Polytopes of Large Diameter

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work started at
AMS-MAA-SIAM
Joint Summer Research Conference, Mt.Holyoke 1996

Background

A *d-polytope* P is the convex hull of a finite set of points whose affine hull is $\mathbb{R}^d$. We denote the set of k-faces of P by $f^k(P)$; in particular, the vertices are $f^0(P)$ and the facets are $f^{d-1}(P)$.

A d-polytope P is *simple* iff each vertex is incident to exactly d facets; that is, each vertex-figure is a $(d-1)$-simplex.

We denote by (d, n) the set of all simple d-polytopes with n facets.

Given $x, y \in f^0(P)$, an *edge-path of length* k from x to y is a sequence of $k+1$ vertices

$$[x = v_0, v_1, \ldots, y = v_k]$$

such that $[v_{i-1}, v_i] \in f^1(P)$ for $i = 1, \ldots, k$.

The *distance* $\delta_P(x, y)$ is the minimum length of an edge-path between x and y.

The *diameter* $\delta(P)$ is the maximum of $\delta_P(x, y)$ over all pairs of vertices x, y.

$$\begin{aligned}\Delta(d, n) &= \max\left\{\delta(P) \;:\; \left|f^d(P)\right| = 1,\; \left|f^{d-1}(P)\right| = n\right\} \\ &= \max\left\{\delta(P) \;:\; P \in (d, n)\right\}\end{aligned}$$

The still open *Hirsch conjecture* asserts that

$$\Delta(d, n) \leq n - d$$

for all $n > d \geq 2$, where $\Delta(d, n)$ denotes the maximum edge-diameter of (convex) d-polytopes with n facets.

We add to the list of pairs (d, n) that are known to be *H-sharp* in the sense that $\Delta(d, n) \geq n - d$. In particular, we prove that

$$\Delta(d, n) \geq n - d \text{ for all } n > d \geq 8.$$

Family of Hirsch conjectures

Hirsch conjecture	
$\Delta(d,n) \leq n-d$	$\Delta(8,n) \geq n-8$
unbounded version (F)	
$\Delta_u(d,n) \leq n-d$	$\Delta_u(4,8) = 5$
d-step conjecture	
$\Delta(d,2d) = d$	$\Delta(5,10) = 5$
strong d-step conjecture (F)	
$\#^d(d,2d) = 2^{d-1}$	$\#^d(d,2d) \leq \frac{1}{2}24^{\lfloor d/5 \rfloor}2^{d \bmod 5}$
monotone Hirsch conjecture (F)	
$\Delta_m(d,n) \leq n-d$	$\Delta_m(4,8) = 5$
strict monotone Hirsch conjecture	
$\Delta_{sm}(d,n) \leq n-d$	$\Delta_{sm}(4,8) = 4$

Setting

(d, n): the set of all simple d-polytopes with n facets.

$\mathcal{S}$: the set of all H-sharp pairs (d, n).

A (d, n)-polytope P is *H-sharp* if and only if $\delta(P) \geq n - d$. The pair (d, n) is H-sharp ($\in \mathcal{S}$) if and only if $\Delta(d, n) \geq n - d$ (equivalently there is an H-sharp (d, n)-polytope).

H-pair: subsets of vertices which are at distance $\geq n-d$.

$X, Y \subset f^0(P)$ are an *H-pair* if and only if $\delta_P(x, y) \geq n - d$ for every pair $(x, y) \in X \times Y$.

$(d, n : h_1, h_2)$: the set of all triples (P, X, Y) such that P is an H-sharp (d, n)-polytope, (X, Y) is an H-pair in P, X holds an h_1-face and Y holds an h_2-face.

$\mathcal{T}$: the set of nonempty quadruples $(d, n : h_1, h_2)$.

Essential Observations

- Membership in $\mathcal{S}$ is implied by membership in $\mathcal{T}$.

- Operations (truncation, wedging, blending, product) act on polytopes, but we can view these as operations on $\mathcal{T}$.

- On polytopes, these operations lead to natural images of faces (especially vertices, edges, and facets), of short paths and of nonrevisiting paths.

- We work with the combinatorial description $M(P)$ of a simple polytope and not with imbedded descriptions ($H^T V \leq \langle 0 \rangle$).

Membership in $\mathcal{S}$ and in $\mathcal{T}$

$$\left.\begin{array}{r} P \in (d,n) \\ \text{and} \qquad \\ \delta(P) \geq n-d \end{array}\right\} \implies (d,n) \in \mathcal{S}$$

$$\begin{aligned} (P,X,Y) \in (d,n:h,k) &\implies (d,n:h,k) \in \mathcal{T} \\ &\implies (d,n) \in \mathcal{S} \end{aligned}$$

$$\begin{aligned} (d,n) \in \mathcal{S} &\implies (d,n:0,0) \in \mathcal{T} \\ &\implies \exists (P,\{x\},\{y\}) \in (d,n:0,0) \end{aligned}$$

What we know about $\Delta(d,n)$

$\Delta(2,n) = \lfloor \frac{n}{2} \rfloor$

$\Delta(3,n) = \lfloor \frac{2n}{3} \rfloor - 1.$

$\Delta(4,n) = n - 4$ for $n = 5, \ldots, 9$

$\Delta(4,10) = 5$

$\Delta(5,n) = n - 5$ for $n = 6, \ldots, 11$

$\Delta(d, d+5) = 5$ for all $d \geq 4$

$\Delta(d, 2d) \geq d$ for all d

...and about $\mathcal{T}$

(4,9 : 0,0)

(5,10 : 1,1) (5,11 : 0,1) (5,12 : 0,0)

(6,11 : 2,2) (6,12 : 1,2) (6,13 : [0,2]) (6,14 : 0,1) (6,15 : 0,0)

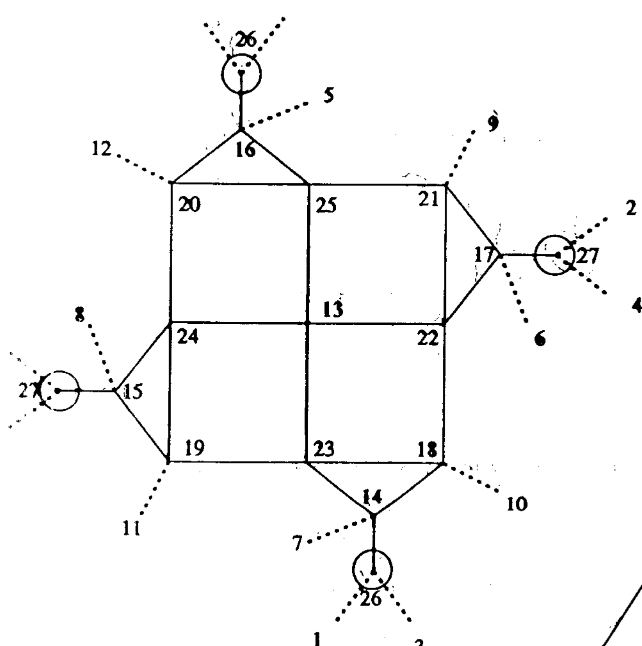
(7,12 : 3,3) (7,13 : 2,3) (7,14 : [1,3]) (7,15 : [0,3]) (7,16 : [0,2]) (7,17 : 0,1) (7,18 : 0,0)

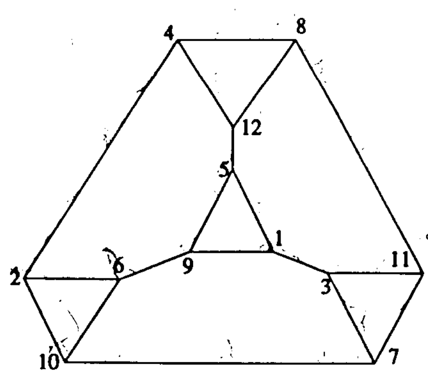
$$(8, 8+5k \ : \ 4,4)$$

$Q_4 \in (4, 9)$

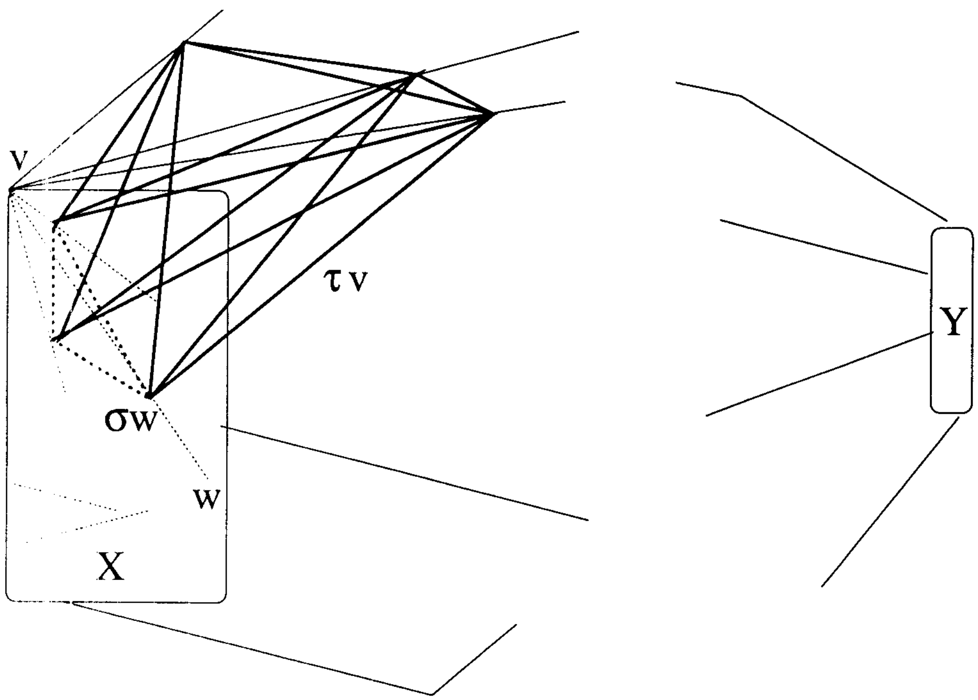
Of the 1142 simple 4-polytopes with 9 facets, exactly *one*, Q_4, has diameter 5; only one pair of the 27 vertices in Q_4 are at distance 5.

$$M(Q_4) := \left(\begin{array}{cccc|cccc|cccc|c|cccc|cccc|cccc|cc}
1&&& & 1&&& & 1&&& & 1 & 1&&1& & 1&&&1 & 1&1&&1 & 1& \\
&1&& & &1&& & &1&& & 1 & &1&&1 & 1&1&& & 1&1&1& & &1 \\
&&1& & &&1& & &&1& & 1 & 1&&1& & &1&1& & &1&1&1 & 1& \\
&&&1 & &&&1 & &&&1 & 1 & &1&&1 & &&1&1 & 1&&1&1 & &1 \\
\hline
1&&1& & &1&1& & 1&1&& & & 1&&&1 & 1&&&1 & 1&&& & 1& \\
&1&&1 & &&1&1 & &1&1& & & 1&1&& & 1&1&& & &1&& & &1 \\
1&&1& & 1&&&1 & &&1&1 & & &1&1& & &1&1& & &&1& & 1& \\
&1&&1 & 1&1&& & 1&&&1 & & &&1&1 & &&1&1 & &&&1 & &1 \\
\hline
1&1&1&1 & 1&1&1&1 & 1&1&1&1 & & &&& & &&& & &&& & &
\end{array}\right)_{(9\times 27)} = \begin{pmatrix} A & B \\ \langle 1\rangle & \langle 0\rangle \end{pmatrix},$$





Truncation (at a vertex)



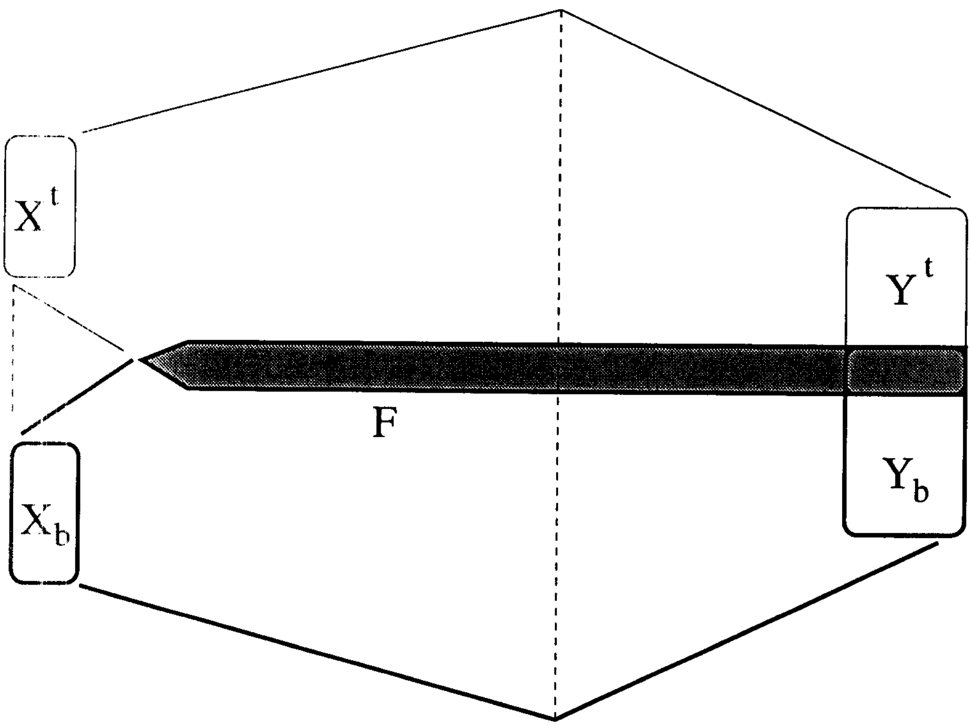
$$(d,n) \xrightarrow{\tau} (d,n+1)$$

Truncation and H-sharpness

1. If $h_1 > 0$, then
   $(d, n : h_1, h_2) \stackrel{\mathcal{T}}{\longrightarrow} (d, n+1 : h_1 - 1, h_2)$.

2. If $(d, n : h_1, h_2) \in \mathcal{T}$,
   then $(d, n+k) \in \mathcal{S}$ for $0 \leq k \leq h_1 + h_2$.

Wedging (over a facet)



$$(d, n) \xrightarrow{\omega} (d+1, n+1)$$

Wedging and H-sharpness

1. If P is H-sharp, then ωP is H-sharp.

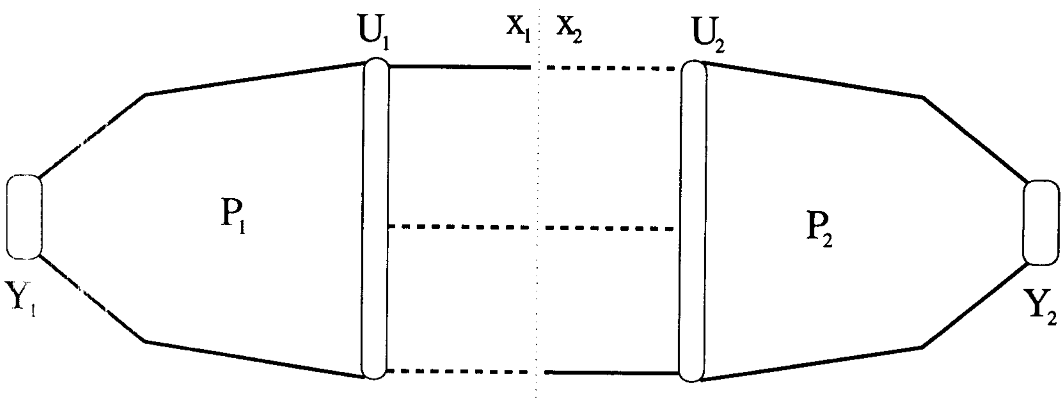
2. If $(d, n : h_1, h_2) \in \mathcal{T}$ with $n > 2d$, then $(d+1, n+1 : h_1+1, h_2+1) \in \mathcal{T}$.

3. If $(d, n) \in \mathcal{S}$ with $n > 2d$, then $(d+1, n+3) \in \mathcal{S}$.

4. If $(d, 2d : h_1, h_2) \in \mathcal{T}$ with $h_1 + h_2 \geq 0$, then $(d+k, 2d+k : h_1+k, h_2+k) \in \mathcal{T}$ for all $k \geq 0$.

Blending (at vertices)

$$\underbrace{(P_1, x_1) \bowtie_\pi (P_2, x_2)}_{P_1 \bowtie P_2}$$



$$(d, n_1), (d, n_2) \xrightarrow{\bowtie} (d, n_1 + n_2 - d)$$

long blends
fast-slow blends

Fast-slow Blending and H-sharpness

1. If $h_1 + h_2 \geq d$, then

$$\left.\begin{array}{l}(d, n_1 : h_1, k_1)\\ (d, n_2 : h_2, k_2)\end{array}\right\} \stackrel{\bowtie}{\longrightarrow} (d, n_1 + n_2 - d : k_1, k_2).$$

Operating on $\mathcal{T}$

(4,9 : 0,0)						
(5,10 : 1,1)	(5,11 : 0,1)	(5,12 : 0,0)				
(6,11 : 2,2)	(6,12 : 1,2)	(6,13 : [0,2])	(6,14 : 0,1)	(6,15 : 0,0)		
(7,12 : 3,3)	(7,13 : 2,3)	(7,14 : [1,3])	(7,15 : [0,3])	(7,16 : [0,2])	(7,17 : 0,1)	(7,18 : 0,0)
(8,13 : 4,4)			$\stackrel{\bowtie}{\longrightarrow}$		(8,18 : 4,4)	$\stackrel{\bowtie}{\longrightarrow}$

- $(Q_4, \{x\}, \{y\}) \in (4, 9 : 0, 0)$.
- $(\omega Q_4, \{x_b, x^t\}, \{y_b, y^t\}) \in (5, 10 : 1, 1)$.
  This is the $(d, 2d : h_1, h_2)$ case with $h_1 + h_2 \geq 1$.
- For all $j \geq 0$

$$(\omega^j Q_4, \omega^j \{x\}, \omega^j \{y\}) \in (4 + j, 9 + j : j, j).$$

- In particular, $(\omega^4 Q_4, \omega^4 \{x\}, \omega^4 \{y\}) \in (8, 13 : 4, 4)$, and thus

$$(8, 13 \ : \ 4, 4) \in \mathcal{T}.$$

All $(8, n)$ are H-sharp

$$\boxed{(8, 13 : 4, 4) \in \dot{\mathcal{T}}}$$

Blending these is like stringing beads!

$$(8, 13 : 4, 4) \xrightarrow{\bowtie} (8, 18 : 4, 4) \xrightarrow{\bowtie} \ldots \xrightarrow{\bowtie} (8, 13{+}5k : 4, 4) \xrightarrow{\bowtie}$$

Truncations fill in the row $d = 8$.

The entire 8^{th} row is H-sharp, and thereby every subsequent row is entirely H-sharp.

Next steps and Wagers

1. Find imbeddings for some of these H-sharp 8-polytopes.

2. (\$50) $\Delta_{sm}(8,16) \geq 9$ and an example can be found among the polytopes constructed by Holt and Klee.

3. (\$50) $\Delta_{sm}(10,20) \geq 11$.

4. **Conjecture**

   SETTING For an edge $[x, y]$ in a simple polytope P, there is a unique facet F_x incident to x and not to y and a unique facet F_y incident to y and not to x.

   CONJECTURE For some (d, n), there exists

   $$(P, \{x, y\}, \{z\}) \in (d, n : 1, 0)$$

   such that $F_x \cap F_y = \emptyset$.

This implies a counterexample to the Hirsch conjecture!

fast-slow *vertex* blend $\Longrightarrow$ H-sharpness

fast-slow *edge* blend $\Longrightarrow$ counterexample