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A PROOF OF THE STRICT MONOTONE 4-STEP CONJECTURE

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Dedicated to Branko Grünbaum for the thirtieth anniversary of his book on Convex Polytopes

ABSTRACT: With $\Delta(d,n)$ denoting the maximum diameter attained by (the graphs of) d-polytopes having n facets, the still unsettled *Hirsch conjecture* asserts that $\Delta(d,n) \leq n-d$ whenever $n > d \geq 2$. Its special case, the *d-step conjecture*, asserts that $\Delta(d,2d) = d$. The present note deals with two related functions, Δ_{sm} and Δ_m, which involve paths along which a given linear objective function is steadily increasing. This note was motivated by Ziegler's *strict monotone Hirsch conjecture*, asserting that always $\Delta_{sm}(d,n) \leq n-d$. (Since $\Delta \leq \Delta_{sm} \leq \Delta_m$, this implies the Hirsch conjecture.) When Γ is any of the functions Δ, Δ_{sm}, and Δ_m, the numbers of the form $\Gamma(k,2k)$ are of special interest because of the fact that $\Gamma(d,n) = \Gamma(n-d, 2(n-d))$ for $d < n \leq 2d$. (In particular, $\Gamma(d,n) \leq n-d$ for all d and n if and only if $\Gamma(d,2d) = d$ for all d.) This note summarizes the present knowledge concerning the functions Δ_{sm} and Δ_m, and proves the *strict monotone 4-step conjecture* asserting that $\Delta_{sm}(4,8) = 4$.

KEYWORDS: d-step conjecture, Dantzig figure, diameter, Hirsch conjecture, linear programming, LP orientation, monotone diameter, monotone path, monotone orientation, strict monotone diameter

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INTRODUCTION

As the term is used here, a linear functional φ is *admissible* for a (convex) polytope P provided that φ does not attain the same value at any two vertices of P. The three functions defined in the next paragraph are all of interest in connection with the behavior of the simplex method of linear programming and the open problem of whether there exists a pivot rule that turns the simplex method into an LP algorithm whose worst-case behavior is polynomially bounded.

Consider all triples (P,x,y) consisting of a d-polytope with n facets ($(d-1)$-faces) and two vertices x and y of P. Let $\Delta(d,n)$ denote the smallest integer k such that for each such triple, x and y can be joined by a path of length $\leq k$ (i.e., one formed from k or fewer edges of P). Let $\Delta_{sm}(d,n)$ denote the smallest integer k such that whenever (P,x,y) is such a triple and φ is an admissible functional for which

$$\min \varphi(P) = \varphi(x) < \varphi(y) = \max \varphi(P),$$

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then x can be joined to y by a path of length $\leq k$ along which the objective function φ is steadily increasing. The function $\Delta_m(d,n)$ is defined similarly, omitting the requirement that $\min \varphi(P) = \varphi(x)$. Evidently,

$$\Delta(d,n) \leq \Delta_{sm}(d,n) \leq \Delta_m(d,n),$$

where the first inequality is proved by an easy application of a projective transformation. Also, the d-cubes demonstrate that $\Delta(d,2d) \geq d$ for all d.

An important property of certain diameter functions Γ is that whenever $1 < d < n$, $\Gamma(d,n) \leq \Gamma(d+1,n+1)$, with equality when $n \leq 2d$. For the choice $\Gamma = \Delta$, that was established in [KW] with the aid of perturbation and wedging. In Section 1 of the present note, it is established for the functions Δ_{sm} and Δ_m. For each such Γ, it is true that

$$\Gamma(d,2d) = \Gamma(d+k,2d+k)$$

for all $k \geq 0$; when $e < n \leq 2e$, the replacement of d and k by $n-e$ and $2e-n$ yields the equality,

$$\Gamma(e,n) = \Gamma(n-e,2(n-e)).$$

Thus it is clear for each such Γ that the d-step conjecture implies the Hirsch conjecture (though not necessarily on a dimension-for-dimension basis), and that, whether the conjectures are true or false, a large share of the information concerning the numbers $\Gamma(d,n)$ is carried by those of the form $\Gamma(d,2d)$. That is the reason for our emphasis here on the numbers $\Delta_{sm}(d,2d)$ and $\Delta_m(d,2d)$.

As reported in [Da1,2], W.M. Hirsch conjectured that always $\Delta(d,n) \leq n-d$. Ziegler [Zi] strengthened this conjecture by stating what he called the ***strict monotone Hirsch conjecture***: $\Delta_{sm}(d,n) \leq n-d$. Its special case, the ***strict monotone d-step conjecture***, asserts that $\Delta_{sm}(d,2d) = d$.

It was shown in [Kl1] that $\Delta(3,n) = \lfloor \frac{2}{3}n \rfloor - 1$ for all n, and in [KW] that $\Delta(d,2d) = d$ for all $d \leq 5$. We note here that $\Delta_{sm}(3,n) = \Delta(3,n)$ for all n, and we show that $\Delta_{sm}(4,8) = 4$. Thus the strict monotone Hirsch conjecture holds for $d = 3$, the d-step conjecture for $d \leq 5$, and the strict monotone d-step conjecture for $d \leq 4$. The precise value of $\Delta(d,2d)$ is unknown when $d \geq 6$, and of $\Delta_{sm}(d,2d)$ when $d \geq 5$.

It was shown in [Kl2] that $\Delta_m(3,n) \leq n-3$, in [To] that $\Delta_m(4,8) \geq 5$. Thus the monotone Hirsch conjecture holds for $d \leq 3$ but the monotone d-step conjecture fails for $d \geq 4$. (However, the monotone Hirsch conjecture has been established without dimensional restrictions for several classes of polytopes that arise in connection with practical optimization problems [BR, Gr, Ma, Ri1, Ri2].) We show here that $\Delta_m(4,8) = 5$, but the precise value of $\Delta_m(d,2d)$ is unknown when $d \geq 5$.

The functions Δ, Δ_m, and Δ_{sm} are all of interest in connection with edge-following algorithms for linear programming. Just as Δ describes, in a sense, the worst possible behavior of the best possible edge-following LP algorithm (see [KK] and [KW]), Δ_m does the same for monotone edge-following algorithms. Similarly, Δ_{sm} applies to monotone edge-following algorithms for those linear programming problems in which an admissible linear objective function φ is to be maximized and there is a natural starting vertex at which φ attains its minimum. That would be the situation in any problem whose (nonempty, bounded) feasible region P is defined by constraints of the form

$$Ax \leq b, x \geq 0$$

where all entries of A are nonnegative and the admissible objective function is of the form $\varphi(x) = c^T x$ with $c \geq 0$. Then the origin is a vertex of P and φ attains its P-minimum at the origin.

1. Equivalence of Hirsch conjectures to d-step conjectures

When P is a polytope, x and y are vertices of P, and φ is an admissible functional such that $\varphi(y) = \max \varphi(P)$, we use $\delta_P^{\varphi}(x, y)$ to denote the smallest integer k such that x is joined to y by a path of length k along which φ is increasing. The *monotone diameter* of P is the maximum of $\delta_P^{\varphi}(x, y)$ over all (x, y, φ) of the indicated sort, and the *strict monotone diameter* of P is similarly defined with respect to (x, y, φ) such that $\varphi(x) = \min \varphi(P)$ and $\varphi(y) = \max \varphi(P)$. Thus $\Delta_m(d, n)$ and $\Delta_{sm}(d, n)$ are respectively the maximum of the monotone diameter and the maximum of the strict monotone diameter as P ranges over all d-polytopes with n facets.

Just as for the diameter function Δ, an easy perturbation argument shows that Δ_m and Δ_{sm} are unchanged when the d-polytopes in question are restricted to those that are *simple* (i.e., each vertex is incident to precisely d edges, or, equivalently, to precisely d facets). As in [HK], we use the term *(d, n)-polytope* to denote a simple d-polytope with n facets.

Now we want to prove the following result, whose consequences were described in the Introduction.

1.1. Proposition. *If Γ is Δ_m or Δ_{sm}, then $\Gamma(d, n) \leq \Gamma(d + 1, n + 1)$ for all $1 < d < n$, with equality when $n \leq 2d$.*

Proof. With $1 < d < n$, consider a (d, n)-polytope $P \subset \mathbb{R}^d$ and a linear functional φ on $\mathbb{R}^d$ such that

$$|\varphi(u) - \varphi(v)| > 1 \text{ for each pair } u, v \text{ of distinct vertices of } P.$$

Choose any facet F of P, and let ψ be a linear functional on $\mathbb{R}^d$ such that

$$\psi(F) = \{0\} \text{ and } \psi(P \setminus F) \subset]0, 1[.$$

Using ψ, construct the $(d + 1, n + 1)$-polytope

$$W = \{(p, \alpha) : p \in P, 0 \leq \alpha \leq \psi(p)\} \subset \mathbb{R}^{d+1},$$

a wedge over P with foot F, and define the linear functional

$$\eta(u, \alpha) = \varphi(u) + \alpha$$

for all $(u, \alpha) \in \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$.

For any vertex u of P, denote the vertices $(u, 0)$ and $(u, \psi(u))$ of W by u_b and u^t respectively. Then each edge of W is of the form $[u_b, v_b]$ or $[u^t, v^t]$ where $[u, v]$ is an edge of P, or of the form $[u_b, u^t]$ where u is a vertex of $P \setminus F$ (these are called *vertical edges*). Note that any two vertices of P have φ-values that differ by more than 1, while on P the range of ψ (and hence of α in the definition of η) is contained in $[0, 1[$. Thus for any two vertices u and v of P, the condition that $\varphi(u) < \varphi(v)$ is equivalent to the condition that

$$\eta(u_b) \leq \eta(u^t) < \eta(v_b) \leq \eta(v^t).$$

Since $\eta(u_b) = \eta(u^t)$ if and only if u_b and u^t coincide, it follows that the linear functional η is admissible for W and that if the minimum and maximum of φ on P are attained at vertices x and y respectively, then the minimum and maximum of η on W are attained at x_b and y^t respectively. Moreover, any η-monotone edge-path in W projects onto a φ-monotone edge-path in P, and the projection of the edges in an η-monotone path is one-to-one except that the vertical edges of W produce vertices rather than edges in P. In any case, the path in W has at least as many edges as its projection in P. Thus we may conclude, for both of the mentioned choices of Γ, that

$$\Gamma(d,n) \leq \Gamma(d+1, n+1).$$

To complete the proof, we assume that $d < n \leq 2d$ and show that then $\Gamma(d+1,n+1) \leq \Gamma(d,n)$. Since $d+1 < 2(n+1)$, any two vertices of a $(d+1,n+1)$-polytope P must lie on a common facet Q of P, and Q is a (d,m)-polytope for some $m \leq n$. For any linear functional φ that is admissible for P, the restriction of φ to Q is admissible for Q. For any pair of vertices x and y of Q, each φ-monotone path from x to y on Q is by inclusion a φ- monotone path on P, and hence the minimum length of such paths on Q is no smaller than the minimum on P. From this it follows that $\Gamma(d+1,n+1) \leq \Gamma(d,m)$. Successive truncation shows that $\Gamma(d,m) \leq \Gamma(d,n)$ and thus completes the proof. □

We also need the following, known from [KW] when Γ is Δ.

1.2. Proposition. *Suppose that Γ is Δ_m or Δ_{sm}, and that $n \geq 2d$. Then the value of $\Gamma(d,n)$ is unaltered if the relevant maximum is restricted to pairs of vertices that do not share a facet.*

Proof. Suppose that $n \geq 2d$. Among the 4-tuples (P,x,y,φ) that satisfy conditions (i), (ii), and (iii$_m$) or (iii$_{sm}$) of the preceding proof, choose one that maximizes the dimension of the smallest face G of P that is incident to both x and y. We want to show that $G = P$. Suppose, to the contrary, that G is contained in some facet F of P, and note that F is a $(d-1,m)$-polytope for some $m \leq n-1$. Note also that

$$\delta_F^{\varphi}(x,y) \geq \delta_P^{\varphi}(x,y).$$

From here on, truncation and wedging are used to produce a contradiction that completes the proof. The details are omitted, for they are virtually identical with those in the proof of 2.8 in [KW]. □

The above facts motivate our focus on the numbers $\Gamma(d,2d)$, and on the triples known as *Dantzig figures*. A d-dimensional Dantzig figure is a triple (P,x,y) such that P is a $(d,2d)$-polytope and x and y are vertices of P that are *estranged* in the sense that they do not share a facet. Such figures play an essential role in all that follows.

2. A STRICTLY MONOTONE VERSION OF d-CONNECTEDNESS

We need a strictly monotone variant of Balinski's observation [Bal] that the graph of a d-polytope is d-connected. The proof below is based on a comment made by David Walkup in the 1960's, and it is somewhat similar in spirit to Barnette's short proof [Bar] of Balinski's theorem.

2.1. Proposition. *Suppose that φ is an admissible functional for a d-polytope P, and x and y are vertices of P such that $\varphi(x) = \min \varphi(P)$ and $\varphi(y) = \max \varphi(y)$. Then among the φ-monotone paths from x to y in P's edge-graph, there are d that are pairwise vertex-disjoint except for having x and y in common.*

Proof. The assertion is obvious when $d = 2$, and we proceed by induction on the dimension. When $d \geq 3$, we apply the lower-dimensional results in conjunction with a directed and vertex-oriented version of Menger's connectivity theorem (e.g., Theorem 11.6 of [BM]).

For each point $q = (q_1, ..., q_d) \in \mathbb{R}^d$, let $\Phi(q) = (q_1, ..., q_{d-1}, 0)$ and $\varphi(q) = q_d$. Then $q = \Phi(q) + \varphi(q)z$, where $z = (0, ..., 0, 1)$. In treating the theorem's d-dimensional case, we may assume that this φ is the admissible functional in question, that P's vertex x is the origin 0, and that $\varphi(y) = 1$. Since φ's level sets are preserved by the linear transformation that takes q into $q - \varphi(q)y$, we may assume also that $y = z$. Now turn each edge of P into a directed edge (u, v) such that $\varphi(u) < \varphi(v)$. To show there are d independent monotone paths from x to y, it suffices, in view of the version of Menger's theorem mentioned above, to show that for each set S of $d - 1$ vertices other than x or y, there is a φ-monotone path from x to y that misses the set S.

Let $J = \{q \in \mathbb{R}^d : q_{d-1} = 0\}$, $J^+ = \{q \in \mathbb{R}^d : q_{d-1} > 0\}$, and $J^- = -J^+$. The orthogonal projection $\Phi(S)$ of S on the hyperplane $H = \{q \in \mathbb{R}^d : q_d = 0\}$ is contained in a $(d-2)$-flat G in H. With G_0 denoting the $(d-2)$-subspace of H that is parallel to G, we may assume (with the aid of a suitable rotation about the line $\mathbb{R}z$, and, if necessary, a reflection across the hyperplane J) that $G_0 = \{q \in \mathbb{R}^d : q_{d-1} = q_d = 0\}$ and that either $S \subset J$ or $S \subset J^-$.

If $S \subset J^-$ and P misses J^+, then the intersection $J \cap P$ is a face of P that misses S and includes the vertices x and y. By the inductive hypothesis it must contain a φ-monotone path from x to y. In the remaining cases, either (a) $S \subset J^-$ and P intersects J^+ or (b) $S \subset J$ and (since P is not contained in J) we may assume that P intersects J^+.

Now let Π denote the transformation that projects P orthogonally onto the 2-dimensional plane $\{q \in \mathbb{R}^d : q_1 = ... = q_{d-2} = 0\}$. The projection $\Pi(P)$ is a convex polygon K that intersects J^+, and it follows from the 2-dimensional result that the boundary of K contains a φ-monotone path that goes from x to y and lies, except for its endpoints, in J^+. This path can be "lifted" to a φ-monotone path that goes from x to y in P and that misses the set S. It remains only to describe the lifting.

With $x = v_0$ and $v_m = y$, let $[v_0, v_1], [v_1, v_2], ..., [v_{m-1}, v_m]$ be the successive edges of the mentioned φ-monotone path from x to y in $K \cap J^+$. Then each of the edges $[v_{j-1}, v_j]$ is of the form $\Pi(F_j)$ for some face F_j of P. Since the projection Π preserves φ-values, there are unique vertices w_{j-1} and w_j of F_j that project into v_{j-1} and v_j respectively, and these are respectively the unique minimizer and the unique maximizer of φ's restriction to F_j. In each F_j there is a φ-monotone path T_j from w_{j-1} to w_j, and stringing these paths together produces a φ-monotone path

$$T = T_1 \cup T_2 \cup \cdots \cup T_m$$

from x to y in P. Except for its end vertices, T is contained in the open halfspace J^+ and hence it misses the set S. That completes the proof. □

Balinski's original d-connectedness theorem is an easy consequence of Proposition 2.1. Consider any two vertices x and y of a d-polytope P in $\mathbb{R}^d$, and let H_x and H_y be hyperplanes whose intersections with P are respectively $\{x\}$ and $\{y\}$. With the aid of a projective transformation (as in [Bar]), we may assume that H_x and H_y are parallel, and then a slight perturbation turns them into the level sets of an admissible function φ whose P-minimum is attained uniquely at x and P-maximum is attained uniquely at y. Then apply Proposition 2.1.

3. Monotone paths on 3-polytopes

Now consider the d independent monotone paths whose existence is asserted by Proposition 2.1. If the shortest of these d paths uses k edges, then each path has at least $k-1$ internal vertices and hence the total number of vertices of P is at least $d(k-1)+2$. When P is a $(3,n)$-polytope, P has precisely $2n-4$ vertices, so $3k-1 \leq 2n-4$ and hence $k \leq \lfloor (2n/3) \rfloor - 1 \leq n-3$. Thus the strict monotone Hirsch conjecture is correct when $d=3$. In fact,

$$\Delta_m(3,n) = \Delta_{sm}(3,n) \geq \Delta(3,n)$$

for all $n \geq 4$, and the "$\geq$" becomes an equality only for $n \leq 6$. The facts that $\Delta_m(3,6)=3$ and $\Delta_m(3,7)=4$ are of particular importance in the work below.

There are five combinatorial types of $(3,7)$-polytopes, denoted in [GS] by d_1, d_2, d_3, d_4, d_5. Each of the d_k has diameter 3 but monotone diameter 4. Moreover, except for the pentagonal prism (d_2), there is an embedding of d_k in $\mathbb{R}^3$ and an admissible linear functional $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that more than one vertex is at monotone distance 4 from the top vertex. The following figures illustrate these conclusions.

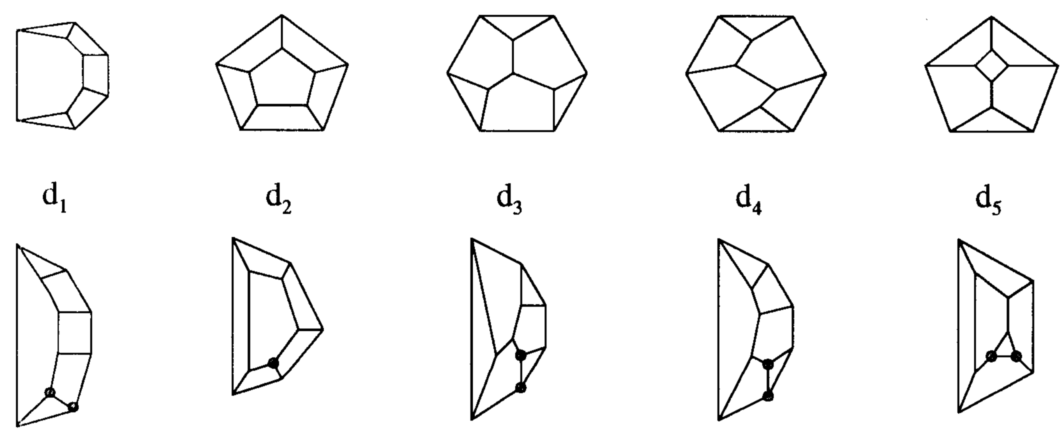


Figure 1: The 2-diagrams in the upper row are those given in [GS] for the $(3,7)$-polytopes $d_1, \ldots, d_5$. The alternate 2-diagrams in the lower row illustrate the fact that each d_k admits a realization that has monotone diameter 4; in fact, each of the highlighted vertices is at monotone distance 4 from the top vertex.

There is danger in jumping to d-dimensional conclusions on the basis of plausible $(d-1)$-dimensional diagrams. That is illustrated by the "Brückner sphere" discussed in [GS] – a 3-diagram that is not combinatorially equivalent to the Schlegel diagram of any 4-polytope. Thus we should state explicitly that, while the diagrams in the lower row of Figure 1 *illustrate* the fact that each of the d_k admits a realization with monotone diameter 4, they do not *prove* this fact. A proof would in each case require an algebraic representation P of d_k in $\mathbb{R}^3$, a specification of an admissible linear functional φ on $\mathbb{R}^3$, and a specification of two vertices x and y of P such that $\varphi(y) = \max \varphi(P)$ and each φ-increasing path from x to y uses at least 4 edges of P. We do not supply such details here because they are not needed for the proof of our main result. The fact that each of the d_k can be geometrically realized so as to have monotone diameter 4 shows that none of the d_k can be neglected in our analysis. However, our analysis does not depend on geometric realizations of the d_k, but only on properties of certain orientations of the graphs of the d_k that are introduced in Section 4 as surrogates for geometric realizations.

4. Proof of the strict monotone 4-step conjecture

A complete catalog of the 37 combinatorial types of (4, 8)-polytopes was produced by Grünbaum and Sreedharan [GS]. We use their tables to exhaust the possibilities for a counterexample to the strict monotone 4-step conjecture. Let us start, then, to develop a profile for a counterexample to the strict monotone 4-step conjecture.

4.1. Lemma. *Suppose that x and y are two vertices of a $(4,8)$-polytope P in $\mathbb{R}^4$, and that $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ is an admissible functional such that $\varphi(y) = \max \varphi(P)$ and $\delta_P^\varphi(x,y) \geq 5$. Then*

(a) *x and y are estranged vertices and hence (P, x, y) is a Dantzig figure.*

Now suppose also that $[x, v]$ is an edge of P, and that F is the (unique) facet of P that is incident to both v and y. If $\varphi(x) < \varphi(v)$, then

(b) *F is a $(3,7)$-polytope and $\delta_F^\varphi(v,y) = 4$.*

Proof. If x and y are incident to a common facet, then that facet is at most a $(3,7)$-polytope and hence

$$\delta_P^\varphi(x,y) \leq \Delta_m(3,7) = 4,$$

contradicting the assumption that $\delta_P^\varphi(x,y) \geq 5$. Hence x and y are estranged, and since P is a $(4,8)$-polytope, (P, x, y) is a Dantzig figure.

Now consider any edge $[x, v]$ such that $\varphi(x) \leq \varphi(v)$. Since (P, x, y) is a Dantzig figure, v and y are incident to a unique common facet F, and we have

$$5 \leq \delta_P^\varphi(x,y) \leq 1 + \delta_F^\varphi(v,y).$$

Of course, v need not be the bottom vertex in F, but in any case the term $\delta_F^\varphi(v,y)$ cannot exceed the monotone diameter of F. From the facts that $\delta_F^\varphi(v,y) \geq 4$, that F has at most 7 2-faces, and that $\Delta_m(3,6) = 3$, it follows that F has exactly 7 2-faces and $\delta_F^\varphi(v,y) = 4$. □

From (b) of Lemma 4.1 it follows that $\delta_P^\varphi(x,y) \leq 5$. In conjunction with Todd's example [To], this shows that $\Delta_m(4,8) = 5$.

4.2. Lemma. *With hypotheses as in Lemma 4.1, assume in addition that $\varphi(x) = \min \varphi(P)$. Then the number of vertices of P is 18, 19, or 20.*

Proof. Since P is a simple 4-polytope with 8 facets, the number m of vertices of P is between 14 and 20. Under the additional assumption that $\varphi(x) = \min \varphi(P)$, there are four independent monotone paths from x to y [Ba]. It follows that the shortest such path in any such set of paths is of length $k \leq \lfloor \frac{m+2}{4} \rfloor$. When $m < 18$, this yields $k \leq 4$, so only the cases $m = 18$, $m = 19$, and $m = 20$ remain. □

It suffices, then, to consider 4-dimensional Dantzig figures (P, x, y) such that P has 18, 19, or 20 vertices, and such that every edge incident to x terminates on a $(3,7)$-facet incident to

y. In [HK] we worked with the duals of the polytopes described in the Grünbaum-Sreedharan [GS] catalog of simplicial 4-polytopes with 8 vertices, and for each of these we found the pairs of vertices that turned these polytopes into Dantzig figures. Only 21 of those Dantzig figures, in 7 of the (4, 8)-polytopes, satisfy the profile, provided in Lemmas 4.1 and 4.2, of a counterexample to the strict monotone 4-step conjecture.

The following table summarizes our study of these 21 Dantzig figures. Let us describe the row corresponding to the first Dantzig figure, D_1.

$D_1 = (P_{25}, N, Q)$	$N = 3456$ $\downarrow OKMJ$ 7272	$Q = 1278$ $\downarrow TRPB$ 3663	$2(d_1)JKAEDPSTBQ$

The first column indicates that the Dantzig figure D_1 is found in the (4, 8)-polytope P_{25} from the [GS] catalog (our P_k is their P_k^8), with estranged vertices N and Q. The next two columns summarize the edge-facet intersections in D_1; the second column records that the vertex N is the intersection of the four hyperplanes/facets indexed $3, 4, 5, 6$, that the edge from N not incident to facet 3 terminates on facet 7 in vertex O, and similarly that the edge from N not incident to facet 4 terminates on facet 2 in vertex K. The third column contains the complementary information about the vertex Q. Similar comments apply to the remaining 20 rows. Thus the first three columns describe, for each of the Dantzig figures $D_1, \ldots, D_{21}$, certain combinatorial aspects that are especially relevant to our analysis; the remaining details of combinatorial structure can be found in [GS] and [HK].

The fourth column in the table outlines the proofs that the various D_i cannot have strict monotone diameter > 4. These proofs are based on properties of the (3, 7)-polytopes d_k with respect to orientations that serve as surrogates for certain sorts of geometric realizations. For example, the fourth column for D_1 indicates that facet 2 is of type d_1, and under the listed combinatorial equivalence, the lemma and corollary for d_1 figures (4.3 and 4.4) applies to D_1. The detailed arguments appear in the proofs of Corollaries 4.4, 4.6, 4.8, and 4.10, and those proofs contain more detailed explanations of the table's fourth column.

We list only one combinatorial equivalence for each Dantzig figure, although there may be others. For example, for D_1, we could also list $3(d_3)BTAFMUSKJN$ and $7(d_4)OMFGHRUTBQ$, demonstrating that D_1 has strict monotone diameter 4 via Lemmas 4.5 and 4.7 respectively. We could establish the strict monotone diameter of D_1 by considering facet 6, of type d_5, as well; however, the analogous lemma for types d_5 is not necessary for our present purposes and so is omitted.

Let G be the graph of a d-polytope P. By a *monotone orientation* of G (or of P) we mean a way of directing all of G's edges so that the resulting digraph satisfies the following conditions:

the digraph is acyclic;
in each k-face P $(1 \le k \le d)$ there is a unique source and a unique sink;
in each k-face, there are k independent paths from source to sink.

Such orientations are called *good orientations* by Kalai [Ka].

Dantzig figures	Edge-facet intersections		Facet for lemmas no.(type)comb.equiv.
$D = (P, x, y)$	$x = h_1 \cap h_2 \cap h_3 \cap h_4$ $\downarrow v_1 \ v_2 \ v_3 \ v_4$ $h_{i_1} \ h_{i_2} \ h_{i_3} \ h_{i_4}$		$h_i(d_j)\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$
$D_1 = (P_{25}, N, Q)$	$N = 3456$ $\downarrow OKMJ$ 7272	$Q = 1278$ $\downarrow TRPB$ 3663	$2(d_1)JKAEDPSTBQ$
$D_2 = (P_{29}, V, K)$	$V = 4568$ $\downarrow UOTE$ 1212	$K = 1237$ $\downarrow BCJN$ 6644	$2(d_3)EOMFBQPJNK$
$D_3 = (P_{31}, N, S)$	$N = 3456$ $\downarrow OKMJ$ 7272	$S = 1278$ $\downarrow VTPC$ 6346	$7(d_1)OMHGFTWVCS$
$D_4 = (P_{33}, M, P)$	$M = 3467$ $\downarrow OLFN$ 5215	$P = 1258$ $\downarrow RUQE$ 6464	$5(d_3)ONWTRKJEUP$
$D_5 = (P_{33}, J, S)$	$J = 2345$ $\downarrow NEKA$ 6161	$S = 1678$ $\downarrow TVQC$ 5422	$1(d_3)EAPUVFBCQS$
$D_6 = (P_{33}, T, A)$	$T = 5678$ $\downarrow SWRO$ 1424	$A = 1234$ $\downarrow JFEB$ 5757	$4(d_3)OWNMFVUEJA$
$D_7 = (P_{33}, L, U)$	$L = 2367$ $\downarrow MCBK$ 4115	$U = 1458$ $\downarrow WPVE$ 7272	$1(d_3)BCAFVSQPEU$
$D_8 = (P_{35}, Y, A)$	$Y = 5678$ $\downarrow XVWO$ 1414	$A = 1234$ $\downarrow JSEP$ 5858	$1(d_1)XWQCDEUSPA$
$D_9 = (P_{35}, T, D)$	$T = 3478$ $\downarrow VRSM$ 5216	$D = 1256$ $\downarrow KWCE$ 3874	$1 \cup 2(d_1 \cup d_1)$ $RLKJEAPQCD$ $SUWX$
$D_{10} = (P_{35}, J, X)$	$J = 2345$ $\downarrow NEKA$ 6161	$X = 1678$ $\downarrow YQWC$ 5252	$1(d_1)AEPSUWDCQX$
$D_{11} = (P_{35}, L, U)$	$L = 2367$ $\downarrow MCRK$ 4185	$U = 1458$ $\downarrow VWSE$ 7632	$1 \cup 8(d_1 \cup d_1)$ $RTVYWXQPSU$ $CDEA$
$D_{12} = (P_{35}, N, Q)$	$N = 3456$ $\downarrow OKMJ$ 7272	$Q = 1278$ $\downarrow RXPC$ 3636	$2(d_1)JKAEDCLRPQ$
$D_{13} = (P_{35}, O, P)$	$O = 4567$ $\downarrow YMVN$ 8383	$P = 1238$ $\downarrow RSQA$ 7474	$3(d_1)NMJKLRTSAP$

Table 1: Candidate Dantzig figures from P_{25}, P_{29}, P_{31}, P_{33}, and P_{35}.

Dantzig figures	Edge-facet intersections		Facet for lemmas no.(type)comb.equiv.
$D = (P, x, y)$	$x = h_1 \cap h_2 \cap h_3 \cap h_4$ $\downarrow v_1 \quad v_2 \quad v_3 \quad v_4$ $h_{i_1}\ h_{i_2}\ h_{i_3}\ h_{i_4}$		$h_i(d_j)\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$
$D_{14} = (P_{36}, J, Q)$	$J = 2345$ $\downarrow NEKA$ 6161	$Q = 1678$ $\downarrow RXPH$ 2525	$1(d_4)AEWUSPDHXQ$
$D_{15} = (P_{36}, O, S)$	$O = 4567$ $\downarrow HMYN$ 1383	$S = 1238$ $\downarrow TUPA$ 7464	$3(d_1)NMJKLTVUAS$
$D_{16} = (P_{36}, D, V)$	$D = 1256$ $\downarrow KHPE$ 3784	$V = 3478$ $\downarrow YTUM$ 5216	$7 \cup 8(d_1 \cup d_1)$ $PSUWYXQRTV$ $HOML$
$D_{17} = (P_{36}, L, W)$	$L = 2367$ $\downarrow MRTK$ 4885	$W = 1458$ $\downarrow YXUE$ 7732	$8(d_1)RTQPSUVYXW$
$D_{18} = (P_{37}, A, R)$	$A = 1234$ $\downarrow JWEU$ 5858	$R = 5678$ $\downarrow TQPO$ 2114	$8(d_1)WUXYVTSPQR$
$D_{19} = (P_{37}, O, U)$	$O = 4567$ $\downarrow RMGN$ 8313	$U = 1238$ $\downarrow VWSA$ 7464	$3(d_1)NMJKLVYWAU$
$D_{20} = (P_{37}, X, K)$	$X = 1478$ $\downarrow YQWG$ 3535	$K = 2356$ $\downarrow NDLJ$ 4174	$3(d_1)WYAUVLMNJK$
$D_{21} = (P_{37}, D, Y)$	$D = 1256$ $\downarrow KPSE$ 3884	$Y = 3478$ $\downarrow XVWM$ 1216	$8(d_1)PSQRTVUWXY$

Table 1 (*cont'd*): Candidate Dantzig figures from P_{36} and P_{37}. The first column in the table identifies the Dantzig figure; $D = (P, x, y)$ indicates that the Dantzig figure is found in the $(4, 8)$-polytope P from the [GS] catalog, with estranged vertices x and y. The next two columns summarize the edge-facet intersections in the Dantzig figure; for example, the vertex $v_1 = h_{i_1} \cap h_2 \cap h_3 \cap h_4$ is the neighbor of x along the edge not incident to h_1. The last column in the table provides a specific combinatorial equivalence between a facet or pair of facets in the Dantzig figure and one of the graphs considered in the following lemmas.

A monotone orientation is a combinatorial phenomenon. In contrast, if P is a d-polytope with m vertices, an *LP-imbedding* (X, ϕ) of P is an assignment of coordinates $X_{d\times m}$ to the vertices of P and of a linear functional φ such that conv X is combinatorially equivalent to P and ϕ is admissible for X. An LP-imbedding induces an orientation on the edges of P; an edge is directed $v_i \to v_j$ iff $\phi(v_i) < \phi(v_j)$. An *LP-orientation* of P is an orientation of the edges of P which is induced by some LP-imbedding (X, ϕ) of P. Every LP-orientation is a monotone orientation of P.

In the proofs of the following four lemmas, we orient edges as necessary to avoid cycles, to preserve one-source/one-sink per face, and to maintain the required monotone distances. In each case, we arrive at a contradiction; none of the four graphs considered in the following lemmas can be given a monotone orientation that preserves the required monotone distances. Each of the Dantzig figures listed above is covered by at least one of these lemmas, as indicated by the fourth column in the table.

4.3 Lemma. *In the d_1 polytope $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$ of Figure 2, with sink ω, either α or β is at monotone distance 3 from ω.*

Proof. Since ω is the sink, we have $\iota \to \omega$, $\zeta \to \omega$, and $\theta \to \omega$. To keep β at monotone distance 4, we must take $\eta \to \beta$.

If $\delta \to \epsilon$, then $\epsilon \to \zeta$, $\epsilon \to \beta$, $\beta \to \alpha$, $\delta \to \alpha$, $\alpha \to \gamma$, $\delta \to \gamma$, $\gamma \to \iota$; but now $[\alpha, \gamma, \iota, \omega]$ is a monotone path from α to ω of length 3 .

On the other hand, if $\epsilon \to \delta$, then $\delta \to \gamma$, $\gamma \to \iota$; $\gamma \to \alpha$, $\delta \to \alpha$, $\alpha \to \beta$; $\epsilon \to \beta$, and β is a second sink. □

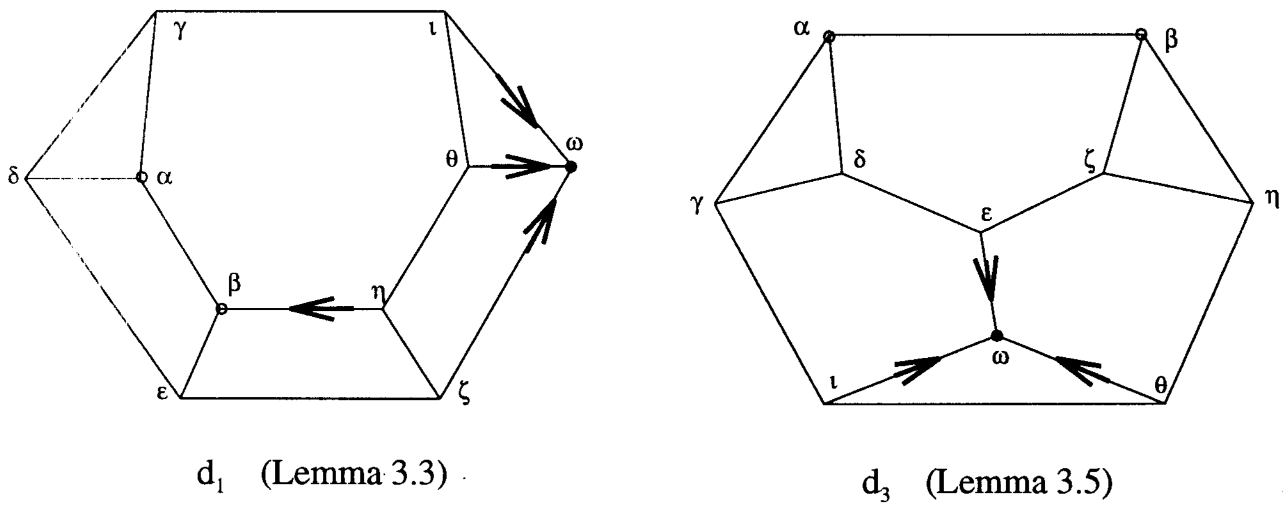


Figure 2: The diagrams for Lemmas 4.3 and 4.5. In each, the vertex ω is the sink, and for any monotone orientation, the monotone distance from one of the highlighted vertices to the sink is less than 4.

4.4 Corollary. *The Dantzig figures $D_1, D_3, D_8, D_{10}, D_{12}, D_{13}, D_{15}, D_{17}, D_{18}, D_{19}, D_{20}, D_{21}$ have strict monotone diameter 4.*

Proof. For each of these Dantzig figures, the last column in its entry in the table above gives a combinatorial equivalence between a d_1-facet of the Dantzig figure and the d_1 polytope $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$ of Lemma 4.3.

For example, in the table entry for D_1, the last column indicates that facet 2 is of type d_1, and the listed map of vertices yields a combinatorial equivalence between facet 2 in D_1 and $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$. Note that under this equivalence $Q(\Leftrightarrow \omega)$ is the sink, and by Lemma 4.3 either $J(\Leftrightarrow \alpha)$ or $K(\Leftrightarrow \beta)$ is at monotone distance 3 from Q. In D_1, if Q is the sink, then N is the source by Lemma 4.1. Since either J or K is at monotone distance 3 from Q, at least one of these directed edges $N \to K$ and $N \to J$ starts a monotone path from N to Q of length 4. Thus the strict monotone diameter of D_1 is 4. □

4.5 Lemma. *In the d_3 polytope $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$ of Figure 2, with sink ω, either α or β is at monotone distance 3 from ω.*

Proof. Since ω is the sink, we have $\iota \to \omega$, $\epsilon \to \omega$, and $\theta \to \omega$. We derive contradictions from all four possible orientations of the pair of edges $\gamma\delta$ and $\eta\zeta$.

If $\eta \to \zeta$ and $\gamma \to \delta$, then $\delta \to \epsilon$, $\zeta \to \epsilon$; $\delta \to \alpha$, $\zeta \to \beta$, but now the face $\alpha\beta\zeta\epsilon\delta$ has two sources.

If $\zeta \to \eta$ and $\delta \to \gamma$, then $\gamma \to \iota$, $\eta \to \theta$; $\eta \to \beta$, $\gamma \to \alpha$, and the face $\alpha\beta\eta\theta\iota\gamma$ has two sources.

If $\zeta \to \eta$ and $\gamma \to \delta$ (or $\eta \to \zeta$ and $\delta \to \gamma$ by symmetry), then $\delta \to \epsilon$, $\eta \to \theta$; $\eta \to \beta$, $\zeta \to \beta$; $\delta \to \alpha$, $\gamma \to \alpha$, and either β or α is a sink. □

4.6 Corollary. *The Dantzig figures D_2, D_4, D_5, D_6, D_7 have strict monotone diameter 4.*

Proof. For each of these Dantzig figures, the last column in its entry in the table above gives a combinatorial equivalence between a d_3-facet of the Dantzig figure and $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$.

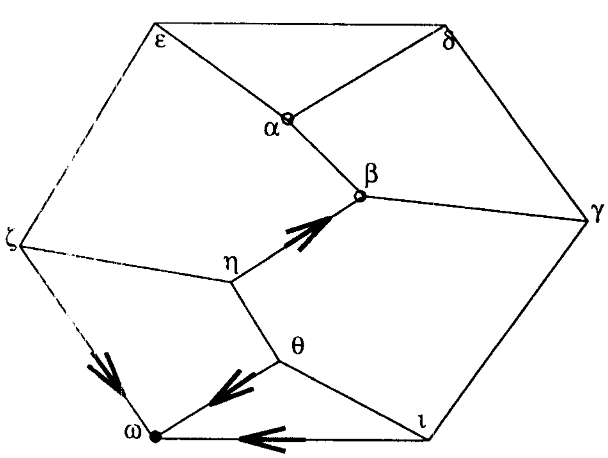
For example, in the table entry for D_2, we see that facet 2 is of type d_3. A combinatorial equivalence between facet 2 of D_2 and the d_3 polytope of Lemma 4.5 is listed by the images of the vertices; we see that $K(\Leftrightarrow \omega)$ is the sink, and either $E(\Leftrightarrow \alpha)$ or $O(\Leftrightarrow \beta)$ is at monotone distance 3 from K. However, V is the source, and at least one of the directed edges $V \to O$ and $V \to E$ initiates a monotone path of length 4 from V to K. Hence, the strict monotone diameter of D_2 is 4. □

4.7 Lemma. *In the d_4 polytope $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\iota\omega$ of Figure 3, with sink ω, either α or β is at monotone distance 3 from ω.*

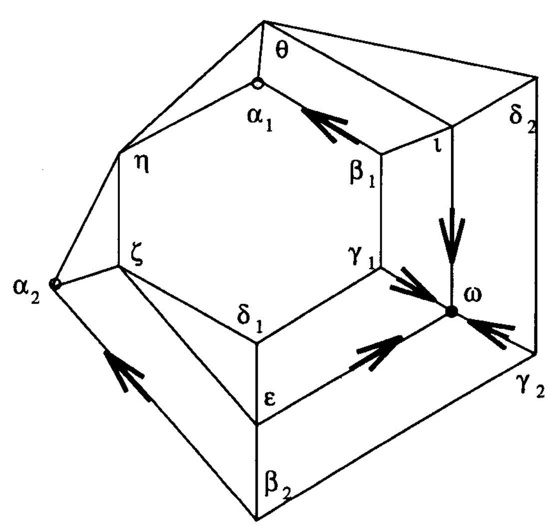
Proof. Since ω is the sink, we have $\iota \to \omega$, $\zeta \to \omega$, and $\theta \to \omega$. To keep β at monotone distance 4, we must take $\eta \to \beta$.

If $\delta \to \epsilon$, then $\epsilon \to \zeta$; $\epsilon \to \alpha$, $\delta \to \alpha$, $\alpha \to \beta$, $\beta \to \gamma$, $\delta \to \gamma$, $\gamma \to \iota$; but now $[\beta, \gamma, \iota, \omega]$ is a monotone path from β to ω of length 3.

On the other hand, if $\epsilon \to \delta$, then $\delta \to \gamma$, $\gamma \to \iota$; $\gamma \to \beta$, $\beta \to \alpha$, $\delta \to \alpha$, $\epsilon \to \alpha$, and α is a second sink. □



d_4 (Lemma 3.7)



$d_1 \cup d_1$ (Lemma 3.9)

Figure 3: The diagrams for Lemmas 4.7 and 4.9. In each, the vertex ω is the sink, and for any monotone orientation, the monotone distance from one of the highlighted vertices to the sink is less than 4.

4.8 Corollary. *The Dantzig figure D_{14} has strict monotone diameter 4.*

Proof. In the table entry for D_{14}, the last column indicates that facet 1 has type d_4. Under the listed combinatorial equivalence between facet 1 and the d_4 of Lemma 4.7, $Q(\Leftrightarrow \omega)$ is the sink, and either $A(\Leftrightarrow \alpha)$ or $E(\Leftrightarrow \beta)$ is at monotone distance 3 from Q. However J is the source, and at least one of the directed edges $J \to A$ or $J \to E$ initiates a monotone path from J to Q of length 4. Hence the strict monotone diameter of D_{14} is 4. □

4.9. Lemma. *For any consistent monotone orientation of the two adjacent d_1 polytopes $\alpha_1\beta_1\gamma_1\delta_1\epsilon\zeta\eta\theta\iota\omega$ and $\alpha_2\beta_2\gamma_2\delta_2\epsilon\zeta\eta\theta\iota\omega$ of Figure 3, with ω as sink, at least one of α_1 and α_2 is at monotone distance 3 from ω.*

Proof. Since ω is the sink, we have $\gamma_1 \to \omega$, $\gamma_2 \to \omega$, $\epsilon \to \omega$, and $\iota \to \omega$. To keep α_1 and α_2 at monotone distance 4, we must take $\beta_1 \to \alpha_1$ and $\beta_2 \to \alpha_2$.

If $\eta \to \zeta$, then $\zeta \to \epsilon$, $\zeta \to \alpha_2$, $\eta \to \alpha_2$, and α_2 is a sink.

If $\zeta \to \eta$, then $\eta \to \theta$, $\theta \to \iota$, $\theta \to \alpha_1$, $\eta \to \alpha_1$, and α_1 is a sink. □

4.10. Corollary. *The Dantzig figures D_9, D_{11}, D_{16} have strict monotone diameter 4.*

Proof. The proof here is similar to those of the previous corollaries, with the exception that here we must exhibit a combinatorial equivalence between a union of two d_1 facets of the Dantzig figure and the graph considered in Lemma 4.9. These equivalences are listed in the fourth column of the table entries for D_9, D_{11}, D_{16}, with the first line identifying the two facets, the second line listing the images of $\alpha_1\beta_1\gamma_1\delta_1\epsilon\zeta\eta\theta\iota\omega$ under this equivalence, and the third line listing the images of $\alpha_2\beta_2\gamma_2\delta_2$.

For example, from the table entry for D_9, we see that facets 1 and 2 are both of type d_1, and their union is combinatorially equivalent to the union of two d_1 polytopes as in Lemma 4.9. Under the listed equivalence, $D(\Leftrightarrow \omega)$ is the sink, and either $R(\Leftrightarrow \alpha_1)$ or $S(\Leftrightarrow \alpha_2)$ is at monotone distance 3 from D. In D_9, T is the source, and so at least one of the directed edges $T \to R$ or $T \to S$ initiates a monotone path from T to D of length 4. □

The above sequence of lemmas and corollaries leads to the following, our main result.

4.11. Theorem. $\Delta_{sm}(4,8) = 4$.

Proof. The 37 combinatorial types of $(4,8)$-polytopes are listed in [GS]. The strict monotone diameter of the 4-cube is equal to 4. The purpose of the lemmas and corollaries is to show that no $(4,8)$-polytope has strict monotone diameter greater than 4. This involves geometric realizations and LP-orientations of the various combinatorial types.

First, lemmas 4.1 and 4.2 show that it suffices to consider LP-orientations of Dantzig figures $D = (P, x, y)$ in which

the estranged vertices x and y are source and sink in the LP-orientation;
every edge from from x terminates on a $(3,7)$-facet incident to y and *vice versa*;
the $(4,8)$-polytope P has at least 18 vertices.

These requirements rule out all but seven of the 37 $(4,8)$-polytopes, leaving as candidates only the 21 Dantzig figures listed in Table 1.

For each candidate Dantzig figure $D = (P, x, y)$, Lemmas 4.3, 4.5, 4.7, and 4.9 served to identify a pair of edge-neighbors of x such that any monotone orientation of P with x as source and y as sink leaves at least one of these two neighbors at monotone distance 3 from y. The fourth column of Table 1 provides a key to our specific arguments along these lines, showing that each of the 21 Dantzig figures is covered by one of the lemmas. It follows that under any monotone orientation of P with x as source and y as sink, the monotone distance from x to y is 4.

The strict monotone 4-step conjecture is a statement about LP-orientations. Since every LP-orientation is a monotone orientation, we conclude that the strict monotone diameter of each of these 21 Dantzig figures is 4, and since these 21 Dantzig figures exhaust the possibilities for a higher strict monotone diameter, $\Delta_{sm}(4,8) = 4$. □

5. Comment

We suspect that the strict monotone d-step conjecture is false when d is sufficiently large, and that it may therefore eventually be added to [KK]'s list of strengthenings of the d-step conjecture that hold for $d \leq 3$ but fail for some larger d. More specifically, we suspect that the failure of the strict monotone d-step conjecture can be shown by means of a polytope combinatorially equivalent to the polytope P_d that was used in [HK] to provide a counterexample to the Lagarias-Prabhu-Reeds strengthening [LPR] of the d-step conjecture. For the $(5,10)$-polytope P_5 of [HK], we have produced a monotone orientation for which the strict monotone diameter is 6. If this monotone orientation is an LP-orientation, then $\Delta_{sm}(5,10) \geq 6$.

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