 10 11, 4 7 11 12, 5 8 12 13, 6 9 13 14, 3 10 14 11, 4 8 11 12, 5 9 12 13, 6 10 13 14, 3 7 14 11, 11 12 13 14, 7 11 12 13, 8 12 13 14, 9 13 14 11, 10 14 11 12, 3 7 12 13, 4 8 13 14, 5 9 14 11, 6 10 11 12, 3 9 12 13, 4 10 13 14, 5 7 14 11, 6 8 11 12, 1 3 9 13, 2 4 10 14, 1 5 7 11, 2 6 8 12, 1 3 7 13, 2 4 8 14, 1 5 9 11, 2 6 10 12, 1 7 11 13, 2 8 12 14, 1 9 13 11, 2 10 14 12.

The smallest known example of an unshellable 3-ball, due to Grünbaum (unpublished), has 14 vertices and only 29 facets. The vertex-sets of its facets are 1 2 3 7, 1 2 4 8, 1 2 7 8, 1 3 5 7, 1 4 8 10, 1 5 6 13, 1 5 7 13, 1 6 11 13, 1 7 8 9, 1 7 11 13, 2 3 7 9, 2 4 6 8, 2 5 6 14, 2 5 12 14, 2 6 8 14, 2 7 8 9, 2 8 12 14, 3 5 7 9, 4 6 8 10, 5 6 13 14, 5 7 9 13, 5 12 13 14, 6 8 10 14, 6 11 13 14, 7 8 9 13, 7 8 10 14, 7 8 13 14, 7 11 13 14, 8 12 13 14.

Another good test problem is provided by the nonspherical 3-manifold with 9 vertices and 27 facets described in [7]. The vertex-sets of its facets are 1236, 1238, 1245, 1247, 1258, 1267, 1345, 1346, 1358, 1469, 1479, 1679, 2345, 2346, 2359, 2389, 2467, 2589, 3578, 3579, 3789, 4678, 4689, 4789, 5678, 5679, 5689.

(11) As the term is used here, a *Poincaré d-sphere* is a d-manifold that is not a d-sphere but has the same homology groups as a d-sphere. The *suspension* $\Sigma \boldsymbol{F}$ of a $(d+1)$-family $\boldsymbol{F}$ is the $(d+2)$-family

$$\{F \cup \{p\} : F \in \boldsymbol{F}\} \cup \{F \cup \{q\} : F \in \boldsymbol{F}\},$$

where p and q are distinct points not in $\boldsymbol{F}$, and the *double suspension* of $\boldsymbol{F}$ is the $(d+3)$-family $\Sigma(\Sigma \boldsymbol{F})$. [25] announced that the double suspension of a certain Poincaré 3-sphere is a 5-sphere, and stated in a later letter that for $d \geq 4$ the double suspension of an arbitrary Poincaré d-sphere is a $(d+2)$-sphere. Though they are spheres, these double suspensions are not combinatorial manifolds and hence are not shellable. For background material on the double suspension problem, see [27] and some of its references.

(10)–(14) Following [22, 23], we define a *partial semishelling* of a $(d+1)$-family $\boldsymbol{F}$ as a sequence $(F_1, \ldots, F_k)$ of facets that satisfies condition (a) of the definition of shelling. (When $\boldsymbol{F}$ is a pseudomanifold, the partial semishellings are identical with the partial shellings.) *Semishellings* and *semishellability* are defined in the obvious way. It would be of interest to study the relationship of semishellability to the notion of constructibility employed by [34, 35, 62, 63]. A complex is *constructible* if it belongs to the smallest class K of complexes such that

(a) if a complex consists of a simplex and all its faces, or is the boundary of a simplex, then it belongs to K;

(b) if C_1 and C_2 are d-dimensional members of K, and $C_1 \cap C_2$ is a member of K of dimension $d-1$, then $C_1 \cup C_2$ belongs to K.

The following questions were suggested by R. Stanley. (The first has been answered affirmatively in [74].)

(i) If $\boldsymbol{S}$ is the collection of all linearly independent subsets of a finite subset of a vector space (more generally, if $\boldsymbol{S}$ is the collection of all independent sets in a matroid), then $\boldsymbol{S}$ is constructible. Must $\boldsymbol{S}$ be semishellable?

(ii) Are the known examples of unshellable balls and spheres constructible?

(iii) Is every sphere constructible?

(iv) Is every constructible complex semishellable? (In view of (11), the answers to (iii) and (iv) cannot both be affirmative.

5. An algorithm that finds all maximal extensions of a partial shelling

The backtrack algorithm described here has been used to settle specific shelling problems, and may serve as a starting point for any reader who wants to continue the algorithmic study of shelling. It is presented first by means of a pidgin ALGOL program and then, in order to clarify certain aspects and to facilitate its actual use and comparison with other algorithms for the same purpose, by means of a complete program written in ALGOL W, the version of ALGOL developed at Stanford University. Starting from a given partial shelling $(F_1, \ldots, F_k)$ of a d-pseudomanifold $\boldsymbol{F}$ with (possibly empty) boundary, the algorithm finds all maximal partial shellings $(F_1, \ldots, F_m)$ that have $(F_1, \ldots, F_k)$ as an initial segment. It seems to be a fairly efficient tool for that purpose and also, when suitably modified, for finding a single maximal partial shelling. Of course it can also be used to test shellability *per se*, but probably is inefficient for that purpose except in settings where shellability implies extendable shellability. A goal of future research should be to find a shellability test which, by means of a clever idea or a deeper understanding of shellability, avoids the direct confrontation of a large number of permutations that is implicit in the approach used here. (See the comments under (7) in the preceding section.)

In the programs, SHELL is the current partial shelling and |SHELL| is its length. CAND consists of all facets which, though not in SHELL, are candidates for addition to SHELL by virtue of being adjacent to some member of SHELL, and ACTIVE consists of all members of CAND which have not yet been tested for addition to the current SHELL. In the ALGOL W program, SHELL is maintained as a stack with pointer SHELLEND, CAND as a doubly linked list with forward linkage FLINK and backward linkage BLINK, and ACTIVE as a terminal segment of CAND accessed from a variable NEXT. There is no output when the initial partial shelling is already maximal.

```
1.  begin
2.    SHELL ← (F_1, ..., F_k);
3.    CAND ← set of all facets of the pseudomanifold F adjacent to SHELL but not in it;
4.    ACTIVE ← CAND;
5.    NEWSTART ← START ← |SHELL|;
6.    while NEWSTART ≥ START do
7.      if ACTIVE is not empty
8.        then begin
9.              TRY ← first member of ACTIVE;
10.             if TRY fails the shelling test relative to SHELL
11.               then ACTIVE ← ACTIVE ~ {TRY}
12.               else begin
13.                     SHELL ← SHELL ∪ {TRY};
14.                     NBRS ← set of all facets adjacent to TRY but
15.                           not in SHELL ∪ CAND;
16.                     CAND ← (CAND ~ {TRY}) ∪ NBRS;
17.                     ACTIVE ← CAND
18.                   end
19.            end
20.       else begin
21.             if |SHELL| > NEWSTART then
22.               print SHELL as a new maximal extension;
23.             DROP ← last member of SHELL;
24.             SHELL ← SHELL ~ {DROP};
25.             NBRS ← set of all facets adjacent to DROP but not to SHELL;
26.             CAND ← (CAND ∪ {DROP}) ~ NBRS;
27.             ACTIVE ← ACTIVE ~ NBRS;
28.             NEWSTART ← |SHELL|
29.            end
30. end
```

To find a single maximal extension of $(F_1,\ldots,F_k)$, replace 6–8 by 6.1 **while** ACTIVE is not empty **do begin**, omit 20–21, and omit 23–29. To find all maximal partial shellings of F, insert 0.1. **begin for** $i \leftarrow 1$ **until** $|F|$ **do**, 31 **end**, and replace 2 by 2.1 SHELL $\leftarrow (F_i)$. To test $\boldsymbol{F}$ for shellability, modify the program for finding all

maximal partial shellings by replacing 21–22 with 21.1. **if** |SHELL| = $|\boldsymbol{F}|$ **then begin write** ("F is shellable"); **goto** EXIT **end** and insert 29.1. **write** ("F is not shellable") **end**, 29.2 EXIT:.

We are primarily interested in pseudomanifolds, and the partial shellings of a pseudomanifold are identical with its partial semishellings. Hence the program below tests only condition (a) while ignoring condition (b). In general, the program applies to a $(d+1)$-family in which each facet is adjacent to at most $d+1$ other facets, and it then finds all maximal partial semishellings that extend a given initial partial semishelling.

In the basic step of the program, a partial semishelling $(F_1,\ldots,F_{j-1})$ is at hand, a facet X has been chosen such that X is adjacent to at least one F_i but not equal to any F_i, and it is desired to know whether $(F_1,\ldots,F_{j-1},X)$ is a partial semishelling. Let

$$H=\{h : h \leqslant j \quad \text{and} \quad |F_h \cap X| = d\}$$

and for each $h \in H$ let p_h denote the sole point of $X \sim F_h$. Condition (a) requires that for each $i<j$ there exist $h \in H$ such that $F_h \cap X \supset F_i \cap X$ or, equivalently, $p_h \notin X$. To test this directly for all $i \notin H$ involves checking at least $j-1-|H|$ and perhaps as many as $|H|(j-1-|H|)$ inclusions. However, one may instead form the set

$$X_H = X \sim \bigcap_{h\in H} F_h = \{p_h : h \in H\}$$

and test the equivalent requirement that for each $i<j$, $X_H \not\subset F_i$. Once Y has been formed, this can be tested for all $i \notin H$ by checking only $j-1-|H|$ inclusions. Since $|H| \leqslant d+1$, this device offers little advantage when d is small, but it is advantageous for large d and in a modified form is incorporated in the ALGOL W program.

Input for the ALGOL W program is assumed to consist of a string of positive integers subject to certain restrictions indicated below. The input string is processed as if it were partitioned into segments as follows:

$$D\,|\,F\,|\,\text{VERTEXSETS}\,|\,\text{START}\,|\,\text{SHELLSTART}.$$

Here D is the dimension, F the number of facets, and START the number of facets in the initial partial shelling that is to be extended. Vertices are represented by positive integers not exceeding the computer's word-length. With $C=D+1$, the segment VERTEXSETS is of length CF and lists the vertex-sets of the successive facets. The facets are regarded as indexed successively from 1 to F. The segment SHELLSTART is of length START and lists the indices of the facets in the initial partial shelling.

```
1.  BEGIN
2.    INTEGER D, F, C, I;
3.    BITS ARRAY MASK (1::9); COMMENT 9 MAY BE REPLACED (HERE AND IN 14, 17)
4.        BY ANY POSITIVE INTEGER W NOT EXCEEDING THE COMPUTER'S WORD-
5.        LENGTH. THE PROGRAM THEN HANDLES (D+1)-FAMILIES IN WHICH EACH
6.        VERTEX IS REPRESENTED BY A POSITIVE INTEGER < = W AND NO FACET IS
7.        ADJACENT TO MORE THAN D+1 OTHER FACETS. IT THEN FINDS ALL MAXIMAL
8.        PARTIAL SEMISHELLINGS THAT HAVE THE GIVEN PARTIAL SEMISHELLING AS
9.        AN INITIAL SEGMENT. LATER COMMENTS REFER TO PARTIAL SHELLINGS
10.       RATHER THAN PARTIAL SEMISHELLINGS BECAUSE THE TWO ARE
11.       IDENTICAL FOR PSEUDOMANIFOLDS AND THAT IS THE CASE OF GREATEST
12.       INTEREST.;
13.   BITS ZERO; ZERO:=BITSTRING(0);
14.   FOR I:=1 UNTIL 9 DC MASK(I):=BITSTRING(ENTIER(2**(I-1)));
15. NEXTCASE:
16.   READON (D,F); C:=D+1;
17.   INTFIELDSIZE:= -ENTIER(-LOG(1+(IF 9<F THEN F ELSE 9))); COMMENT
18.       THIS IS USED TO COMPACTIFY THE OUTPUT.;
19. BEGIN
20.   INTEGER ARRAY ADJ(1::F,1::C); COMMENT ADJ(I,1::C) IS FIRST
21.       USED TO READ IN THE VERTICES OF THE I-TH FACET, BUT DURING
22.       MOST OF THE COMPUTATION IT LISTS THE INDICES OF ALL FACETS
23.       ADJACENT TO THE I-TH ONE.;
24.   BITS ARRAY EM(1::C); BITS INTERSECT; COMMENT THESE ARE USED IN
25.       SETTING UP THE ADJACENCY LISTS.;
26.   INTEGER ARRAY SHELL, CHECK, NEW(1::F); COMMENT SHELL RECORDS THE
27.       INDICES OF THE SUCCESSIVE FACETS IN THE CURRENT PARTIAL
28.       SHELLING. CHECK(I) IS THE LOCATION IN SHELL AT WHICH
29.       TESTING MUST BEGIN TO DETERMINE WHETHER THE I-TH FACET CAN
30.       BE ADDED AT THE END OF THE CURRENT PARTIAL SHELLING. (THE
31.       FIRST TEST INVOLVES OMADJ(I) AND FACET(SHELL(CHECK(I))).)
32.       FOR START<I<=F, NEW(I) IS THE NUMBER OF NEW FACETS ADDED TO THE
33.       CANDIDATE LIST WHEN THE I-TH MEMBER IS ADDED TO THE PARTIAL
34.       SHELLING.;
35.   BITS ARRAY FACET, OMADJ(1::F); COMMENT FACET(I) IS A BITSTRING
36.       OF WEIGHT C, THE POSITIONS OF THE I'S INDICATING THE C
37.       VERTICES OF THE I-TH FACET. OMADJ(I) CONSISTS OF 0'S
38.       EXCEPT FOR A 1 CORRESPONDING TO EACH VERTEX OF THE I-TH
39.       FACET THAT IS OMITTED BY A FACET IN THE CURRENT PARTIAL
40.       SHELLING WHICH IS ADJACENT TO THE I-TH ONE AND APPEARS IN THE
41.       SHELLING BEFORE THE I-TH ONE. OMADJ IS USED IN THE ALTERNATE
42.       FORM OF THE SHELLING TEST DESCRIBED IN THE TEXT.;
43.   LOGICAL ARRAY USED, CAND(1::F); COMMENT THESE INDICATE RESPECT-
44.       IVELY FACETS THAT ARE USED IN THE CURRENT PARTIAL SHELLING
45.       AND THOSE THAT ARE NOT USED BUT ARE CANDIDATES FOR USE BY
46.       VIRTUE OF BEING ADJACENT TO USED FACETS.;
47.   INTEGER ARRAY FLINK,BLINK(0::F); COMMENT FLINK IS A FORWARD
48.       LINKAGE WHICH, WHEN ACCESSED FROM 0, LEADS TO THE INDICES
49.       OF CANDIDATE FACETS, AND WHEN ACCESSED FROM THE VARIABLE
50.       NEXT LEADS TO CANDIDATES THAT ARE CURRENTLY ACTIVE. BLINK
51.       IS A BACKWARD LINKAGE USED IN UPDATING FLINK.;
52.   INTEGER START, NEWSTART, SHELLEND, NEXT, CTR, J, K, L, M; COMMENT
53.       START IS THE NUMBER OF FACETS IN THE INITIAL PARTIAL
54.       SHELLING THAT IS TO BE EXTENDED. NEWSTART, USED IN BACK-
55.       TRACKING, PLAYS A SOMEWHAT SIMILAR ROLE. SHELLEND IS THE
```



```
56.       LENGTH OF THE CURRENT PARTIAL SHELLING. FLINK(NEXT) IS
57.       (WHEN NOT 0) THE INDEX OF THE NEXT FACET TO BE TESTED FOR
58.       ADDITION TO THE CURRENT PARTIAL SHELLING. CTR IS A COUNTER
59.       USED IN SETTING UP THE ADJACENCY LISTS.;
60.   COMMENT FIRST THE VERTEX-SETS ARE READ IN, THE FACETS ARE
61.       RECORDED AS BITSTRINGS, AND THE ADJACENCY LISTS ARE SET
62.       UP.;
63.   FOR I:=1 UNTIL F DO FOR J:=1 UNTIL C DO READON (ADJ(I,J));
64.   FOR I:=1 UNTIL F DO
65.     BEGIN
66.       FACET(I):=ZERO;
67.       FOR J:=1 UNTIL C DO FACET(I):=FACET(I) OR MASK(ADJ(I,J))
68.     END;
69.   FOR I:=1 UNTIL F DO
70.     BEGIN
71.       L:=0;
72.       FOR K:=1 UNTIL C DO EM(K):=MASK(ADJ(I,K));
73.       FOR J:=1 UNTIL F DO
74.         BEGIN
75.           CTR:=0;
76.           INTERSECT:=FACET(I) AND FACET(J);
77.           FOR K:=1 UNTIL C DO
78.             IF EM(K)=(EM(K) AND INTERSECT) THEN CTR:=CTR+1;
79.           IF CTR=0 THEN BEGIN L:=L+1; ADJ(I,L):=J END
80.         END;
81.       WHILE L<0 DO BEGIN L:=L+1; ADJ(I,L):=0 END
82.     END;
83.   COMMENT NEXT THE INITIAL PARTIAL SHELLING IS READ IN, CERTAIN
84.       ARRAYS ARE INITIALIZED, THE CANDIDATE LINKAGES ARE SET UP,
85.       AND THE INITIAL ADJUSTMENTS OF OMADJ ARE MADE.;
86.   READON (START);
87.   FOR I:=1 UNTIL START DO READON (SHELL(I));
88.   FOR I:=1 UNTIL F DO BEGIN
89.                           USED(I):=CAND(I):=FALSE;
90.                           CHECK(I):=1; OMADJ(I):=ZERO
91.                         END;
92.   FOR I:=1 UNTIL START DO USED(SHELL(I)):=TRUE;
93.   SHELLEND:=NEWSTART:=START;
94.   NEW(START):=0;
95.   FLINK(0):=BLINK(0):=NEXT:=0;
96.   FOR I:=1 UNTIL START DO
97.     BEGIN
98.       K:=SHELL(I);
99.       L:=1;
100.      WHILE (L<=C) AND (ADJ(K,L)¬=0) DO
101.        BEGIN
102.          J:=ADJ(K,L);
103.          L:=L+1;
104.          IF ¬USED(J) THEN
105.            BEGIN
106.              OMADJ(J):=OMADJ(J) OR (FACET(J) AND ¬FACET(K));
107.              IF ¬CAND(J) THEN BEGIN
108.                                  CAND(J):=TRUE;
109.                                  FLINK(BLINK(0)):=J; FLINK(J):=0;
110.                                  BLINK(J):=BLINK(0); BLINK(0):=J
```

```
                    END
        END
      END
  END;
COMMENT NOW THE MAIN PART OF THE COMPUTATION BEGINS. FOR EASIER
    UNDERSTANDING, COMPARE THE STEPS BELOW WITH THOSE IN THE
    EARLIER PIDGIN ALGOL PROGRAM.;
WHILE NEWSTART> =START DO
  IF FLINK(NEXT)¬=0
    THEN BEGIN
      K:=FLINK(NEXT); COMMENT NOW TEST THE KTH FACET FOR
          POSSIBLE ADDITION AT THE END OF THE CURRENT
          PARTIAL SHELLING.;
      WHILE (CHECK(K)< =SHELLEND) AND
          ((OMADJ(K) AND ¬FACET(SHELL(CHECK(K))))¬=ZERO)
          DO CHECK(K):=CHECK(K)+1;
      IF CHECK(K)< =SHELLEND THEN NEXT:=FLINK(NEXT)
        ELSE
          BEGIN COMMENT ADD THE KTH FACET TO THE PARTIAL
              SHELLING AND DROP IT FROM THE CANDIDATE
              LIST.;
            SHELLEND:=SHELLEND+1;
            SHELL(SHELLEND):=K;
            USED(K):=TRUE; CAND(K):=FALSE;
            FLINK(NEXT):=FLINK(K); BLINK(FLINK(K)):=NEXT;
            NEXT:=0;
            COMMENT ENLARGE THE CANDIDATE LIST BY ADDING
                FACETS THAT ARE ADJACENT TO THE KTH ONE
                BUT ARE NOT ALREADY CANDIDATES. THE
                NUMBER OF NEW CANDIDATES IS RECORDED IN
                NEW FOR USE IN BACKTRACKING.;
            L:=1; M:=0;
            WHILE (L< =0) AND (ADJ(K,L)¬=0) DO
              BEGIN
                J:=ADJ(K,L);
                L:=L+1;
                IF ¬USED(J) THEN
                  BEGIN
                    OMADJ(J):=OMADJ(J) OR
                        (FACET(J) AND ¬FACET(K));
                    IF ¬CAND(J) THEN
                      BEGIN
                        M:=M+1;
                        CAND(J):=TRUE;
                        FLINK(BLINK(0)):=J; FLINK(J):=0;
                        BLINK(J):=BLINK(0); BLINK(0):=J
                      END
                  END
              END;
            NEW(SHELLEND):=M
          END
    END
  ELSE BEGIN
      IF SHELLEND>NEWSTART THEN
        BEGIN
```



```
              WRITE ("A MAXIMAL PARTIAL SHELLING THAT EXTENDS THE
ORIGINAL ONE IS"); WRITE (" ");
              FOR I:=1 UNTIL SHELLEND DO WRITEON (SHELL(I))
            END;
          IF SHELLEND=START THEN GO TO NEXTCASE;
          COMMENT IN BACKTRACKING, THE LAST FACET K=SHELL(SHELLEND)
              IS DROPPED FROM THE CURRENT PARTIAL SHELLING.;
          NEXT:=K:=SHELL(SHELLEND);
          USED(K):=FALSE;
          CHECK(K):=SHELLEND;
          COMMENT FOR FACETS J IN THE CANDIDATE LIST ADJACENT TO
              FACET K, CHECK(J) AND OMADJ(J) ARE ALTERED TO TAKE
              ACCOUNT OF THE REMOVAL OF K FROM THE CURRENT PARTIAL
              SHELLING.;
          L:=1;
          WHILE (L< =C) AND (ADJ(K,L)¬=0) DO
            BEGIN
              J:=ADJ(K,L);
              L:=L+1;
              IF CAND(J) THEN
                BEGIN
                  CHECK(J):=1;
                  OMADJ(J):=OMADJ(J) AND ¬(FACET(J) AND ¬FACET(K));
                END
            END;
          COMMENT FACETS ARE DROPPED FROM THE CANDIDATE LIST IF THEY
              ARE NOT ADJACENT TO ANY MEMBER OF THE CURRENT PARTIAL
              SHELLING OTHER THAN K.;
          FOR M:=1 UNTIL NEW(SHELLEND) DO
            BEGIN
              J:=BLINK(0);
              CAND(J):=FALSE;
              BLINK(0):=J:=BLINK(J);
              FLINK(J):=0
            END;
          COMMENT FINALLY, K IS RETURNED TO THE CANDIDATE LIST.;
              CAND(K):=TRUE;
              FLINK(BLINK(K)):=K; BLINK(FLINK(K)):=K;
              NEWSTART:=SHELLEND:=SHELLEND-1
        END
END
END.
```

6. Computational results

Let us say that a $(d+1)$-family $\boldsymbol{F}$ is *strongly shellable* if for each pair of facets $X, Y \in \boldsymbol{F}$ there is a shelling of $\boldsymbol{F}$ that starts with X and ends with Y. As was mentioned earlier, all convex spheres are strongly shellable [20]. Since all 2-spheres are convex [65, 28], and for $d \geq 3$ all d-spheres with at most $d+4$ vertices are convex [45], the simplest spheres whose shellability is of interest are the nonconvex 3-spheres with 8 vertices. Computations of [19, 30, 13, 4, 5] show that up to combinatorial equivalence there are 39 3-manifolds with 8 vertices, all are spheres and 37 of them are convex. Three of the convex spheres and one of the nonconvex spheres are neighborly (each pair of vertices joined by an edge). The two nonconvex spheres are the neighborly 4-family M of [30] and the 4-family M' of [13]. They were coded as follows and both were found to be strongly shellable.

Nonconvex 3-sphere M (8 vertices, 20 facets):

1234	1237	1248	1267	1268	1347	1478	1567	1568	1578
2345	2358	2367	2368	2458	3456	3467	3568	4567	4578

Nonconvex 3-sphere M' (8 vertices, 19 facets):

1237	1238	1245	1247	1258	1346	1348	1367	1458	1467
2356	2357	2368	2457	2568	3468	3567	4567	4568	

As with 8 vertices, the 3-manifolds with 9 vertices were determined in several stages. First [6, 61] the 23 neighborly convex spheres were found, and later [7] the remaining neighborly 3-manifolds with 9 vertices were found to consist of 27 nonconvex spheres and one nonsphere. Finally ([8] and later additions) there were found to be 1246 non-neighborly 3-manifolds with 9 vertices; all were spheres, 1057 convex, 115 nonconvex, and 74 undecided. They are not all listed in [8], but Steinberg was kind enough to supply a detailed catalog. Of the 1296 3-spheres with 9 vertices, the 142 nonconvex ones (27 of which are neighborly) and the 74 undecided ones were tested and all were found to be strongly shellable. (Some of the 74 undecided cases were later decided by Steinberg.)

The shelling tests described in [23] used a modification of the backtrack program presented there. However, after the report was written it was discovered that, on the spheres in question, the program was never actually forced to backtrack; in each case it simply added new facets until a shelling was obtained. Also, it was discovered by Altshuler and Steinberg that their original catalog had been incomplete. They supplied the missing spheres, and the nonconvex and undecided ones among them were shown to be strongly shellable by a modification, with no provision for backtracking, of the program that appears in the present paper. Included among the new spheres were 45 nonconvex ones and 22 undecided ones with 9 vertices and 26 facets each. To establish their strong shellability required the computation of 67 different adjacency matrices ADJ(1::26, 1::4) and the discovery of 43,550 shellings, each with specified starting and ending facets. The execution time on the IBM 370/168 was about 4 seconds.

7. Notes added in proof

After this survey article had gone to the printer, we learned of several additional references that should have been included. Some are mentioned briefly in the main

text, in parenthetical comments added in proof. Three of special interest are described below.

[43] studies minimum coverings of combinatorial manifolds by combinatorial balls. A fast algorithm for producing such coverings would be of considerable interest. A relatively small number of covering balls for a pseudomanifold (F) can in many cases be produced quickly by applying our algorithm (without backtracking) to find a maximal partial shelling $(F_1, \ldots, F_k)$ of F whose union is a combinatorial ball, then doing the same to the pseudomanifold $F \sim \{F_1, \ldots, F_k\}$, and continuing the process until all facets of F have been used.

[71] contains interesting results on shelling and several related notions. In particular, it is shown that each d-sphere with $d+3$ vertices is extendably shellable.

[75] contains a new approach to the homeomorphism problem for the 3-sphere, leading its authors to "hope there are sufficient grounds for assuming that the problem of discriminating algorithmically the standard three-dimensional sphere will be solved positively by means of the new topological invariant constructed in the paper". The paper also contains an outline of S.P. Novikov's proof that there is no algorithm for deciding when a 5-manifold is a sphere.

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A REPRESENTATION OF 2-DIMENSIONAL PSEUDOMANIFOLDS AND ITS USE IN THE DESIGN OF A LINEAR-TIME SHELLING ALGORITHM

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A shelling of a d-dimensional pseudomanifold is an arrangement of its d-cells in a sequence such that each cell after the first intersects the union of its predecessors in a $(d-1)$-ball, except that the final intersection may be a $(d-1)$-sphere. When $d \geqslant 3$, it is unknown whether shellability can be tested in polynomial time. However, it is shown here that when $d = 2$, there is a linear-time algorithm that not merely tests for shellability but actually finds a maximal partial shelling $\boldsymbol{S}$; by a basic result, $\boldsymbol{S}$ is a shelling or the 2-pseudomanifold is not shellable. The algorithm is based on a special representation of 2-pseudomanifolds that can be produced in linear time and may be of interest in itself.

1. Introduction

When $\boldsymbol{C}$ is a finite collection of topological d-balls forming a cell-complex, a *partial shelling* of $\boldsymbol{C}$ is defined as a sequence $C_1, \ldots, C_k$ of distinct members of $\boldsymbol{C}$, such that the intersection $C_j \cap (\bigcup_{i=1}^{j-1} C_i)$ is topologically a $(d-1)$-ball for $1 < j \leqslant k$ except that, when $j = k = |\boldsymbol{C}|$ it may instead be a $(d-1)$-sphere. A *shelling* of $\boldsymbol{C}$ is a partial shelling for which $k = |\boldsymbol{C}|$, and $\boldsymbol{C}$ is *shellable* if it admits a shelling. When $\boldsymbol{C}$ is a pseudomanifold and the members of $\boldsymbol{C}$ are convex polytopes, shellability implies that $\boldsymbol{C}$ is a piecewise linear d-ball or d-sphere; however, for $d \geqslant 4$ it is unknown whether the problem of testing $\boldsymbol{C}$ for shellability is recursively solvable. When the members of $\boldsymbol{C}$ are Euclidean simplices, shellability can be tested by a straightforward backtrack algorithm, but even when $\boldsymbol{C}$ is a pseudomanifold it is unknown for $d \geqslant 3$ whether there is a polynomial-time test. For discussions of the 3- and higher-dimensional cases, and of the importance of the notion of shelling (see [3–5, 7]).

The present paper describes a linear-time algorithm, applicable only when $\boldsymbol{C}$ is a 2-dimensional pseudomanifold, that produces a partial shelling $\boldsymbol{S}$ of $\boldsymbol{C}$ which is *maximal* in the sense that $\boldsymbol{S}$ is not an initial segment of any other partial shelling. By a basic result on extendability of partial shellings, either $\boldsymbol{S}$ is a shelling of $\boldsymbol{C}$ or $\boldsymbol{C}$ is not shellable, and when $\boldsymbol{C}$ is not a sphere $\boldsymbol{S}$ is maximal among the subcomplexes of $\boldsymbol{C}$ that span 2-balls. Design of the algorithm is based on the connectedness game of [6] and on a representation of 2-pseudomanifolds that may be of interest in itself.

2. Basic results

Since only 2-dimensional pseudomanifolds are treated here, it is convenient to employ an equivalent graph-theoretic formulation. All that follows is based on the

Standing Hypothesis. *$\mathbf{C}$ is a set of circuits covering a connected graph G in such a way that each edge of G appears in at least one and at most two members of $\mathbf{C}$.*

The collection $\mathbf{C}$ is called a *2-pseudomanifold*, and G is the *graph* of $\mathbf{C}$. A *partial shelling* of $\mathbf{C}$ is a sequence $C_1, \ldots, C_k$ of distinct members of $\mathbf{C}$ such that the intersection $C_j \cap (\bigcup_{i=1}^{j-1} C_i)$ is a path for $1 < j \leq k$ except that, when $j = k = |\mathbf{C}|$ it may instead be a circuit. *Shelling* and *shellable* are then defined in the obvious way.

The following result, though not essentially new (see [9, 8, 11, 2]) is proved here because of its fundamental role in what follows.

Theorem. *Suppose that $\mathbf{C}$ is a 2-pseudomanifold and G is its graph. Then $\mathbf{C}$ is shellable if and only if G can be topologically embedded in a 2-ball or 2-sphere M in such a way that $M \sim G$ is the union of $|\mathbf{C}|$ pairwise disjoint open 2-balls whose boundaries are the members of $\mathbf{C}$. If $\mathbf{C}$ is shellable then every maximal partial shelling of $\mathbf{C}$ is a shelling.*

Proof. The proof uses, without specific mention, some basic results of 2-dimensional topology. For these see [10].

Consider a topological representation of $\mathbf{C}$, so that the members of $\mathbf{C}$ are simple closed curves. Associate with each member C of $\mathbf{C}$ a 2-ball C^* such that the boundary of C^* is C and the balls in the collection $\mathbf{C}^* = \{C^* : C \in \mathbf{C}\}$ have pairwise disjoint interiors. Let M denote the resulting space $\bigcup \mathbf{C}^*$ and let $m = |\mathbf{C}|$.

If $C_1, \ldots, C_m$ is a shelling of $\mathbf{C}$, it follows readily that the subset $\bigcup_1^j C_i^*$ of M is topologically a 2-ball for $1 \leq j < m$, and for $j = m$ is a 2-ball or 2-sphere according as C_m intersects the union of its predecessors in a path or circuit.

To complete the proof it suffices to show that if M is a 2-ball or 2-sphere and $C_1, \ldots, C_k$ is a partial shelling of $\mathbf{C}$ with $k < m$, then there exists $C_{k+1} \in \mathbf{C}$ such that the sequence $C_1, \ldots, C_k, C_{k+1}$ is also a partial shelling. Let B denote the boundary of the 2-ball $\bigcup_1^k C_i^*$, let $\mathbf{P}$ denote the set of all $P \in \mathbf{C} \sim \{C_1, \ldots, C_k\}$ such that at least one edge of P is in B, and let $\mathbf{D}$ denote the set of all $D \in \mathbf{P}$ such that $D \cap B$ is disconnected. Plainly $\mathbf{P}$ is nonempty. To complete the proof it suffices to show $\mathbf{D} \neq \mathbf{P}$, for then any choice of $C_{k+1} \in \mathbf{P} \sim \mathbf{D}$ has the desired property.

For notational convenience, let us fix a 2-sphere $S \supset M$. For each $D \in \mathbf{D}$ let $\mathbf{A}_D$ denote the set of all components of $B \sim D$. Then for each $A \in \mathbf{A}_D$ there is a unique arc D_A in D such that the simple closed curve $A \cup D_A$ is the boundary of a component of the set

$$\left(S \sim \bigcup_1^k C_i^*\right) \sim D^*;$$

let $Q(D, A)$ denote the closure of that component. Since $S \sim M$ is connected, each $D \in \mathbf{D}$ it is true that $Q(D, A) \subset M$ for all but at most one $A \in \mathbf{A}_D$.

Among all choices of $D \in \mathbf{D}$ and $A \in \mathbf{A}_D$ such that $Q(D, A) \subset M$, consider for which the set $Q(D, A)$ is minimal. Plainly there is a member P of $\mathbf{P}$ that sh an edge with A. If $P \in \mathbf{D}$ then some member A_0 of $\mathbf{A}_P$ is a proper subset of A (Fig. 1), whence $Q(P, A_0)$ is a proper subset of $Q(D, A)$ and the minimality of latter is contradicted. It follows that $P \in \mathbf{P} \sim \mathbf{D}$ and the proof is complete.

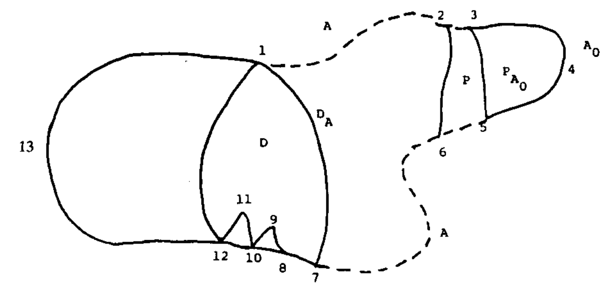


Fig. 1.

In Fig. 1. B is the simple closed curve 1 2 3 4 5 6 7 8 10 12 13 1; $\bigcup_1^k C_i^*$ is closed "outer" region bounded by B; the boundary of the region D is the sin closed curve 1 7 8 9 10 11 12 1; A is the arc 1 2 3 4 5 6 7, D_A the arc 7 1, $Q(D, A)$ is bounded by $A \cup D_A$; the boundary of the region P is the simple clo curve 2 3 5 6; A_0 is the arc 3 4 5, P_{A_0} the arc 5 3, and $Q(P, A_0)$ is bounded $A_0 \cup P_{A_0}$.

Henceforth, the members of $\mathbf{C}$ are called *faces* and the numbers of verti edges and faces are denoted by V, E and F respectively. The algorithm of Secti finds a maximal partial shelling of $\mathbf{C}$, which in view of the Theorem is a shelling is shellable. The algorithm is *linear* in the sense that, relative to the uniform criterion for the RAM model of random access computation (see [1]) its time- space-complexity are both $O(E)$. Since the shellability of $\mathbf{C}$ implies that G is pla and hence (by Euler's theorem) $E \leq 3V - 3$, there is a simple modification of algorithm which in $O(V)$ time either finds a shelling of $\mathbf{C}$ or concludes that n exists.

3. Data conversion for 2-pseudomanifolds

The algorithm of this section converts the list of faces of a 2-pseudomanifold i the more elaborate data structure required as input by the shelling algorithm

Section 4. Input to the conversion algorithm consists of positive integers V and L and an integer array LIST $[1:L]$. The V vertices of the pseudomanifold are represented by the integers from 1 to V, each face is represented in LIST by the sequence of its vertices in a natural order (corresponding to a traversal of the circuit in question), and the edges of such a face $(i_1, \ldots, i_k)$ are the unordered pairs $\{i_1, i_2\}, \ldots, \{i_{k-1}, i_k\}, \{i_k, i_1\}$. Successive faces are separated in LIST by 0. In our graph-theoretic formulation, each face must have at least three vertices, but that is a minor restriction in view of the possibility of adding vertices in the middles of edges. Faces may have more than three vertices, vertices of valence two are permitted, and intersections of faces need not be connected. It is assumed each edge is incident to at least one and at most two faces, but the pseudomanifold may be with or without boundary and the graph G need not be planar.

The conversion algorithm outputs the numbers V, E and F of vertices, edges and faces respectively, and integer arrays START$[1:E]$, TERM$[1:E]$, FACE $[-E:E]$, SED$[-E:E]$ and TED$[-E:E]$ whose significance is indicated in Fig. 2.

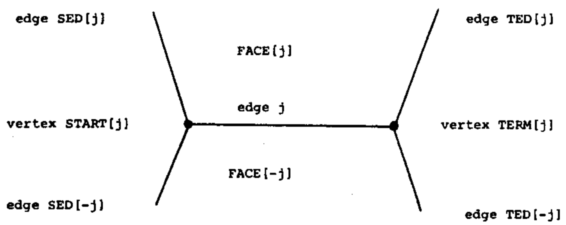
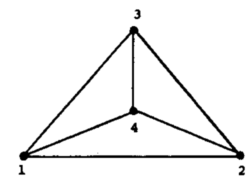


Fig. 2.

The vertices, edges and faces are indexed from 1 to V, 1 to E and 1 to F respectively. For $1 \leq j \leq E$, START$[j]$ and TERM$[j]$ (resp. FACE$[j]$ and FACE$[-j]$) are the indices of the two vertices (resp. faces) incident to edge j. The values of FACE[0], SED[0] and TED[0] are immaterial. For $-E \leq h \leq E$ with $h \neq 0$, SED$[h]$ and TED$[h]$ are the indices of the edges of FACE$[h]$ that are different from edge $j = \text{abs}(h)$ and incident respectively to START$[j]$ and TERM$[j]$. (Think of SED and TED as "starting edge" and "terminal edge".) When edge j is in the boundary of the pseudomanifold (incident to only one face), either FACE$[j]$ = SED$[j]$ = TED$[j]$ = 0 or FACE$[-j]$ = SED$[-j]$ = TED$[-j]$ = 0. When edge j is not on the boundary but the vertex START$[j]$ (resp. TERM$[j]$) is of valence two, SED$[j]$ = SED$[-j]$ (resp. TED$[j]$ = TED$[-j]$).

Fig. 3 and 4 show two acceptable sets of input data for the conversion algorithm, pseudomanifolds from which they might have come, and the complete output data in the first case.

Two versions of the data conversion algorithm are described, both of time-complexity $O(E)$. The first version is simpler, but it employs an auxiliary integer



Input $V = 4$, $L = 15$, LIST$[1:L]$ = 4 3 1 0 1 4 2 0 2 4 3 0 2 1 3.
Output $V = 4$, $E = 6$, $F = 4$ and data below.

j	START$[j]$	TERM$[j]$	FACE$[j]$	FACE$[-j]$	SED$[j]$	SED$[-j]$	TED$[j]$	TED$[j]$
1	4	3	1	3	3	4	2	6
2	3	1	1	4	1	6	3	5
3	1	4	1	2	2	5	1	4
4	4	2	2	3	3	1	5	6
5	2	1	2	4	4	6	3	2
6	3	2	3	4	1	2	4	5

Fig. 3.

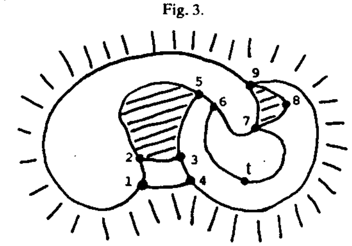


Fig. 4.

Input $V = 10$, $L = 28$, LIST$[1:L]$ = 1 2 5 6 7 9 0 1 2 3 4 0 6 7 t 0 5 3 4 9 8 7 t 6 0 6 7 t.
(Shaded portion not included)
Output $V = 10$, $E = 15$, $F = 4$ and additional data not shown.

array AUX$[1:V, 1:V]$ and hence is of space-complexity $O(V^2)$. The second version, which incorporates a radix sort suggested by Robert Tarjan, is of space-complexity $O(E)$.

In its preliminary phase, the first version runs through LIST and sets AUX$[h, i]$ and AUX$[i, h]$ to 0 for each edge $\{h, i\}$ that is encountered. The main phase runs through LIST again, in the manner described below.

Let $(i_1, \ldots, i_k)$ be the sequence from LIST representing the current face, f the index of that face, and e the number of edges previously encountered in the main phase of the conversion. The successive pairs

$$(h,i) \in \{(i_1,i_2),\ldots,(i_{k-1},i_k),(i_k,i_1)\}$$

are processed as follows:

(a) If AUX$[h,i]=0$, $\{h,i\}$ is recognized as a "new" edge and the following assignments are executed: $e \leftarrow e+1$; START$[e] \leftarrow h$; TERM$[e] \leftarrow i$; FACE$[e] \leftarrow f$.

(b) If AUX$[h,i] \neq 0$, $\{h,i\}$ is recognized as an "old" edge whose index AUX$[h,i]$, START and TERM have already been assigned; then FACE$[-\text{AUX}[h,i]] \leftarrow f$.

Now suppose that $j_1,\ldots,j_k$ have been assigned as the indices of the successive edges $\{i_1,i_2\},\ldots,\{i_{k-1},i_k\},\{i_k,i_1\}$ of the current face. The sequence is extended by setting $j_{k+1} \leftarrow j_1$ and $j_{k+2} \leftarrow j_2$, and values of SED and TED are then assigned as follows.

```
for r ← 2 until k + 1 do
  if j_r is a new edge
    then begin SED[j_r] ← j_{r-1}; TED[j_r] ← j_{r+1} end
  else if the current orientation of edge j_r agrees
      with its first orientation
    then begin SED[-j_r] ← j_{r-1}; TED[-j_r] ← j_{r+1} end
  else begin SED[-j_r] ← j_{r+1}; TED[-j_r] ← j_{r-1} end
```

The above version of the conversion algorithm is described in full detail in the ALGOL 60 program of [5]. The version below, whose time- and space-complexity are both $O(E)$, replaces AUX by arrays HAND, PLACE and EARLIER of length L and arrays NUM, SUIT, ESS, KAY and WHERE of length V. It is assumed for simplicity that all of these arrays are initialized at 0, though for some the initial values are immaterial.

Suppose that h and i are the successive vertices of an edge encountered in traversing a face represented in LIST, and let p be the location in LIST of this particular h. Thus LIST $[p]=h$, and either LIST$[p+1]=i$ or h and i are respectively the last and the first vertex of the face in question. In either case, the *suit* of the edge $\{h,i\}$ and the *place* of this occurrence of the edge are defined to be $\min(h,i)$ and p respectively. In a first pass through LIST, it is determined how many edges (counted according to multiplicity) appear in each suit and the results are recorded in NUM. In a second pass through LIST, representatives of these edges are recorded in HAND and the places of the edges are recorded in PLACE. Then the arrays HAND and PLACE are used to construct the array EARLIER such that, for each edge $\{h,i\}$ with place p encountered in LIST, it is true that

(a) if p is the first place at which $\{h,i\}$ occurs, then EARLIER$[p]=0$, and

(b) if p is the second place at which $\{h,i\}$ occurs, then EARLIER$[p]$ is the first place at which $\{h,i\}$ occurs.

With the aid of the array EARLIER it is easy, in a final pass through LIST, to produce the arrays START, TERM, FACE, SED and TED: The details are very similar to those in the first version of the conversion algorithm.

Below is a pidgin ALGOL program for the construction of EARLIER:

```
begin
  for each face represented in LIST do
    run once around the face and
      for each pair {h, i} of successive vertices of the face do
        NUM[min(h, i)] ← NUM[min(h, i)] + 1;
  SUIT[1] ← 1;
  for i ← 1 until V do SUIT[i + 1] ← SUIT[i] + NUM[i];
  for each face represented in LIST do
    run once around the face and
      for each pair {h, i} of successive vertices of the face do
        begin
          s ← min(h, i);
          k ← max(h, i);
          record k in the next available location in
            HAND[SUIT[s] : SUIT[s + 1] - 1];
          record h in the next available location in
            PLACE[SUIT[s] : SUIT[s + 1] - 1]
        end;
  for s ← 1 until V - 1 do
    for h ← SUIT[s] until SUIT[s + 1] - 1 do
      begin
        k ← HAND[h];
        if ESS[k] = s and KAY[s] = k then
          EARLIER[PLACE[h]] ← WHERE[k]
        else begin
              ESS[k] ← s;
              KAY[s] ← k;
              WHERE[k] ← PLACE[h]
            end
      end
end
```

4. A linear-time shelling algorithm

In addition to the arrays START, TERM, FACE, SED and TED mentioned earlier, the shelling algorithm employs an integer array SHELL$[1:F]$ to record the indices of the successive faces of the partial shelling and boolean arrays FUSED$[1:F]$, EUSED$[1:E]$ and VUSED$[1:V]$ to indicate which faces have been used and which edges and vertices are covered by those faces. There are also a "forward" and a "backward" linkage, FLINK$[0:E]$ and BLINK$[0:E]$, which serve to maintain a linked list RELEDGE of those signed edge-indices h that are relevant to the attempt to extend the partial shelling. The list RELEDGE consists of all integers h such that

(a) $1 \leq \mathrm{abs}(h) \leq E$,

(b) the face with index FACE[h] has been used in the partial shelling (whence FUSED[FACE[h]] = *true*), and

(c) the index $-h$ has not yet been tested to see whether the face C with index FACE[$-h$] can be added to the partial shelling. (The face C may have been tested, and either added to the partial shelling or temporarily rejected, but not in association with the edge-index $-h$.)

The changes in RELEDGE specified in the program below are effected by adjustments in FLINK and BLINK. Starting with an arbitrary signed edge-index e_1, the shelling algorithm proceeds as shown in the program below. Several comments follow the program.

A pidgin ALGOL program that finds a maximal partial shelling of a 2-dimensional pseudomanifold:

```
begin
  e ← e1; f ← FACE[e];
  if f = 0 then begin e ← − e; f ← FACE[e] end;                    (1)
  s ← 1; SHELL[s] ← f; FUSED[f] ← true;
  update EUSED and VUSED;                                            (2)
  RELEDGE ← {h : FACE[h] = f};                                       (2)
  while RELEDGE not empty do
    begin
      e ← first edge-index in RELEDGE;
      RELEDGE ← RELEDGE ~ {e};
      f ← FACE[− e];
      if ¬FUSED[f] and                                               (3)
        the face with index f has the proper sort of                 (4)
        intersection with the union of all faces
        previously used
          then begin
                 s ← s + 1; SHELL[s] ← f; FUSED[f] ← true;
                 update EUSED and VUSED;                             (2)
                 RELEDGE ←
                   RELEDGE ∪ {h : h ≠ e and FACE[h] = f}             (2)
               end
    end;
  print SHELL:
  if s = F
    then write "SHELL represents a shelling of the pseudomanifold."
    else write "The pseudomanifold is not shellable but SHELL
      represents a maximal partial shelling."
end
```


Comments

(1) The first face in the partial shelling has index FACE[e_1] unless FACE[e_1] = 0 (which can happen when abs(e_1) is the index of a boundary edge), in which case the first face has index FACE[$-e_1$].

(2) Since the new face is associated with a specific edge-index, the desired updating of EUSED, VUSED and RELEDGE can be accomplished by running once around the face with the aid of the appropriate arrays. For example, the first updating of EUSED and VUSED proceeds as follows:

```
begin
  a ← abs(e);
  VUSED[START[a]] ← EUSED[a] ← true;
  NEXTVERT ← TERM[a];
  NEXTEDGE ← TED[a];
  while NEXTEDGE ≠ a do
    begin
      VUSED[NEXTVERT] ← EUSED[NEXTEDGE] ← true;
      if START[NEXTEDGE] = NEXTVERT
        then begin
               NEXTVERT ← TERM[NEXTEDGE];
               NEXTEDGE ← if FACE[TED[NEXTEDGE]]
                 = FACE[NEXTEDGE] then
                 TED[NEXTEDGE] else TED[− NEXTEDGE]
             end
        else begin
               NEXTVERT ← START[NEXTEDGE];
               NEXTEDGE ← if FACE[SED[NEXTEDGE]]
                 = FACE[NEXTEDGE] then
                 SED[NEXTEDGE] else SED[− NEXTEDGE]
             end
    end
end
```

For the simplicial case, the details of updating RELEDGE by means of adjustments in FLINK and BLINK may be found in the ALGOL 60 program of [5]. Note, however, that the array FACE is not used there, its role being played by the integer procedure APX defined as follows.

APX: = **if** START[SED[j]] = START[abs(j)] **then**
TERM[SED[j]] **else** START[SED[j]].

The three vertices of FACE[j] are then START[abs(j)], TERM[abs(j)] and APX(j), which facilitates several programming shortcuts in the simplicial case that are not available for the general case.

(3) In order that the condition $\neg$FUSED[f] shall here imply FACE[$-e$] $\neq 0$, the range of the array FUSED is actually $[0:F]$, with FUSED[0] $\leftarrow$ **true**.

(4) In the simplicial case it is feasible to determine these intersections completely, as in [5], without destroying the linearity of the algorithm. The need for a subtler approach in the general case led to the connectivity game of [6], familiarity with which is assumed in what follows.

For each face $C \in \mathbf{C}$ the **total graph** T_C is a circuit whose vertices correspond alternately to the vertices of C and the edges of C. At the start of the shelling algorithm there is produced, for each $C \in \mathbf{C}$, a representation of T_C by means of adjacency lists. That is done in linear time and space by using the output of the data conversion algorithm, and then we are ready to play the connectedness game $\Gamma(T_C)$ in each T_C. Note that C has the proper sort of intersection with the union of all faces previously used if and only if

(a) at least one edge of C has been used,

(b) unless $F-1$ faces have been used, there is at least one unused edge of C, and

(c) the used edges and vertices of C form a connected set of vertices of T_C.

The shelling algorithm involves the "simultaneous play" (as the term is used in chess) of several games $\Gamma(T_C)$, one for each C except the first one, though when $\mathbf{C}$ is not shellable some of the games may never start. Consider an arbitrary face C with index i. Whenever it happens, after an assignment $f \leftarrow$ FACE[$-e$] in the program for the shelling algorithm, that $f = i$ and FUSED[f] = **false**, then it is our turn to move in the game $\Gamma(T_C)$. Our opponents' enlargement of their set X of vertices of T_C (see Section 1 of [6]) is indicated by changes in the arrays EUSED and VUSED. We compute and move in $\Gamma(T_C)$ in the manner described in Section 3 of [6], in order to determine whether the set X is connected. If

(a) X is connected, and

(b) X is not the entire vertex-set of T_C or

(c) X is the entire vertex-set of T_C and all faces other than C have already appeared in the partial shelling,

then C is added to the partial shelling and the play of $\Gamma(T_C)$ has ended, though under (a) $\wedge$ (b) the computation may later involve the play of $\Gamma(T_B)$ for various $B \in \mathbf{C} \sim \{C\}$. When (a) fails, C is rejected for the time being and the computation continues, perhaps to return to the game $\Gamma(T_C)$. When (a) holds but (b) and (c) both fail, the partial shelling is maximal. If $\mathbf{C}$ is shellable every one of the games $\Gamma(T_C)$ is eventually played to completion.

Though the algorithm is complicated, its property of linear space is obvious. To establish linear time it suffices, in conjunction with [6]'s bound on the computational c-complexity of circuits, to note that no edge is added to RELEDGE more than once.

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