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THE d-STEP CONJECTURE AND ITS RELATIVES*†

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The d-step conjecture arose from an attempt to understand the computational complexity of edge-following algorithms for linear programming, such as the simplex algorithm. It can be stated in terms of diameters of graphs of convex polytopes, in terms of the existence of nonrevisiting paths in such graphs, in terms of an exchange process for simplicial bases of a vector space, and in terms of matrix pivot operations. First formulated by W. M. Hirsch in 1957, the conjecture remains unsettled, though it has been proved in many special cases and counterexamples have been found for slightly stronger conjectures. If the conjecture is false, as we believe to be the case, then finding a counterexample will be merely a small first step in the line of investigation related to the conjecture. This report summarizes what is known about the d-step conjecture and its relatives. A considerable amount of new material is included, but it does not seem to come close to settling the conjecture. Of special interest is the first example of a polytope that is not vertex-decomposable, showing that a certain natural approach to the conjecture will not work. Also significant are the quantitative relations among the lengths of paths associated with various forms of the conjecture.

0. Introduction. Stimulated since the early 1950's by its relationship to linear programming, and more recently by connections with other computational areas, the combinatorial study of convex polytopes has advanced greatly in the past 30 years. In 1957 Motzkin [Mo′] conjectured that the maximum number of vertices of d-polytopes with n facets (dually, of facets of d-polytopes with n vertices) is

$$\binom{n - \lfloor (d+1)/2 \rfloor}{n-d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n-d}.$$

In 1961–64 Fieldhouse [Fi], Gale [Ga1] and Klee [Kl2] went far toward proving this upper bound conjecture, and in 1970 McMullen [Mc1] proved the result for all d and n by developing a new approach based on the shelling technique of Bruggesser and Mani [BM]. In 1975 Stanley [St1] used the theory of Cohen-Macaulay rings to extend the upper bound theorem to arbitrary triangulated spheres.

In 1909 Brückner [Br] conjectured that the minimum number of vertices of simple d-polytopes with n facets (dually, of facets of simplicial d-polytopes with n vertices) is

$$(n-d)(d-1)+2.$$

In 1970 this was proved by Walkup [Wa1] for $d \leqslant 5$, and in 1971 Barnette [Ba′3] developed a new inductive approach to prove the result for all d and n. Later the lower bound theorem was extended to arbitrary triangulated manifolds [Ba′4, 5] and even pseudomanifolds [Kl6].

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The *f-vector* of a polytope lists the numbers of faces of various dimensions. In 1971 McMullen [Mc2] proposed a purely combinatorial characterization of the f-vectors of simple polytopes (dually, of simplicial polytopes), subsuming the upper bound conjecture and the lower bound conjecture. In 1980 the sufficiency of his conditions was proved by Billera and Lee [BL] and the necessity by Stanley [St3], the latter using tools from algebraic geometry.

In 1972 Klee and Minty [KM] disproved the long-standing conjecture that the *worst-case* behavior of the simplex algorithm of linear programming (LP) is polynomially bounded. Their examples, using the pivot rule of Dantzig [Da1], were followed by similar examples based on other pivot rules ([Za1, 2], [Je], [AC], [GS], [Cu], [Mu2], [Cl], [Bl], [Go]), but the possibility remains that the simplex method can be turned into a polynomially bounded edge-following LP algorithm by use of a suitable pivot rule. Zadeh [Za3] formulates a pivot rule that he conjectures to have this property. In any case, after preliminary studies by others [Li], it was proved in 1978–83 by Borgwardt [Bo1, 2, 3] and Smale [Sm] that the *average-case* behavior of the simplex method is polynomially bounded under reasonable assumptions on the pivot rule and the distribution of input data. There has been much recent work in this area. (See [AM], [AMT] and their references.)

In 1979 Khachian [Kh] showed that the ellipsoid method of Shor [Sh″], Judin and Nemirovskii [JN1, 2] provides an LP algorithm whose *worst-case* behavior is bounded by a polynomial in the length of the binary encoding required to present an LP problem to a Turing machine. Though the ellipsoid method has not proved to be useful for actual LP computations, its polynomial boundedness has led to many new results on the computational complexity of optimization problems [BGT], [GLS]. The 1983 method of Karmarkar [Ka] provides a better polynomial bound (still in terms of the binary encoding) and seems computationally promising as well. However, it is still unknown whether there is an LP algorithm whose worst-case behavior is polynomially bounded in a way that is independent of the method of encoding the input data, and Megiddo [Me1] argues that undue importance has been given to the binary encoding. The earlier speculation on the simplex algorithm had implicitly assumed infinite-precision real arithmetic and conjectured that the number of arithmetic operations required to solve an LP problem involving d real variables and n linear inequalities is bounded by a polynomial in d and n. Traub and Wozniakowski [TW] argue that for measuring usefulness in practical computation, the infinite-precision model is more appropriate than the Turing-machine model. For the pivot rules that have been successfully analyzed, the worst-case behavior of the simplex method is not polynomially bounded in either model. However, since the simplex method usually works so well in practice [Da2], there is continuing practical as well as theoretical interest in studying the computational complexity of edge-following LP algorithms. The d-step conjecture and its relatives play a central role in this study.

The *d-step conjecture* was formulated by W. M. Hirsch in 1957, and reported in the 1963 book of Dantzig [Da2] and his 1964 article [Da3] on unsolved problems from mathematical programming. One of its several equivalent forms concerns $\Delta(d, n)$, the maximum diameter of (the graphs of) d-polytopes with n facets; it asserts $\Delta(d, 2d) = d$. For $d \leqslant 5$, this was proved by Klee and Walkup [KW] in 1967, and the dual form of their result was extended by Adler and Dantzig [AD] in 1974 to a class of simplicial complexes that includes (for $d \leqslant 5$) all triangulated $(d-1)$-manifolds with $2d$ vertices. The d-step conjecture is open for all $d \geqslant 6$.

The *Hirsch conjecture*, also reported in [Da2], asserts that $\Delta(d, n) \leqslant n - d$ for all d and n (we always assume implicitly that $n > d \geqslant 2$). It is known [KW] that $\Delta(d, n) \leqslant n - d$ for all d and n if and only if $\Delta(d, 2d) = d$ for all d; thus the two conjectures are equivalent, though not necessarily on a dimension-for-dimension basis. The Hirsch conjecture holds for $d = 3$ and all n ([Kl1]), even in unbounded or monotone versions ([Kl3, 4]) and it holds whenever $n - d \leqslant 5$ [KW]; however, the unbounded version fails for $(d, n) = (4, 8)$ [KW] and the monotone version fails for $(d, n) = (4, 9)$ ([To]). The Hirsch conjecture is open for all (d, n) with $d \geqslant 4$ and $n \geqslant \max\{12, d + 6\}$.

Indeed, when d and $n - d$ are both large the sharpest known lower and upper bounds are as follows, due respectively to Adler ([Ad]) and Larman ([La]):

$$\left\lfloor (n-d) - \frac{n-d}{\lfloor 5d/4 \rfloor} \right\rfloor - 1 \leqslant \Delta(d, n) \leqslant 2^{d-1}n.$$

These bounds imply that, for each fixed d, $\Delta(d, n)$ grows only linearly with increasing n, but they don't tell whether $\Delta(d, 2d)$ grows exponentially or algebraically with increasing d. Any polynomial upper bound on $\Delta(d, 2d)$ would be of great interest.

As formulated in [Da2], the d-step conjecture and the Hirsch conjecture did not assume boundedness of the polyhedron in question. However, since it is now known that the unbounded forms are false for $d \geqslant 4$, the terms as used here refer only to the bounded case. Nevertheless, we use the term *Hirsch polyhedron* to describe any polyhedron, bounded or not, that is of dimension d, has n facets, and is of diameter $\leqslant n - d$. Thus not all polyhedra are Hirsch polyhedra, but the Hirsch conjecture asserts that all polytopes are Hirsch polytopes. Several special classes of polyhedra, including those arising as feasible regions of some important classes of LP problems, have been shown to be Hirsch polyhedra, but even for transportation polytopes the Hirsch conjecture has not been fully settled.

In research on the d-step conjecture and its relatives, the following *nonrevisiting conjecture* of Klee and P. Wolfe (also called the *W_v conjecture*) has played an important role: *Any two vertices of a simple polytope P can be joined by a path that does not revisit any facet of P.* This implies that P is a Hirsch polytope, and the general nonrevisiting conjecture is known [KW] to be equivalent to the Hirsch conjecture, though not necessarily on a dimension-for-dimension basis. For $d = 3$, some strengthened forms of the nonrevisiting conjecture are proved in [Kl3, 4] and [Ba′1].

As formulated for simple polytopes, the d-step, Hirsch, and nonrevisiting conjectures all admit dual (or polar) equivalents that deal with simplicial polytopes and concern the "dual paths" formed by certain sequences of successive facets rather than the "primal paths" formed by sequences of successively adjacent vertices. Since the boundary complex of a simplicial d-polytope is a triangulated $(d-1)$-sphere, it is natural to extend the dual equivalent conjecture to all triangulated spheres. However, the extended dual d-step conjecture fails for a triangulated 11-sphere with 24 vertices and the extended dual nonrevisiting conjecture fails for a triangulated 3-sphere with 16 vertices. These constructions are due to Mani and Walkup ([MW]).

Provan and Billera ([Pr], [PB1]) show that if a pure simplicial complex $\mathscr{C}$ has a property known as vertex-decomposability (which requires that $\mathscr{C}$ can be constructed from smaller complexes in an especially simple way), then the dual form of the nonrevisiting conjecture is valid for $\mathscr{C}$. It was thought that this might lead to a proof of the dual Hirsch conjecture for simplicial polytopes. However, it is shown here that for each $d \geqslant 4$ there is a simplicial d-polytope with $d + 6$ vertices whose boundary complex is not vertex-decomposable.

As is clear from the brief summary just provided, progress on the d-step conjecture and its relatives has been slight in comparison with other advances in understanding the combinatorial structure of convex polytopes. Because of intrinsic interest, because of connections with questions of computational complexity, and because full understanding may well require the development of deep new methods, the d-step conjecture and its relatives are probably the most important open problems in the combinatorial study of polytopes. This applies especially to the problem of finding sharp asymptotic estimates for $\Delta(d, n)$, which may turn out to be much more difficult than "'merely" settling the d-step conjecture.

The present report contains a comprehensive survey of what is known about the d-step conjecture and its relatives. Its main purpose is to survey the present state of knowledge, and in many cases this is done by merely stating a result and giving a reference. In other cases, arguments are outlined as an indication of the methods that have been used. We also include several new results, and sharpened versions or better proofs of old ones. The most important new results are the quantitative relations among the lengths of paths associated with various relatives of the d-step conjecture (§2), and the examples of simplicial polytopes that are not vertex-decomposable (§6). We also consider (in §8) the conjecture of Saigal [Sa] that if a d-polytope Q in $\mathbb{R}^d$ is the intersection of a cube with a Hirsch polyhedron, then Q itself is a Hirsch polytope. This is shown to be equivalent to the general Hirsch conjecture.

The section headings are as follows: 1. Equivalent statements; 2. Proofs of equivalence; 3. Relations to linear programming; 4. Low-dimensional results; 5. Relatives in more general complexes; 6. More counterexamples to stronger statements; 7. General lower and upper bounds; 8. Bounds for special classes of polyhedra.

1. Equivalent statements. Some ambitious readers may prefer, rather than reading our survey, to set right to work to settle the d-step conjecture. It is stated here in several equivalent forms, some (from [KW]) concerning the facial structure of polytopes, one (from [K17]) involving an analogue of the Steinitz exchange process, and one (implicitly from [Da2]) involving matrix pivot operation. Proofs of equivalence appear in §2, which discusses in more detail the interrelations among the diameter functions Δ and Δ_u, the revisit functions R and R_u, the exchange functions E and E_u, and the pivot functions Π and Π_u defined there. As is seen from §3's discussion of the relationship to linear programming, settling the d-step conjecture is only a first step, and perhaps not even a necessary one, toward understanding the worst-case complexity of edge-following algorithms for linear programming. Nevertheless, the d-step conjecture itself is of great interest and has been attacked unsuccessfully by a number of authors.

As the terms are used here, a *polyhedron* is the intersection of a finite collection of closed halfspaces in a finite-dimensional real vector space and a *polytope* is a bounded polyhedron; equivalently, a polytope is the convex hull of a finite set. A *face* of a polyhedron P is the empty set $\varnothing$, P itself, or the intersection of P with a supporting hyperplane. Prefixes indicate dimension, and the 0-, 1-, $(d-2)$- and $(d-1)$-faces of a d-polyhedron are respectively its *vertices*, *edges*, *ridges* and *facets*. A polyhedron is *pointed* if it has at least one vertex. A d-polyhedron is *simple* if it is pointed and each of its vertices is incident to precisely d edges or, equivalently, to precisely d facets. A d-polytope is *simplicial* if each of its facets is a simplex. For the facial and combinatorial structure of polyhedra and polytopes, the basic references are Grünbaum [Gr'1], McMullen and Shephard [MS] and Brøndsted [Br']. See also Bartels [Ba"] and Yemelichev, Kovalev and Kravtsov [YKK].

The *graph* of a polyhedron P is the combinatorial structure formed by P's vertices and bounded edges. Each such graph is connected, and for a d-polytope the graph is d-connected (Balinski [Bal]) and many other properties are known (see Grünbaum [Gr'2] for a survey). When u and v are vertices of a polyhedron P, $\delta_P(u, v)$ denotes the *distance* between u and v in the graph of P, that is, the length (number of edges) of the shortest path from u to v. The *diameter* $\delta(P)$ is defined as the diameter of P's graph, thus the maximum of $\delta_P(u, v)$ over all pairs of vertices (u, v); equivalently, $\delta(P)$ is the least integer l such that any two vertices of P are joined by a path formed from l or fewer edges of P. For each $n > d \geqslant 2$, let $\Delta(d, n)$ $\langle$resp. $\Delta_u(d, n)\rangle$ denote the maximum of the diameters of d-polytopes $\langle d$-polyhedra$\rangle$ with n facets. It is known [KW] that these maxima are attained by simple polytopes and polyhedra.

Now suppose that

(i) X and X' are affine orthants in $\mathbb{R}^d$, with respective vertices $x \in \operatorname{int} X'$ and $x' \in \operatorname{int} X$, and

(ii) the intersection $P = X \cap X'$ is a simple d-polytope.

The d-step conjecture, $\Delta(d, 2d) = d$, implies $\delta_P(x, x') = d$ *whenever* (i) *and* (ii) *hold*, and this seemingly special case is actually equivalent to the d-step conjecture [KW]. For a simple example satisfying (i) and (ii), let X be the positive orthant $\mathbb{R}^d_+$ in $\mathbb{R}^d$, X' the origin $(0, 0, \ldots, 0)$, x' the point $(1, 1, \ldots, 1)$, and $X' = x' - \mathbb{R}^d_+$, a translate of the negative orthant. Then the intersection P is merely the d-cube $[0, 1]^d$, so of course $\delta_P(x, x') = d$. In this example, P has exactly 2^d vertices, but the number of vertices of an intersection P satisfying (i) and (ii) ranges from $d^2 - d + 2$ to $2\binom{3k-1}{k}$ when $d = 2k$ and to $2\binom{3k+1}{k}$ when $d = 2k + 1$. The lowest dimension for which the d-step conjecture is open is $d = 6$. For the simple 6-polytopes with 12 facets, the number of different f-vectors is 235 and it would not surprise us if the number of combinatorial types exceeds 10^9. (It follows from a theorem of Goodman and Pollack [GP] that the number of combinatorial types does not exceed $12e^{504}$.)

A path on a polyhedron, formed by successively adjacent vertices $v_0, \ldots, v_l$, is *nonrevisiting* if for each facet F and triple (i, j, k) such that $i < j < k$ and $v_i, v_k \in F$, the vertex v_j also belongs to F. For each fixed d, the Hirsch conjecture, $\Delta(d, n) \leqslant n - d$ for all $n > d$, is implied by the assertion that *any two vertices of a simple d-polytope are joined by a nonrevisiting path*; the d-step conjecture is equivalent to the same assertion for simple d-polytopes with $2d$ facets [KW]. (Nonrevisiting paths were originally called W_v paths [K13, 4].)

There are polar equivalents of all forms of the d-step conjecture stated thus far. Recall that when the origin of $\mathbb{R}^d$ is interior to a d-polytope P, and the polar polytope $Q = P^0$ is given by $P^0 = \{y \in \mathbb{R}^d : \langle x, y\rangle \leqslant 1 \text{ for all } x \in P\}$, then Q's face-lattice is anti-isomorphic to that of P. The anti-isomorphism preserves incidence and complements dimension, so that the k-faces of Q correspond to the $(d - 1 - k)$-faces of P. Vertices of P joined by an edge correspond to facets of Q that intersect in a ridge. Thus, for example, the polar equivalent of the above conjecture concerning intersections of affine orthants is as follows: *Whenever F and G are disjoint facets of a simplicial d-polytope with* $2d$ *vertices in all (so that each vertex belongs to F or G), there is a sequence* $F_0, F_1, \ldots, F_d = G$ *of facets such that* $F_i \cap F_{i-1}$ *is a ridge for* $1 \leqslant i \leqslant d$.

A set $B \subset \mathbb{R}^{d-1}$ is a *simplicial basis* (also called a *minimum positive basis*) for $\mathbb{R}^{d-1}$ if it is the vertex-set of a $(d-1)$-simplex whose interior includes the origin. Equivalently, B is affinely independent, $|B| = d$, and the origin 0 is a strictly positive combination of the points of B. An equivalent of the d-step conjecture is reminiscent of the exchange argument used to show that all linear bases of a vector space are of the same cardinality. It asserts that *if B and B′ are disjoint simplicial bases of* $\mathbb{R}^{d-1}$ *and every* $(d-1)$*-set in the union* $U = B \cup B'$ *is linearly independent, then there is a sequence* $B = B_0, B_1, \ldots, B_d = B'$ *of simplicial bases such that for* $1 \leqslant i \leqslant d$, B_i *is obtained from* B_{i-1} *by replacing a point of* B_{i-1} *with a point of* $U \sim B_{i-1}$. In the example below, $d = 3$ and $0 < \epsilon < 1$. The rows represent points of B_i:

$$\underset{B_0 = B}{\begin{matrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{matrix}} \to \underset{B_1}{\begin{matrix} 1 & 0 \\ 0 & 1 \\ -\epsilon & -1 \end{matrix}} \to \underset{B_2}{\begin{matrix} 1 & 1-\epsilon \\ 0 & 1 \\ -\epsilon & -\epsilon \end{matrix}} \to \underset{B_3 = B'}{\begin{matrix} 1 & 1-\epsilon \\ -1 & -\epsilon \\ -\epsilon & -1 \end{matrix}}$$

We close this section with an equivalent of the d-step conjecture that is closely related to linear programming methods: *If the real $d \times (2d+1)$ matrices $A = (I, B, c)$ and $A' = (B', I, c')$ are row-equivalent, where $d \geqslant 2$, I is the $d \times d$ identity matrix, and the columns c and c' are > 0, and if the polyhedron $P = \{x \in \mathbb{R}^{2d}_{+}\colon (I, B)x = c\}$ is bounded, then it is possible to pass from A to A' by a sequence of at most d feasible pivots followed if necessary by a permutation of rows.* Here a *pivot*, as applied to an $m \times (n+1)$ matrix $S = [s_{ij}]$, is the operation of choosing (i, j) with $j \leqslant n$ and $s_{ij} \neq 0$, then dividing the ith row of S by s_{ij} so as to obtain 1 in position (i, j), and finally subtracting appropriate multiples of the ith row from other rows so as to obtain 0 in all positions (h, j) for $h \neq i$. A pivot is *feasible* if the last column of the matrix is nonnegative both before and after the pivot. In the example below, $d = 2$ and the pairs (i, j) under the arrows indicate the positions of the pivot entries:

$$\begin{array}{ccccc} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 2 & 2 \end{array} \xrightarrow[(1,3)]{} \begin{array}{ccccc} 1/2 & 0 & 1 & -1/2 & 1 \\ 1/2 & 1 & 0 & 3/2 & 3 \end{array} \xrightarrow[(2,4)]{} \begin{array}{ccccc} 2/3 & 1/3 & 1 & 0 & 2 \\ 1/3 & 2/3 & 0 & 1 & 2 \end{array}$$

$$I \quad B \quad c \qquad\qquad\qquad\qquad\qquad\qquad B' \quad I \quad c'$$

2. Proofs of equivalence. This section establishes the equivalence of the forms of the d-step conjecture mentioned in §1. More significantly, since we believe the conjecture is false, it establishes close quantitative relations among the step, revisit, exchange and pivot notions that appear in §1. The unbounded case is also considered.

Our functions Δ and Δ_u are denoted respectively by Δ_b and Δ in [KW], whose results include the following.

2.1. *For $2 \leqslant d < n$, $\Delta(d, n)$ $\langle$resp. $\Delta_u(d, n)\rangle$ is realized as the distance $\delta_p(u, v)$ between two vertices u and v of a simple d-polytope $\langle$simple d-polyhedron$\rangle$ with n facets, and when $n \geqslant 2d$ the requirement may be added that u and v do not lie on the same facet of P. For Δ_u it may be required also that u and v are incident to unbounded edges of P.*

2.2. *For $2 \leqslant d < n$ it is true that:*
$\Delta(d, n) \leqslant \Delta(d, n+1)$, $\Delta(d, n) \leqslant \Delta(d+1, n+1)$ *and* $\Delta(d, n) < \Delta(d+1, n+2)$;
$\Delta(d, n) \leqslant \Delta(n-d, 2(n-d))$ *with equality when* $n \leqslant 2d$;
$\Delta_u(d, n) < \Delta_u(d, n+1)$ *and* $\Delta_u(d, n) < \Delta_u(d+1, n+1)$;
$\Delta_u(d, n) \leqslant \Delta_u(n-d, 2(n-d))$ *with equality when* $n \leqslant 2d$.

In particular, for the complete determination of Δ and Δ_u it suffices to consider the cases in which $n \geqslant 2d$.

For a path $v_0, \ldots, v_l$ formed from successively adjacent vertices of a polyhedron P, a *revisit* is a triple (F, i, k) such that F is a facet of P, $0 \leqslant i < i+1 < k \leqslant l$, v_i and v_k are incident to F, and for $i < j < k$ the vertex v_j is not incident to F. For two vertices u and v of P, let $\rho_P(u, v)$ denote the minimum number of revisits among paths from u to v on P. The following remarks extend ones in [Kl3, 4] and [KW].

2.3. *If u and v are vertices of a d-polyhedron P with n facets then*

$$\delta_P(u, v) \leqslant n - d + \rho_P(u, v).$$

PROOF. Let $u = v_0, \ldots, v_l = v$ be a path from u to v that has r revisits. Let $F_1, \ldots, F_d$ be facets of P that are incident to v_0, and for $1 \leqslant k \leqslant l$ let F_{d+k} be a facet that is incident to v_k but not to v_{k-1}. Let $s = \sum_{F \text{ facet of } P} |\{i\colon F_i = F\}|$. Then it is easy to verify that

$$s - (d + l) \leqslant r \quad \text{and} \quad 2(d + l) - s \leqslant n$$

whence $d + l \leqslant n + r$ and the stated conclusion follows. ∎

2.4. *If u and v are vertices of a simple d-polyhedron P with $2d$ facets, and no facet of P is incident to both u and v, then $\delta_P(u, v) = d + \rho_P(u, v)$.*

PROOF. Let the notation be as in the proof of 2.3. By simplicity, for $1 \leqslant k \leqslant l$ the facet F_{d+k} is the unique facet that is incident to v_k but not to v_{k-1}. From this it follows that each facet of P appears among the F_i's $(1 \leqslant i \leqslant d + l)$ and the three inequalities of the preceding argument become equalities. ■

The *revisit number* $\rho(P)$ of a polyhedron P is the maximum of $\rho_P(u, v)$ over all pairs (u, v) of vertices of P. For $n > d$, $\overline{R}(d, n)$ $\langle$resp. $\overline{R}_u(d, n)\rangle$ is defined as the maximum of $\rho(P)$ over all d-polytopes $\langle d$-polyhedra$\rangle$ P with n facets; the functions R and R_u are similarly defined with respect to simple polytopes and polyhedra. Two of the assertions of 2.2 may be strengthened as follows, by extending the reasoning of [KW].

2.5. $\Delta(d, n) \leqslant n - d + R(d, n) \leqslant \Delta(n - d, 2(n - d))$ *and*

$$\Delta_u(d, n) \leqslant n - d + R_u(d, n) \leqslant \Delta_u(n - d, 2(n - d)),$$

with equalities when $n \leqslant 2d$.

PROOF. For the inequalities $\Delta(d, n) \leqslant \cdots$ and $\Delta_u(d, n) \leqslant \ldots$, use 2.3 and the fact that the maxima Δ and Δ_u are attained for simple polytopes and polyhedra. To establish the remaining inequalities, consider two vertices u and v of a simple d-polyhedron P with n facets, let c denote the dimension of the smallest face F of P that is incident to both u and v, and let k denote the number of facets of F that are incident to neither u nor v. Then F is a simple c-polyhedron whose number of facets is $2c + k \leqslant n - (d - c)$. It follows from the simplicity of F that each of its facets is the intersection with F of a unique facet of P, and this implies $\rho_F(u, v) \geqslant \rho_P(u, v)$.

Assume without loss of generality that $F \subset \mathbb{R}^c$; let $Q_0 = F$, $u^0 = u$ and $v^0 = v$. Then the following conditions are satisfied when $i = 0$:

Q_i is a simple $(c + i)$-polyhedron in $\mathbb{R}^{c+i}$, bounded when P is bounded;

u^i and v^i are vertices of Q_i such that $\rho_{Q_i}(u^i, v^i) = \rho_P(u, v)$;

Q_i has $c + i$ facets incident to u^i, $c + i$ other facets incident to v^i, and an additional $k - i$ facets, thus $2c + k + i$ facets in all.

As long as $i < k$, the construction may be continued inductively as follows, using the "wedging" process of [KW]:

let G_i be a facet of Q_i that is incident to neither u^i nor v^i;

in the space $\mathbb{R}^{c+i+1} = \mathbb{R}^{c+i} \times \mathbb{R}$, let J_i be a closed halfspace with bounding hyperplane H_i such that $H_i \cap (\mathbb{R}^{c+i} \times \{0\}) = G_i \times \{0\}$, and for each vertex v of $Q_i \sim G_i$, H_i intersects the open ray $\{v\} \times]0, \infty[$ in a point v';

let $Q_{i+1} = J_i \cap (Q_i \times [0, \infty[)$, $u^{i+1} = (u^i, 0)$, $v^{i+1} = (v^{i+1})'$.

This leads to a simple $(c + k)$-polyhedron Q_k, bounded when P is bounded, and two vertices u^k and v^k of Q_k such that Q_k has $2(c + k)$ facets in all, with $c + k$ incident to u^k and $c + k$ others incident to v^k. It follows from 2.4 that

$$c + k + \rho_{Q_k}(u^k, v^k) = \delta_{Q_k}(u^k, v^k).$$

Now note that

$$(n - d) - (c + k) = t \geqslant 0.$$

and for $k \leqslant i \leqslant k + t$ let

$$Q_{i+1} = Q_i \times [0,1], \quad u^{i+1} = (u^i, 0), \quad v^{i+1} = (v^i, 1).$$

Since, for i in this range,

$$\rho_{Q_{i+1}}(u^{i+1}, v^{i+1}) = \rho_{Q_i}(u^i, v^i) \quad \text{and} \quad \delta_{Q_{i+1}}(u^{i+1}, v^{i+1}) = \delta_{Q_i}(u^i, v^i) + 1,$$

it follows that

$$n - d + \rho_P(u, v) \leqslant n - d + \rho_F(u, v) = \delta_{Q_{k+t}}(u^{n-d}, v^{n-d})$$

But Q_{k+t} is a simple $(n-d)$-polyhedron, bounded when P is bounded, and has $2(n-d)$ facets. That completes the proof. ∎

We turn now to the formulation of the d-step conjecture in terms of the exchange process for simplicial bases. Both Δ and Δ_u are considered. In the case of Δ, the following result asserts $\Delta(d, n) = E(n-d-1, n)$ for $2 \leqslant d < n-1$; in particular, $\Delta(d, 2d) = E(d-1, 2d)$. A subset of $\mathbb{R}^d$ is a *Haar set* if each of its subsets of cardinality $\leqslant d$ is linearly independent.

2.6. *For $1 \leqslant h < h+3 \leqslant n \leqslant 2k+2$, let $E(h, n)$ and $E_u(h, n)$ be defined as follows:*

$E(h, n)$ is the smallest integer r such that whenever B and B' are simplicial bases of $\mathbb{R}^h$ whose union $U = B \cup B'$ is a Haar set of cardinality n, there is a sequence $B = B_0, B_1, \ldots, B_k = B'$ of simplicial bases with $k \leqslant r$ such that for $1 \leqslant i \leqslant k$, B_i is obtained from B_{i-1} by replacing a point of B_{i-1} with a point of $U \sim B_{i-1}$;

$E_u(h, n)$ is the smallest integer s such that whenever B and B' are simplicial bases of $\mathbb{R}^{h+1}$ whose union $U = B \cup B'$ is a Haar set of cardinality $n+1$, and $q \in B \cap B'$, there is a sequence $B = B_0, B_1, \ldots, B_l = B'$ of simplicial bases with $l \leqslant s$ such that each B_i includes q and, for $1 \leqslant i \leqslant k$, B_i is obtained from B_{i-1} by replacing a point of $B_{i-1} \sim \{q\}$ with a point of $U \sim B_{i-1}$.

Then $E(h, n) = \Delta(n-h-1, n)$ and $E_u(h, n) = \Delta_u(n-h-1, n)$.

Proof. We first show

$$E(h, n) \leqslant \Delta(n-h-1, n) \quad \text{and} \quad E_u(h, n) \leqslant \Delta_u(n-h-1, n).$$

Let the set of all real functions x on U such that $\Sigma_{u \in U} x(u)u = 0$ be denoted by L; and such that $x \geqslant 0$ and $\Sigma_{u \in U} x(u) = 1$ be denoted by S; and let $P = L \cap S$. Then L is the subspace of $\mathbb{R}^{|U|}$ consisting of all linear relations on U, S is the standard $(|U|-1)$-simplex in $\mathbb{R}^{|U|}$, and P is the set of all convex relations on U. Since each of B and B' is the support of a convex relation, so is U and hence L intersects the relative interior of S. Since U is a Haar set, L is of dimension $n-h$ and P of dimension $d = n-h-1$. The facets of P are whichever of the sets $F_u = \{x \in P: x(u) = 0\}$ are of dimension $d-1$, so P has at most $|U|$ facets. By a theorem of Davis [Da'2], the vertices of P are precisely the minimal convex relations of U—that is, those with minimal support. But U is a Haar set in $\mathbb{R}^t$, where t is h or $h+1$ according as E or E_u is under discussion, so each minimal convex relation on U has support of cardinality $t+1$ and two such relations are adjacent as vertices of P if and only if their supports have precisely t points in common.

It is now clear that there is a natural correspondence between the sequences of bases $B = B_0, B_1, \ldots, B_k = B'$, of the sort involved in the definition of $E(h, n)$, and certain paths on the polytope P. Hence there is such a sequence for which

$$k \leqslant \delta(P) \leqslant \Delta(n-h-1, n)$$

and it follows that

$$E(h, n) \leqslant \Delta(n-h-1, n).$$

Turning now to E_u, we note that the simplicial bases of $\mathbb{R}^{h+1}$ which include the specified point q correspond to minimal convex relations on U whose support includes q, hence to relations which, as vertices of the polytope P, do not belong to the face F_q of P. Thus the sequences $B = B_0, B_1, \ldots, B_l = B'$ of the sort involved in the definition of $E_u(h, n)$ correspond to paths on P that miss F_q. Let H be a hyperplane in $\mathbb{R}^{|U|}$ whose intersection with P is F_q, and let P^* be the unbounded polyhedron that is the image of P under a projective transformation that carries H into the hyperplane at infinity. Then there is a sequence $B_0 = B, \ldots, B_l = B'$ for which $l \leqslant \delta(P^*) \leqslant$

$\Delta_u(n-h-1, n)$ and it follows that

$$E_u(h,n) \leqslant \Delta_u(n-h-1,n).$$

To complete the proof by establishing the reverse inequality for E ⟨resp. E_u⟩, set $d = n - h - 1$ and consider an arbitrary simple d-polytope P ⟨unbounded simple d-polyhedron P^*⟩ with n facets. In the latter case, note that the facial structure of P^* is isomorphic to that of $P \sim F$ for a suitable simple d-polytope P with $n+1$ facets and facet F of P. Let $u = n\langle n+1\rangle$. By a well-known result [Da'1], P is affinely equivalent to a d-dimensional section of a $(u-1)$-simplex S, and since P has u facets each facet of S gives rise to a facet of P. It is therefore possible to represent (an affine equivalent of) P in the form described in the first paragraph of this proof, and from the discussion there it follows that $\delta(P) \leqslant E(h,n)$ and $\delta(P^*) \leqslant E_u(h,n)$. That completes the proof of 2.6. ∎

An alternative proof of 2.6 can be based on the machinery of Gale-transforms as developed in Grünbaum's book [Gr'1]. For example, the inequality $E(d-1, 2d) \leqslant \Delta(d, 2d)$ is a consequence of the following observations concerning simplicial bases B and B' of $\mathbb{R}^{d-1}$ whose union is a Haar set of cardinality $2d$:

(i) At least one point of B and at least one point of B' belongs to each open halfspace in $\mathbb{R}^{d-1}$ whose boundary contains the origin. Thus it follows from Theorem 2 (on p. 88 of [Gr'1]) that the set $B \cup B'$ is a Gale-transform of a d-polytope Q with $2d$ vertices.

(ii) Since each of $d-1$ points in $B \cup B'$ is linearly independent, it follows from Theorem 4 that the polytope Q is simplicial.

(iii) For any two simplicial bases X and Y contained in $B \cup B'$, it follows from Theorem 1 that the sets $(B \cup B') \sim X$ and $(B \cup B') \sim Y$ are the vertex-sets of facets of Q; hence if $|X \cap Y| = d-1$, passing from X to Y corresponds to passing from a facet F of Q to another facet that shares a ridge with F.

The following result justifies §1's formulation of the d-step conjecture in terms of matrix pivot operations.

2.7. *For $1 \leqslant m \leqslant n \leqslant 2m$, let $\Pi_u(m,n)$ ⟨resp. $\Pi(m,n)$⟩ denote the smallest integer k that has the following property:*

whenever the real $m \times (n+1)$ matrices $A = (I, B, c)$ and $A' = (B', i, c')$ are row-equivalent, where I is the $m \times m$ identity matrix and the columns c and c' are > 0 ⟨and where, for Π, the polyhedron $\{x \in \mathbb{R}^n_+ : (I,B)x = c\}$ is bounded⟩, it is possible to pass from A to A' by a sequence of $\leqslant k$ feasible pivots followed if necessary by a permutation of rows.

Then $\Pi(m,n) = \Delta(n-m,n)$ and $\Pi_u(m,n) = \Delta_u(n-m,n)$.

Proof. We show first that

$$\Pi(m,n) \leqslant \Delta(n-m,n) \quad \text{and} \quad \Pi_u(m,n) \leqslant \Delta_u(n-m,n).$$

To handle the case in which the polyhedron $\{x \in \mathbb{R}^n_+ : (I,B)x = c\}$ is not simple, the notion of the *expanded graph* $EG(P)$ of a pointed d-polyhedron P is useful. The nodes of $EG(P)$ are the d-sets $\mathscr{V}$ of facets of P such that some vertex v of P is incident to all members of $\mathscr{V}$, whence $\bigcap\mathscr{V} = \{v\}$. Two nodes $\mathscr{V}$ and $\mathscr{W}$ of $EG(P)$ form an edge of $EG(P)$ if and only if $|\mathscr{V} \cap \mathscr{W}| = d-1$. The natural mapping μ of the nodes of $EG(P)$ onto the vertices of P is seen to have the following two properties:

(i) if nodes $\mathscr{V}$ and $\mathscr{W}$ are adjacent in $EG(P)$ then vertices $\mu(\mathscr{V})$ and $\mu(\mathscr{W})$ are the same or adjacent in $G(P)$;

(ii) if vertices v and w are adjacent in $G(P)$ there are adjacent nodes V and W in $EG(P)$ such that $\mu(\mathscr{V}) = v$ and $\mu(\mathscr{W}) = w$.

In particular, P is simple if and only if μ is an isomorphism of $EG(P)$ onto $G(P)$.

Now suppose that P is a pointed polyhedron with interior point z, and $\mathscr{H}$ is a set of closed halfspaces such that $\bigcap\mathscr{H} = P$. Suppose also that $\mathscr{H}$ is minimal with respect to this property, so that the bounding hyperplanes of the members of $\mathscr{H}$ are precisely the affine hulls of the facets of P. By slightly perturbing $\mathscr{H}$, translating each of its members toward z by a small amount, there arises a set $\mathscr{H}'$ of halfspaces such that the intersection $P' = \bigcap H'$ is a simple polyhedron near P and the bounding hyperplanes of the members of $\mathscr{H}'$ are precisely the affine hulls of the members of P'. The natural correspondence between the facets of P' and those of P induces an adjacency-preserving injection of the node-set of $G(P')$ into that of $EG(P)$. (To verify this, note that the perturbation of P's facets is dual, with respect to polarity based on the point z as origin, to the vertex-pulling described in [EGK], and then use the results of [EGK].) Since $\Delta(d, n)$ is the maximum of the diameters of simple d-polytopes with n facets, we conclude that if v and w are vertices of a d-polytope P with n facets, $\mathscr{V}$ ⟨resp. $\mathscr{W}$⟩ is the set of all facets incident to v ⟨w⟩, and $|\mathscr{V}| = |\mathscr{W}| = d$, then $\mathscr{V}$ and $\mathscr{W}$ are joined in $EG(P)$ by a path formed from at most $\Delta(d, n)$ edges of $EG(P)$. When P is merely a polyhedron the same is true with $\Delta(d, n)$ replaced by $\Delta_u(d, n)$.

Now consider two row-equivalent $m \times (n + 1)$ matrices $A = (I, B, c)$ and $A' = (B', I, c')$ as described in the hypotheses of 2.7. Since the m rows of the matrix (I, B) are linearly independent, the flat $S = \{x \in \mathbb{R}^n\colon (I, B)x = c\}$ is of dimension $d = n - m$. Let v ⟨resp. v'⟩ denote the point of $\mathbb{R}^n_+$ whose first ⟨resp. last⟩ m coordinates are those of c ⟨resp. c'⟩ and remaining $n - m$ coordinates are 0. Then v and v' are vertices of the polyhedron $P = S \cap \mathbb{R}^n_+$. Since $n \leqslant 2m$, the point $\frac{1}{2}(v + v')$ of S is interior to $\mathbb{R}^n_+$ and consequently P is of dimension d. The facets of P are precisely those sets $F_K = \{x = (x_1, \ldots, x_n) \in P\colon x_k = 0\}$ that are $(d - 1)$-dimensional, and hence P has $\bar{n}$ facets for some $\bar{n} \leqslant n$. Note that $\Delta(d, \bar{n}) \leqslant \Delta(d, n)$ and $\Delta(d, \bar{n}) \leqslant \Delta_u(d, n)$ by 2.2.

Let V ⟨resp. V'⟩ denote the set of all $k \in N = \{1, \ldots, n\}$ such that F_k is a facet of P incident to v ⟨v'⟩. Then $V = \{m + 1, \ldots, n\}$ and $V' = \{1, \ldots, d\}$ because $c > 0$ and $c' > 0$. From the third paragraph of this proof it follows that for some $l \leqslant \Delta_u(d, n)$ ⟨$l \leqslant \Delta(d, n)$ when P is bounded⟩ there is a sequence $V = V_0, V_1, \ldots, V_{l-1}, V_l = V'$ of d-sets in N and there is a sequence $v = v_0, v_1, \ldots, v_l = v'$ of vertices of P such that for $0 \leqslant i \leqslant l$, F_k is incident to v_i for all $k \in V_i$, and for $1 \leqslant i \leqslant l$, $|V_{i-1} \cap V_i| = d - 1$.

Let c_i denote the column vector whose m coordinates are, in order, the coordinates of v_i in positions belonging to $N \sim V_i$. We claim that for $0 \leqslant i \leqslant l$ there is a unique $m \times (n + 1)$ matrix A_i such that the last column of A_i is c_i, the columns of A_i in positions belonging to $N \sim V_i$ form the $m \times m$ identity matrix, and $P = \{x \in \mathbb{R}^n_+\colon A_i^* x = c_i\}$ where A_i^* is formed from A_i by omitting the last column c_i. To prove this, let L_i denote the vector space of all affine functionals on the flat S that vanish at the point $v_i \in S$. Since S is d-dimensional, so is L_i. From the definition of the F_k's and from their relationship to the d-set V_i it follows that if ξ_j is the restriction to S of the coordinate functional x_j, then the ξ_t's for $t \in V_i$ form a linear basis for L_i.

Now consider an arbitrary $j \in N \sim V_i$ and suppose j is the hth member of $N \sim V_i$. Since the functional $\xi_j - \xi_j(v_j)$ belongs to V_i there are unique numbers α_t such that $\xi_j - \xi_j(v_i) = \sum_{t \in V_i} \alpha_t \xi_t$, and these numbers α_t form the hth row of A_i in positions corresponding to V_i. The remaining entries of the hth row are c_j in the last column, 1 in the jth column, and 0 in columns of index neither equal to j nor belonging to V_i.

For each i, the flat $S_i = \{x \in \mathbb{R}^n\colon A_i^* x = c_i\}$ and the polyhedron $S_i \cap \mathbb{R}^n_+$ are both d-dimensional, so the former is the affine hull of the latter. But the intersections $S_i \cap \mathbb{R}_n^+$ are all equal to P, so the flats S_i are all equal and the matrices A_i all have the same row space. With $|V_{i-1} \cap V_i| = d - 1$, it then follows readily that A_i can be obtained from A_{i-1} by a feasible pivot followed if necessary by a permutation of rows. This completes the proof that $\Pi(m, n) \leqslant \Delta(n - m, n)$ and $\Pi_u(m, n) \leqslant \Delta_u(n - m, n)$.

To show $\Delta(n-m, n) \leqslant \Pi(m, n)$ and $\Delta_u(n-m, n) \leqslant \Pi_u(m, n)$ let $d = n - m$, whence $n \geqslant 2d$. By 2.1 there exists a simple d-polytope ⟨resp. polyhedron⟩ Q with n facets and two vertices y and z of Q such that $\delta_Q(y, z) = \Delta(d, n)$ $\langle = \Delta_u(d, n)\rangle$ and no facet of Q is incident to both y and z. As noted in [Da'1], [Kl2], Q is affinely equivalent to an intersection $P = S \cap \mathbb{R}^n_+$ for some d-flat S in $\mathbb{R}^n_+$. The facets of P are the sets $F_k = \{x \in P\colon x_k = 0\}$ and each vertex of P belongs to precisely d facets. Let v and w be the vertices of P corresponding to y and z respectively. Since no facet of P is incident to both v and w, the coordinates in R^n can be numbered so that

$$\{j\colon v_j = 0\} = \{m+1, \ldots, n\} \quad \text{and} \quad \{j\colon w_j = 0\} = \{1, \ldots, d\}.$$

Let $A_0 = (H, c_0)$ be an arbitrary $m \times (n+1)$ matrix such that $S = \{x \in \mathbb{R}^n\colon Hx = c_0\}$. Let $A_0 = (H, c_0)$ be an arbitrary $m \times (n+1)$ matrix such that $S = \{x \in \mathbb{R}^n\colon Hx = c_0\}$. By a fundamental result relating extreme points to basic feasible solutions (p. 98 of [Mu1] has a clear statement; see also [PS]), the square matrix formed by the first ⟨resp. last⟩ m columns of H is nonsingular and hence the row space of A_0 includes matrices $A = (I, B, c)$ and $A' = (B', I, c')$ of the sort involved in the definition of $\Pi(m, n)$. The basic feasible solutions corresponding to A and A' are the vertices v and w respectively. Thus A' can be obtained from A by a sequence of $k \leqslant \Pi_u(m, n)$ feasible pivots $\langle \leqslant \Pi(m, n)$ when Q is bounded⟩, followed if necessary by a permutation of rows. This sequence of pivots generates a path of length $\leqslant k$ from v to w in $G(Q)$, thus completing the proof of 2.7. ∎

3. Relations to linear programming. If the nonempty feasible region P of an LP problem is defined by $n - d$ irredundant linear equalities in n nonnegative variables, or by $n - d$ linear inequalities in d nonnegative variables, or by n linear inequalities in d real variables, then P is a c-polyhedron with m facets for some $c \leqslant d$ and $m \leqslant n$. When $(c, m) = (d, n)$ there are arbitrarily slight perturbations of the defining equalities or inequalities for which P is a pointed simple d-polyhedron with n facets. Thus it is appropriate to focus on such polyhedra in studying the computational complexity of LP; and of course the bounded case is of special interest.

In seeking to maximize a linear function φ on a simple polyhedron P, an edge-following LP algorithm starts with a vertex u and then constructs, edge by edge, a path that leads from u to a vertex v such that $\varphi(v) = \max \varphi P$ or v is the end of an unbounded edge E for which $\sup \varphi E = \infty$. (We are concerned here only with "primal" algorithms.) These properties of v are easily recognized computationally, and the various edge-following algorithms differ principally in the pivot rule by which the sequence of edges is chosen. Each pivot decision amounts (in the nondegenerate case that we are considering) to deciding which edge of P to follow next in seeking a target vertex v.

While the computational effort per pivot varies from one rule to another, the distance $\delta_P(u, v)$ is an obvious lower bound for the total effort of applying any edge-following algorithm to P with initial vertex u and target vertex v. It thus seems reasonable, since there is generally little control over the choice of the initial vertex u and since any vertex v may be the unique target vertex, to regard $\delta(P)$ as a lower bound for the worst-case behavior of edge-following algorithms applied to P as feasible region. (One might expect $\delta(P)$ to be a very weak bound when P is highly "degenerate" (i.e., far from being simple), but perhaps a fairly sharp bound when P is simple.)

(We note in passing that for each pair of distinct vertices u, v of a polyhedron $P \subset \mathbb{R}^d$, there is a projective transformation π of $\mathbb{R}^d$ and there is a linear function ψ on $\mathbb{R}^d$ such that (P, u, v) is carried by π onto a triple (P', u', v') for which the minimum ⟨resp. maximum⟩ of ψ on P' is attained uniquely at u' ⟨resp. v'⟩. Indeed, let H_u and H_v be hyperplanes that intersect P only at u and v respectively, let H be a hyperplane that contains $H_u \cap H_v$ but misses P, and let π carry H onto the hyperplane at infinity. In $\mathbb{R}^d$ there are parallel hyperplanes $H_{u'}$ and $H_{v'}$ that contain the π-images of $H_u \sim H_v$ and $H_v \sim H_u$ respectively; $H_{u'}$ and $H_{v'}$ are level sets of a linear function ψ that has the stated property.)

It is clear from the above discussion that $\Delta(d, n)$ ⟨resp. $\Delta_u(d, n)$⟩ is a lower bound for the worst-case behavior of edge-following LP algorithms over all simple d-polytopes ⟨resp. d-polyhedra⟩ with n facets. Since this applies to *all* edge-following algorithms, we may say that Δ and Δ_u estimate the worst possible behavior of the best possible edge-following algorithm.

Most of the edge-following LP algorithms used in practice are monotone in the sense that, in leaving a vertex v_i to continue the path toward the target vertex v, the next vertex v_{i+1} is (in the "nondegenerate" φ-maximizing case being considered) always such that $\varphi(v_{i+1}) > \varphi(v_i)$. Since the "monotone Hirsch conjecture" fails in the (d, n) case for each $d \geqslant 4$ and $n \geqslant d + 5$ (see 4.2 (vii) and [To]), no monotone LP algorithm can provide a proof of the Hirsch conjecture. In fact, all of the edge-following algorithms that have been successfully analyzed have been shown to produce exponentially long paths in their worst-case behavior and hence cannot even be used to establish polynomial bounds on Δ. However, if Δ is indeed polynomially bounded the best way of proving this may be by means of some not-yet-analyzed pivot rule such as the one of Zadeh [Za3] described below. A purely combinatorial proof of a polynomial upper bound for Δ might or might not lead to a practical pivot rule that turns the simplex method into an algorithm whose worst-case behavior is polynomially bounded. However, if Δ could be shown to grow exponentially this would of course imply that all edge-following LP algorithms are exponentially bad in their worst-case behavior. For large values of d and n, the best lower and upper bounds on Δ and Δ_u are given in §7 below. They imply that for each *fixed* d, $\Delta(d, n)$ and $\Delta_u(d, n)$ both grow linearly in n, but they do not tell how $\Delta(d, 2d)$ and $\Delta_u(d, 2d)$ increase as $d \to \infty$. Linearly? Quadratically? Exponentially?

For several LP pivot rules it has been shown that, when the rule is applied to maximize a linear function over a simple d-polytope with n facets, and when a bad choice of polytope, function, and initial vertex is made, exponentially many pivots may be required to reach a target vertex and thus the number of edges in the resulting path is not bounded by any polynomial in d and n. These examples do not require "exotic" polytopes—for example, those in the simplest construction of Klee and Minty [KM] are projectively equivalent to cubes and can even be made metrically arbitrarily close to cubes. The pivot rule studied by [KM] is the original rule of Dantzig [Da1], which maximizes the gradient in the space of nonbasic variables. (See [AS] for a more abstract formulation of the [KM] construction.) Similar results were established by Jeroslow [Je] for the greatest increment rule, by Avis and Chvatál [AC] for the pivot rule of Bland [Bl], by Goldfarb and Sit [GS′] for the all-variable gradient rule, and by Murty [Mu2] for the parametric programming rule of Gass and Saaty [Ga′], [GS]. (The [GS] rule is also discussed by [Go], who attributes it to Borgwardt.) The various examples have been constructed so as to behave badly for specific rules, but some unification of the constructions and some especially simple numerical examples have been produced by Zadeh [Za3], Clausen [Cl] and Blair [Bl].

Interesting numerical experience with various pivot rules has been reported by Gotterbarm [Go′] and Clausen [Cl]; also by Lindberg and Olafsson [LO], [Ol1, 2, 3], [OL], who are concerned primarily with assignment and transportation problems.

Some general LP pivot rules whose worst-case behavior has not yet been analyzed are those of Zadeh [Za3] and Cirinà [Ci1], [Ci2]. Cirinà reports good computational experience and shows that his rule, like that of Bland, is noncycling. Since Cirinà's rule takes the increment of the objective function into account, it is less "combinatorial" than Bland's.

Zadeh [Za1, 2] was one of the first to construct LP problems requiring exponentially many pivots for certain pivot rules (his are network problems), and his pivot rule in [Za3] is designed explicitly to defeat various constructions. For each edge $[x, y]$ of a simple polytope P, there is a unique facet F of P that is incident to x but not y. When P is the feasible region of a nondegenerate LP, the move from x to y is accomplished by bringing into the basis the variable that is associated with F. (See [Da2], [KM] for the relationship between geometry and linear algebra that is involved here.) Thus Zadeh's rule, "Enter the improving variable which has been entered least often", produces a monotone path which at each step keeps the maximum of the "leaving indices" as small as possible where the *leaving index* of a facet F, with respect to a path $x_0, x_1, \ldots, x_k$, is the number of indices i such that $x_i \in F$ but $x_{i+1} \notin F$. Zadeh [Za3] conjectures that the lengths of the paths produced by his rule are bounded by a polynomial in d and n (this would of course establish a polynomial upper bound for $\Delta(d, n)$), and in a letter he offers $1000 for a proof or disproof of his conjecture. Note the close relationship between Zadeh's rule and the revisit function ρ_p of 2.3.

In addition to general LP algorithms, those for important problems with special structure are also of great interest. In 1972 Edmonds and Karp [EK] asked whether minimum cost network flow problems admit a simplex algorithm that is *genuinely polynomial*, meaning that the total number of arithmetic steps is bounded by a polynomial in the number n of nodes of the network, independent of both costs and capacities. Some progress was made by Ikura and Nemhauser [IN], and in 1984 Orlin [Or] applied the [EK] scaling technique to produce a dual simplex algorithm that requires $O(n^3 \log n)$ pivots (thus providing a bound on the diameters of the feasible regions of the dual problems) and whose total number of arithmetic steps is also polynomially bounded. It would be of interest to find simpler pivot rules that have these same desirable properties for minimum cost flow problems.

For assignment problems of size n, the primal simplex method of Roohy-Laleh [Ro] requires $O(n^3)$ pivots, independent of cost, while Hung's estimate [Hu] of the number of pivots in his method is $O(n^3)$ times the logarithm of a constant that depends on costs. The dual simplex method of Balinski [Ba5, 6] has a strikingly simple pivot rule that requires at most $(n-1)(n-2)/2$ pivots and $O(n^3)$ arithmetic operations to solve the problem. Balinski's method is based on his proof of the Hirsch conjecture for dual transportation polyhedra [Ba3, 4], discussed in §8 below. His algorithm is extended by Kleinschmidt and Lee [KL] to some classes of transportation problems with constant demands; in particular, a transportation problem with m sources and n destinations having unit demands can be solved with $O(mn^2)$ arithmetic operations.

As far as worst-case behavior is concerned, it seems that dual pivoting methods are better suited to flow problems than are primal methods. This may be because, as noted by [BR3], the number of vertices is considerably smaller for the dual polyhedron than for the primal polyhedron. However, Cunningham [Cu] does have a primal pivot rule for network problems for which the number of consecutive degenerate pivots is polynomially bounded.

* * * * * * *

Until now, all the material of this section has been related to diameters of polytopes and to the worst-case complexity of edge-following algorithms for linear programming, measured with respect to infinite-precision real arithmetic. Some other important aspects of LP complexity will now be mentioned in order to round out the picture, but will be touched upon only briefly because they do not appear to be closely related to the d-step conjecture or its relatives.

The work of Borgwardt [Bo1, 2, 3], Smale [Sm] and others shows that, with respect to various pivot rules and various averaging procedures, the average-case complexity of the simplex algorithm is polynomially bounded (for d-dimensional feasible regions with n facets, the average number of computational steps is bounded by a polynomial in d and n) or close to that (see, especially Adler and Megiddo [AM]). These results do of course say something about diameters of polytopes. The result of Haimovich [Ha], in particular, can be interpreted as saying that, in a sense, the expected diameter of a random d-polytope with n facets does not exceed $n - d + 1$. However, the averaging process is such that various combinatorial types of such polytopes appear with varying frequencies, and it is unclear what (if anything) his or other results on average-case complexity say about the distribution of diameters over the finite probability space in which each of the k combinatorial types has weight $1/k$. Nevertheless, it would seem to be worthwhile to study the average-case results with the functions Δ and Δ_u in mind. For a good overview of results on average-case complexity of the simplex method, see Adler, Megiddo and Todd [AMT] and Shamir [Sh].

The papers of Megiddo [Me2, 3] and Dyer [Dy] contain LP algorithms that do not follow edges of the feasible region. For each fixed dimension d of the feasible region, the algorithm of [Me3] is linear in the number n of facets. While this has an intriguing resemblance to the inequality, $\Delta(d, n) \leqslant 2^{d-1}n$ (see [La] and 7.3), we are not aware of any direct relationship between [Me3] or [Dy] and diameters of polytopes.

The LP algorithms of Khachian [Kh] and Karmarkar [Ka] are polynomially bounded with respect to a different model of computation (the "Turing machine" model), in which the measurement of input size takes account not only of the number of variables and number of constraints (or, in the irredundant case, the dimension and number of facets of the feasible region), but also of the lengths of the binary encoding of the rational coefficients that form part of the input data. Although this model is much favored by workers in the theory of computation, its appropriateness for linear programming has been questioned by Megiddo [Me1, 2] and by Traub and Wozniakowski [TW]. As far as we are aware, computational experience with the ellipsoid method of [Kh] has been dismal, but it has had a powerful influence on the theory of combinatorial optimization (see Grötschel, Lovász and Schrijver [GLS], [GLS′]). An excellent general survey of the ellipsoid method is that of Bland, Goldfarb and Todd [BGT]. Reports of the practical efficiency of Karmarkar's algorithm [Ka] are very favorable. In any case, these methods, while generally related to the geometry of polytopes, seem to have no relationship to the diameters of polytopes. The same may be said of the recent work of Tardos [Ta1, 2] and Frank and Tardos [FT], which shows how to modify various algorithms to strengthen the sense in which they are "polynomial."

In ending this section, we mention the papers of Dobkin, Lipton and Reiss [DLR], Dobkin and Reiss [DR], Chandrasekaran, Kabadi and Murty [CKM], and Karp and Papadimitriou [KP], as being of special interest because of the way in which they relate the computational complexity of linear programming or closely related problems to that of other computational problems. Telgen [Te2] has references to other results of this sort.

4. Low-dimensional results. The following statement covers all pairs (d, n) for which it is known that $\Delta(d, n) \leqslant n - d$ or the precise value of $\Delta(d, n)$ or $\Delta_u(d, n)$ is known to the authors.

4.1. $\Delta(d, n) = \lfloor((d-1)/d)n\rfloor - d + 2$ *if* $d \leqslant 3$ *or* $n \leqslant d+4$;
$\Delta(d, d+k) = k$ *if* $k \leqslant d \leqslant 5$;
$\Delta(4,10) = 5$; $\Delta(4,11) = 6$ *or* 7; $\Delta(5,11) = 6$;
$\Delta_u(d, d+k) = k$ *if* $d \leqslant 3$ *or* $k \leqslant 3$;
$\Delta_u(d, d+4) = 5$ *if* $d \geqslant 4$.

PROOFS. The assertions are obvious for $d = 2$. To see that $\Delta(3, n) \leqslant \lfloor 2n/3\rfloor - 1$, note that the maximum is attained by a simple 3-polytope with n facets and each such polytope has precisely $2n - 4$ vertices. Since its graph is 3-connected, any two vertices u and v can be joined by 3 independent paths. If l is the length of the shortest of 3 independent paths joining u and v, then the paths use at least $3(l-1)+2$ vertices in all, whence $3l \leqslant 2n - 3$. This argument appears in [GM] and [Kl1], and the maximizing 3-polytopes are constructed in the latter paper.

Easily constructed polyhedra [Kl1, 4] show that $\Delta(d, d+k) \geqslant k$ when $k \leqslant d$ and $\Delta_u(d, d+k) \geqslant k$ for all d and k. That $\Delta_u(3, 3+k) \leqslant k$ is proved in [Kl3, 4] by constructing nonrevisiting paths. Consider, for example, an unbounded pointed polyhedron P in 3-space such that the interior of P contains the positive z-axis A and P is supported by the xy-plane H. Since P has only finitely many vertices, there is a point a of A that is above (has greater z-coordinate than) all vertices of P. Let $\mathscr{K}$ denote the complex that is formed by the intersections of P's proper faces with the half-open strip $S = H + [0, a[$ that is bounded below by $H \subset S$ and above by $H + a \not\subset S$. By projection π along the rays that issue from the point a and intersect the plane H, the complex $\mathscr{K}$ is carried onto a polyhedral subdivision $\pi\mathscr{K}$ of H. For any two vertices u and v of $\mathscr{K}$, the vertices πu and πv of $\pi\mathscr{K}$ are joined by a path formed from edges of $\pi\mathscr{K}$, and since the cells of $\pi\mathscr{K}$ are convex it is easily seen that when such a path is of minimum Euclidean length it does not revisit any cell of $\pi\mathscr{K}$. The corresponding path in $\mathscr{K}$ is a nonrevisiting path from u to v on P.

Now suppose that u and v are vertices of a simple d-polyhedron P and no facet is incident to both u and v; then there are d facets incident to u and d other facets incident to v. If P has $2d$ facets in all then each edge that issues from u terminates on a facet incident to v, and this facet is a simple $(d-1)$-polyhedron with at most $2d-1$ facets of its own. When $d \in \{3,4\}$ it is true that $\Delta_u(d-1, 2d-1) = d$ and $\Delta(d-1, 2d-1) = d-1$; thus each edge from u is the start of a path of length $\leqslant d+1$ (of length d when P is bounded) from u to v, whence $\Delta(4,8) = 4$ and $\Delta_u(4,8) \leqslant 5$. If P has $2d+1$ facets in all then at least one edge issues from u and terminates on a facet incident to v; since $\Delta(3,8) = 4$, it follows that $\Delta(4,9) \leqslant 5$.

In [KW] there is constructed an unbounded simple 4-polyhedron with 8 facets and diameter 5; a simple 4-polytope with 9 facets and diameter 5 is obtained by intersecting this with a suitable halfspace. Hence $\Delta_u(4,8) = 5 = \Delta(4,9)$.

For each d-polytope Q with at most $2d+2$ facets, let $\alpha(Q)$ denote the smallest integer k that has the following property:

for each partition of Q's facets into two classes $\mathscr{X}$ and $\mathscr{Y}$ of at most $d+1$ facets each, if X $\langle$resp. $Y\rangle$ denotes the set of all vertices of Q that are incident to no member of $\mathscr{Y}$ $\langle$resp. $\mathscr{X}\rangle$, then X or Y is empty or there is a path of length $\leqslant k$ joining a member of X and a member of Y.

For a simple d-polytope P with $2d$ facets, easy arguments of [KW] show that if vertices u and v are not on the same facet then there are edges from u and v that terminate on the same ridge R of P; and $\delta_P(u, v) \leqslant \alpha(R) + 2$ for each such R. Hence

$$\delta(P) \leqslant \max\{\max\{\delta(F)\colon F \text{ facet of } P\}, 2 + \max\{\alpha(R)\colon R \text{ ridge of } P\}\}.$$

Now let $A(d)$ denote the maximum of $\alpha(Q)$ over all simple d-polytopes Q with $2d$,

$2d + 1$ or $2d + 2$ facets. Then it follows with the aid of 3.1 that $\Delta(d, 2d) \leqslant A(d-2) + 2$. This implies $\Delta(5, 10) = 5$, for it is shown in [KW] that $A(3) = 3$.

Except for those on its third line, the remaining assertions of 4.1 follow from those already established, in conjunction with 3.2.

It is proved in [KW] that $\Delta(4, 11) \in \{6, 7\}$. The method is extended by Larman [La] and then by Goodey [Go1] to show $\Delta(4, 10) = 5$ and $\Delta(5, 11) = 6$. ∎

In dual form, the equality $\Delta(5, 10) = 5$ is extended by Adler and Dantzig [AD] to the ridge-diameters of certain simplicial complexes that they call "abstract polytopes". See §5 for relevant definitions.

Surprisingly little beyond 4.1 is known about the functions Δ and Δ_u. For example, while the Hirsch conjecture asserts only $\Delta(d, n) \leqslant n - d$, it may be that for each fixed d, $n - \Delta(d, n) \to \infty$ as $n \to \infty$. This does occur when $d \in \{2, 3\}$, and $(4, 9)$ and $(5, 11)$ are the only pairs (d, n) for which it is known that $n > 2d$ and $\Delta(d, n) \geqslant n - d$. Among the 1142 different combinatorial types of simple 4-polytopes with 9 facets, the one constructed in [KW] is the only one that has diameter 5; all the others are of diameter 3 or 4. (See §5 for more information on this.)

As noted in [KW], from $\Delta_u(4, 8) = 5$ it follows that for each d and n,

$$\Delta_u(d, n) \geqslant n - d + \min\{\lfloor d/4 \rfloor, \lfloor (n-d)/4 \rfloor\}.$$

It is nevertheless conceivable that for each fixed d and fixed $\mu > 1$, $\mu n - \Delta_u(d, n) \to \infty$ as $n \to \infty$.

The smallest pairs (d, n) for which the Hirsch conjecture may conceivably fail are $(4, 12)$, $(5, 12)$ and $(6, 12)$, and failure in at least one of these cases would not surprise us. In particular, we strongly suspect the d-step conjecture fails when the dimension is as large as 12. Here are some plausible strengthenings of the conjecture that hold only for small d. (The last three conditions are discussed in §6, after the relevant definitions are provided in §5.)

4.2. *Each of the following statements implies $\Delta(d, 2d) \leqslant d$. Each holds for $d \leqslant 3$ but fails for all sufficiently large d.*

(i) $\Delta_u(d, 2d) \leqslant d$;

(ii) $\Delta(d, 2d + 1) \leqslant d$;

(iii) $A(d-2) \leqslant d - 2$;

(iv) *if two vertices of a simple d-polytope lie in an open halfspace they are joined by a nonrevisiting path that lies entirely in the halfspace*;

(v) *if two vertices of a simple d-polytope do not lie on the same facet they are joined by d independent nonrevisiting paths*;

(vi) *if two vertices u and v of a simple d-polytope do not lie on the same facet then each edge from u starts a nonrevisiting path to v*;

(vii) *if a linear function φ on a simple d-polytope attains its maximum uniquely at a vertex v then from each vertex u there is a nonrevisiting path to v along which φ is increasing*;

(viii) *the boundary complex of each simplicial d-polytope is vertex decomposable*;

(ix) *in each triangulated $(d-1)$-sphere, each pair of facets is joined by a ridge-path that does not revisit any vertex*;

(x) *in each triangulated $(d-1)$-sphere with $2d$ vertices, each pair of facets is joined by a ridge-path of length $\leqslant d$.*

Proof. The first two conditions have already been discussed; each holds for $d \leqslant 3$ and fails for $d = 4$. Condition (iii) is obvious when $d \leqslant 4$; that is, $A(d) = d$ for $d \leqslant 2$. [KW] shows $A(3) = 3$, uses this to prove $\Delta(5, 10) = 5$, and shows $A(d) > d$ for $d \in \{4, 5\}$ by constructing a simple d-polytope Q_d with $d + 6$ facets such that

$\alpha(Q_d) = d + 1$. This suggests there may exist a simple 6-polytope Q_6 with 12 facets and $\alpha(Q_6) = 7$, and that would show $\Delta(6, 12) \geqslant 7$.

Note that (vi) is implied by (iv) and also by (v). Condition (v) is obvious for $d = 2$, and for $d = 3$ follows from a theorem of Barnette [Ba'1] stating that if two vertices of a 3-polytope do not lie on the same facet $\langle$resp. edge$\rangle$ they are joined by three $\langle$two$\rangle$ independent nonrevisiting paths.

To sharpen (iv) when $d = 3$, suppose that P is a 3-polytope in $\mathbb{R}^3$ and J is a plane bounding two open halfspaces J^+ and J^- each of which intersects P. We show that if two vertices u and v of P lie in the closed halfspace $J \cup J^+$ then they are joined by a nonrevisiting path which, with the possible exception of the points u and v, lies entirely in J^+. Let P' be the unbounded polyhedron obtained from $P \cap J^+$ by a projective transformation ξ that sends J to the plane at infinity, and let π be a projection of P' as described in the proof of 4.1, so that $\pi P'$ is a complex subdividing an appropriate plane. If $u \in J^+$ let $u' = \pi\xi(u)$, and if $u \in J$ let u' be a point in the relative interior of an unbounded edge of $\pi P'$ incident to u. Define v' similarly. Then u' and v' are joined by a path of minimum Euclidean length in the edge-graph of $\pi P'$, and since the cells of $\pi P'$ are convex this path does not revisit any facet of $\pi P'$. It is easily seen that this path induces the desired path on P.

As was shown in the discussion of 4.1, condition (vi) holds for all simple 4-polytopes with 8 facets. To show that (iv), (v) and (vi) do not apply to all simple 5-polytopes with 10 facets, we construct such a polytope P with two vertices u and v that do not share a facet and are such that a certain edge from u does not start a nonrevisiting path to v. Let Q be a simple 4-polytope with 9 facets having two vertices s and t such that $\delta_Q(s, t) = 5$, and let F be a facet not incident to s or t. Let P be the wedge over Q with foot F, s' and t' the new vertices of P that are over s and t respectively, and let $u = s$, $v = t'$. If a path from u to v is nonrevisiting in P its length is 5, and if it also starts along the edge $[u, s']$ its projection on Q is a path of length $\leqslant 4$ from s to t; but there is no such path.

For $d = 3$, condition (vii) is established by [Kl3] without assuming simplicity. Todd [To] shows that (vii) fails for a certain simple 4-polytope with 9 facets. ∎

We have seen that in a simple d-polytope with n facets, each nonrevisiting path is of length $\leqslant n - d$. However, even when $d = 3$, a shortest path joining two vertices of a simple d-polytope may revisit some facets. Minimum examples of this and related phenomena are described by [Ba'1] and [Go2].

5. Relatives in more general complexes. As the term is used here, a *complex* is a finite collection $\mathscr{C}$ of pointed polyhedra in a real vector space such that each face of a member of $\mathscr{C}$ also belongs to $\mathscr{C}$ and the intersection of any two members of $\mathscr{C}$ is a face of each. The members of $\mathscr{C}$ are its *faces*, the maximum of their dimensions is the *dimension* of $\mathscr{C}$, and $\mathscr{C}$ is *pure* if its maximal faces are all of the same dimension. The 0-faces and 1-faces of a complex are its *vertices* and *edges*, and the s-faces and $(s - 1)$-faces of a pure s-complex are its *facets* and *ridges* respectively. When P is a pointed d-polyhedron, the faces of P other than P itself form a pure $(d - 1)$-complex $\mathscr{B}(P)$, the *boundary complex* of P, and the vertices, edges, facets and ridges of $\mathscr{B}(P)$ are those of P as defined earlier.

At least when its members are bounded, a complex in our sense is sometimes called a "cell-complex" and its members "cells". However, we prefer the term "faces" to emphasize that our primary interest is in the boundary complexes of polyhedra.

The *graph* of a complex $\mathscr{C}$ is the combinatorial structure formed by its vertices and bounded edges, and when $\mathscr{C}$ is pure the *dual graph* is formed by its facets and ridges; that is, the nodes of the dual graph are $\mathscr{C}$'s facets and two nodes are *adjacent* (joined by an edge of the dual graph) if and only if their intersection is a ridge. Paths in $\mathscr{C}$'s graph are sometimes called *edge-paths*, and we refer to paths in the dual graph as *ridge-paths*. Just as an edge-path of length l is a sequence $v_0, \ldots, v_l$ of $l+1$ distinct vertices such that each $[v_{i-1}, v_i]$ is an edge, a ridge-path of length l is a sequence $F_0, \ldots, F_l$ of $l+1$ distinct facets such that each $F_{i-1} \cap F_i$ is a ridge. The *ridge-distance* $\delta^*_{\mathscr{C}}(F, G)$ between two facets F and G is defined in the natural way, as is the *ridge-diameter* $\delta^*(\mathscr{C})$, and when P is a pointed polyhedron we write $\delta^*(P)$ or $\delta^*(\mathscr{B}(P))$. If P is a polytope and Q its polar then $\delta^*(P) = \delta(Q)$; hence $\Delta(d, n)$ is the maximum ridge-diameter of d-polytopes with n vertices, and the maximum is attained by simplicial polytopes.

The revisiting notions of §3 can also be dualized so as to apply to ridge-paths $F_0, \ldots, F_l$ in a pure complex $\mathscr{C}$. A *revisit* is a triple (v, i, k) such that v is a vertex of $\mathscr{C}$, $0 \leqslant i < i+1 < k \leqslant l$, F_i and F_k are incident to v, and for $i < j < k$ the facet F_j is not incident to v; $\rho^*_{\mathscr{C}}(F, G)$ is the minimum number of revisits among the ridge-paths from F to G in $\mathscr{C}$, and $\rho^*(\mathscr{C})$ is the maximum of this over all pairs (F, G) of facets of $\mathscr{C}$. By polarity, $R(d, n)$ is the maximum of $\rho^*(P)$ $(= \rho^*(\mathscr{B}(P)))$ over all simplicial d-polytopes P with n vertices.

The functions Δ_u and R_u can also be described in a dual manner. For each unbounded simple d-polyhedron Q with n facets there is a simple d-polytope Q' with $n+1$ facets such that n of these lie in the respective facets of Q and the remaining facet Y of Q' is such that the incidence-structure of Q is obtained from that of Q' by deleting Y and all its proper faces. It follows by polarity that $\Delta_u(d, n)$ ⟨resp. $R_u(d, n)$⟩ is equal to the maximum of $\delta^*(\mathscr{B}(P) \setminus v)$ ⟨$\rho^*(\mathscr{B}(P) \setminus v)$⟩ over all pairs (P, v) consisting of a simplicial d-polytope P with $n+1$ vertices, and a particular vertex v of P. (For a face F of a complex $\mathscr{C}$, $\mathscr{C} \setminus F := \{C \in \mathscr{C}: F \text{ is not a face of } C\}$.)

A complex $\mathscr{C}$ is *polytopal* ⟨resp. *polyhedral*⟩ if it is isomorphic to the boundary complex of a polytope ⟨pointed polyhedron⟩. Here the isomorphism is a one-to-one correspondence that preserves incidence and dimension; equivalently, it is an isomorphism betwen the two posets of faces, with set-inclusion as the ordering. Plainly each polytopal complex $\mathscr{C}$ is *spherical*, meaning that the union $\bigcup \mathscr{C}$ of its faces is homeomorphic for some s with the s-sphere $\{x \in \mathbb{R}^{s+1}: \|x\| = 1\}$. A complex is *simplicial* if each of its facets (and hence each face) is a simplex. *Triangulated s-sphere* is another name for a spherical simplicial s-complex, and a *triangulated s-ball* is defined similarly. Whenever $3 \leqslant s \leqslant n-5$ there exists a triangulated s-sphere with n vertices that is not polytopal [GS], [Ma], but when $s \leqslant 2$ or $n \leqslant s+4$ each spherical s-complex with n vertices is polytopal [Ma], [Kl'1].

Associated with a face A of a complex C are the *star*, the *antistar*, and the *link* of A in $\mathscr{C}$, where these subcomplexes are defined as follows:

$$\mathrm{st}(A, \mathscr{C}) := \{B \in \mathscr{C}: \textit{there exists } C \in \mathscr{C} \textit{ such that } A \textit{ and } B \textit{ are faces of } C\};$$

$$\mathrm{ast}(A, \mathscr{C}) := \{B \in \mathscr{C}: A \cap B = \varnothing\};$$

$$\mathrm{link}(A, \mathscr{C}) := \mathrm{st}(A, \mathscr{C}) \cap \mathrm{ast}(A, \mathscr{C}).$$

As the terms are used here, a pure simplicial s-complex $\mathscr{C}$ is:

a *pseudomanifold* ⟨resp. *closed pseudomanifold*⟩ if it is ridge-path-connected and each ridge lies in at most ⟨exactly⟩ two facets;

an *abstract* $(s+1)$*-polytope* (term due to [AD]) if it is a closed pseudomanifold in which the link of each face is ridge-path-connected;

a *triangulated manifold* if $\bigcup \mathscr{C}$ is locally homeomorphic to $\mathbb{R}^s$.

Each of the last three conditions implies its predecessor.

In the next section and the rest of this one, attention is focused on the ridge-diameters of some important classes of pure simplicial complexes. For each such class $\mathfrak{C}$, we denote by $\Delta^*_{\mathfrak{C}}(d,n)$ $\langle$resp. $R^*_{\mathfrak{C}}(d,n)\rangle$ the maximum of $\delta^*(\mathscr{C})$ $\langle\rho^*(\mathscr{C})\rangle$ over all members $\mathscr{C}$ of $\mathfrak{C}$ that are $(d-1)$-dimensional and have n vertices. It is natural to focus on "simplicial" rather than "simple" generalizations of the Hirsch conjecture and related questions, because there is already an extensive language and literature for dealing with simplicial complexes. As is exemplified by 5.1 below, relatives of 2.1 and 2.2 can be proved for any sufficiently rich class of simplicial complexes. This involves forming new simplicial complexes from old ones in ways that are now described.

Whenever simplicial complexes $\mathscr{C}$ and $\mathscr{D}$ are such that $\bigcup\mathscr{C}$ and $\bigcup\mathscr{D}$ are disjoint and the collection

$$\mathscr{C}\cdot\mathscr{D}=\{\operatorname{con}(C\cup D): C\in\mathscr{C}, D\in\mathscr{D}\}$$

is also a simplicial complex, then $\mathscr{C}\cdot\mathscr{D}$ is called the *join* of $\mathscr{C}$ and $\mathscr{D}$. (Here con denotes the convex hull. Note that if $\mathscr{C}$ and $\mathscr{D}$ are pure complexes, then $\mathscr{C}\cdot\mathscr{D}$ is pure of dimension $\dim\mathscr{C}+\dim\mathscr{D}$.) The condition on sets of the form $\operatorname{con}(C\cup D)$ is surely satisfied if $\bigcup\mathscr{C}$ and $\bigcup\mathscr{D}$ lie in two skew flats in a real vector space, and in many other situations as well. It is necessary because we have chosen to treat complexes as geometrically embedded objects in a real vector space. Simplicial complexes can also be considered as purely combinatorial objects whose vertices are the points of a finite set M and whose faces are subsets of M such that each subset of a face is a face and the intersection of any two faces is a face. In that approach, it would be necessary only to assume $\bigcup\mathscr{C}$ and $\bigcup\mathscr{D}$ are disjoint, and $\mathscr{C}\cdot\mathscr{D}$ would be defined as $\{C\cup D: C\in\mathscr{C}, D\in\mathscr{D}\}$. Similar comments apply to the other notions defined below.

When $\mathscr{D}$ is a 0-complex consisting of just two vertices x and y, the join $\mathscr{C}\cdot\mathscr{D}$ is called a *suspension* of $\mathscr{C}$. Its vertices are x, y, and the vertices of $\mathscr{C}$. For a vertex v of $\mathscr{C}$, a *dual wedge* of $\mathscr{C}$ on v is a complex

$$\operatorname{dw}(v,\mathscr{C})=((\mathscr{C}\setminus v)\cdot\{x,y\})\cup\{\operatorname{con}([x,y]\cup C): C\in\operatorname{link}(v,\mathscr{C})\},$$

formed under the assumption that $v\in]x,y[$ and v is the only point common to the segment $[x,y]$ and the affine hull of any member of $\operatorname{st}(v,\mathscr{C})$. The vertices of $\operatorname{dw}(v,\mathscr{C})$ are x, y, and the vertices of $\mathscr{C}$ other than v.

For a relative interior point p of a cell F of a simplicial complex $\mathscr{C}$, the associated *stellar subdivision* is defined succinctly by the formula

$$\operatorname{sub}(p,F,\mathscr{C})=(\mathscr{C}\setminus F)\cup(p\cdot\mathscr{B}(F)\cdot\operatorname{link}(F,\mathscr{C})).$$

The above operations on simplicial complexes are dual, in a sense that can be made precise, to operations on simple polytopes used in [KW] to prove 2.1 and 2.2. Joining two boundary complexes is dual to forming the *product* $P\times Q$ of two simple polytopes P and Q. The faces of $P\times Q$ are precisely the products of a face of P by a face of Q, and plainly $\delta(P\times Q)=\delta(P)+\delta(Q)$. Forming a dual wedge of a boundary complex is dual to the polytope wedging operation already described in §2. Forming a stellar subdivision of a boundary complex is dual to *truncating* a polytope P at a face G, which means intersecting P with a closed halfspace whose bounding hyperplane strictly separates G from the vertices of $P\sim G$. The intersection is a simple polytope Q such that $\delta(Q)\leqslant\delta(P)+1$, and thus truncation, like formation of products and wedges, preserves the property of being a Hirsch polytope.

In 5.1 and 5.2 below, a complex is said to be of *type* (d,n) if it is a pure $(d-1)$-complex that has exactly n vertices.

5.1. *Suppose that $\mathfrak{C}$ is a class of simplicial complexes that has members of type (d, n). If $\mathfrak{C}$ is closed under the formation of*
stellar subdivisions, then $\Delta^*_{\mathfrak{C}}(d, n+1) \geqslant \Delta^*_{\mathfrak{C}}(d, n)$;
dual wedges, then $\Delta^*_{\mathfrak{C}}(d+1, n+1) \geqslant \Delta^*_{\mathfrak{C}}(d, n)$;
suspensions, then $\Delta^*_{\mathfrak{C}}(d+1, n+2) \geqslant \Delta^*_{\mathfrak{C}}(d, n) + 1$.

5.2. *Suppose that $\mathfrak{C}$ is a class of simplicial complexes that has members of type (d, n) and is closed under formation of links, stellar subdivisions, and dual wedges. If $d < n \leqslant 2d$ then $\Delta^*_{\mathfrak{C}}(d, n) = \Delta^*_{\mathfrak{C}}(n-d, 2(n-d))$. If $n \geqslant 2d$ there exist $\mathscr{C} \in \mathfrak{C}$ of type (d, n) and two facets F and G of $\mathscr{C}$ such that $F \cap G = \varnothing$ and $\delta^*_{\mathscr{C}}(F, G) = \Delta^*_{\mathscr{C}}(d, n)$.*

PROOF. The (obvious) assertions of 5.1 are used in proving 5.2. Consider two facets F and G of a $\mathscr{C} \in \mathfrak{C}$ such that $\mathscr{C}$ is of type (d, n) and $\delta^*_{\mathscr{C}}(F, G) = \Delta^*_{\mathscr{C}}(d, n)$. Let $S = F \cap G$, $s = \dim S + 1$, and $\mathscr{L} = \operatorname{link}(S, \mathscr{C})$. Then $\mathscr{L} \in \mathfrak{C}$, $\mathscr{L}$ is of type $(d-s, n-m)$ for some $m \geqslant s$, and

$$\delta^*_{\mathscr{L}}(F \setminus S, G \setminus S) \geqslant \delta^*_{\mathscr{C}}(F, G) = \Delta^*_{\mathfrak{C}}(d, n).$$

Note that the number of vertices of $F \cup G$ is $2d - s \leqslant n$. If $n = 2d - k$ with $k \geqslant 0$ then $k \leqslant s$ and we have

$$\begin{aligned} \Delta^*_{\mathfrak{C}}(n-d, 2(n-d)) &= \Delta^*_{\mathfrak{C}}(d-k, n-k) \geqslant \Delta^*_{\mathfrak{C}}(d-s, n-s) \\ &\geqslant \Delta^*_{\mathfrak{C}}(d-s, n-m) \geqslant \delta^*_{\mathscr{L}}(F \setminus S, G \setminus S) \geqslant \delta^*_{\mathscr{C}}(F, G) \\ &= \Delta^*_{\mathfrak{C}}(d, n) \geqslant \Delta^*_{\mathfrak{C}}(n-d, 2(n-d)). \end{aligned}$$

Here the third inequality is obvious and the first ⟨resp. second, fourth, fifth⟩ follows from closure under dual wedging ⟨resp. stellar subdivision, link-formation, dual wedging⟩.

Now suppose that $n \geqslant 2d$, and among all the triples $(\mathscr{C}, F, G)$ described above choose one for which s is minimum. If $s = 0$ the proof is complete. Suppose, then, that $s > 0$, let $\mathscr{L} = \operatorname{link}(S, \mathscr{C})$, and note that $\mathscr{L}$ must have a vertex v not in $F \cup G$, for otherwise

$$2 = \frac{n-s}{d-s} > \frac{n}{d} \geqslant 2.$$

Let $\mathscr{L}'$ be the result of subdividing $\mathscr{L}$ $m - s$ times, using always a new point p that is relatively interior to a facet other than $F \setminus S$ and $G \setminus S$. With v, x and y as in the definition of dual wedge, let $\mathscr{L}^* = dw(v, \mathscr{L}')$, $F^* = (F \setminus S) \cdot x$ and $G^* = (G \setminus S) \cdot y$. Then $\mathscr{L}^* \in \mathfrak{C}$, $\mathscr{L}^*$ is of type $(d-s+1, n-s+1)$, and F^* and G^* are facets of $\mathscr{L}^*$ such that $F^* \cap G^* = \varnothing$ and $\delta_{\mathscr{L}^*}(F^*, G^*) \geqslant \Delta_{\mathfrak{C}}(d, n)$. The proof is completed by forming dual wedges $s - 1$ more times, starting from $\mathscr{L}^*$. ∎

Theorems 5.1 and 5.2 are the analogues, for general simplicial complexes, of 2.1 and 2.2. Theorem 2.5 can also be extended to general complexes, thus relating the function $\Delta^*_{\mathfrak{C}}$ to the function $R^*_{\mathfrak{C}}$. These extensions are valid not only when $\mathfrak{C}$ is the class of all polytopal simplicial complexes but also when $\mathfrak{C}$ is any of the following classes to be defined later: vertex-decomposable complexes, vertex-decomposable triangulated spheres, combinatorial spheres. For the class $\mathfrak{S}$ of all triangulated spheres, the condition on links fails and thus we don't know whether the extensions are valid. However, since $\mathfrak{S}$ is of special interest because of the close relationship of triangulated spheres to boundary complexes of simplicial polytopes, the functions $\Delta^*_{\mathfrak{S}}$ and $R^*_{\mathfrak{S}}$ are denoted here more simply by Δ_σ and R_σ.

The next two results may be regarded as analogues, for graphs not required to be 3-polytopal, of the fact that $\Delta(3, n)$ is bounded above by a linear function of n.

5.3. *If $\mathscr{C}$ is a triangulated 2-manifold that has n vertices and Euler characteristic χ, then*

$$\delta^*(\mathscr{C}) \leqslant \left\lfloor \frac{2n - 2\chi - 2}{3} \right\rfloor + 1$$

and at least when χ is 2 (sphere), 1 (projective plane) or 0 (torus), $\rho^*(\mathscr{C}) = 0$.

PROOF. Let e and t denote the numbers of edges and triangles of $\mathscr{C}$, whence $2e = 3t$, $n - e + t = \chi$, and hence $t = 2n - 2\chi$. Since the dual graph of a closed d-pseudomanifold is always $(d+1)$-connected [Kl5], $\mathscr{C}$'s dual graph is 3-connected and has t vertices. The inequality for δ^* then follows as in the first paragraph of the proof of 4.1. For ρ^* in the case of 2-spheres, use the known properties of 3-polytopes and the fact [SR], [Gr'1] that all spherical 2-complexes are polytopal. The cases of triangulated projective planes and tori are settled by Barnette [Ba'10, 11]. ∎

Aside from the following, little is known about higher-dimensional analogues of 5.3.

5.4. *If $n \leqslant d + 5$ and $\mathscr{C}$ is an abstract d-polytope with n vertices then $\rho^*(\mathscr{C}) = 0$ and hence $\delta^*(\mathscr{C}) \leqslant n - d$. Thus*

$$\text{if } d \leqslant 3 \text{ or } n \leqslant d + 3 \text{ then } R_\sigma(d, n) = 0 \text{ and } \Delta_\sigma(d, n) \leqslant n - d.$$

However,

$$R_\sigma(4, 16) \geqslant 1 \quad \text{and} \quad \Delta_\sigma(12, 24) \geqslant 13.$$

PROOF. The result on abstract polytopes is due to Adler and Dantzig [AD]. The inequalities of $R_\sigma(4, 16)$ and $\Delta_\sigma(12, 24)$ are due to Mani and Walkup [MW]. They were mentioned also in 4.2 and are discussed in the next section. ∎

We turn now to a class of complexes introduced by Provan and Billera [PB1] and used to prove the Hirsch conjecture for some classes of polytopes that arise in linear programming (see our §8). The inductive definition leads to a recursive algorithm for finding short ridge-paths.

An s-complex $\mathscr{C}$ is *k-decomposable* if it is pure and simplicial and either $\mathscr{C}$ consists of an s-simplex and its faces or there exists a j-face F of $\mathscr{C}$ (called a *shedding face*) with $j \leqslant k$ such that

(i) the complex $\mathscr{C} \setminus F$ is s-dimensional and k-decomposable, and

(ii) the complex *link* $(F, \mathscr{C})$ is $(s - k - 1)$-dimensional and k-decomposable.

The complexes defined by (i) alone are called *weakly k-decomposable*, and the (weakly) 0-decomposable complexes are said to be *(weakly) vertex-decomposable*.

The following result is due to [Pr] and [PB1] for decomposable complexes and slightly sharpens their bound in the case of weak decomposability.

5.5. *If $0 \leqslant k \leqslant s$ and $\mathscr{C}$ is an s-complex that has $f_k(\mathscr{C})$ k-faces then*

$$\delta^*(\mathscr{C}) \leqslant f_k(\mathscr{C}) - \binom{s+1}{k+1} \quad \text{when } \mathscr{C} \text{ is } k\text{-decomposable} \quad \text{and}$$

$$\delta^*(\mathscr{C}) \leqslant 2\left(f_k(\mathscr{C}) - \binom{s+1}{k+1}\right) \quad \text{when } \mathscr{C} \text{ is weakly } k\text{-decomposable}.$$

Rather than discussing 5.5 in general, we focus on the most important case $k = 0$. The argument below is the basis of a recursive algorithm that finds nonrevisiting ridge-paths.

5.6. *Any two facets of a vertex-decomposable complex are joined by a ridge-path that does not revisit any vertex.*

PROOF. The basic idea is to form a ridge-path for which, as the path departs from any vertex, that vertex is shedded away and hence cannot be revisited. The construction below is algorithmic in nature.

Starting from

a vertex-decomposable s-complex $\mathscr{C}_0$ with n vertices in all
shedding vertices $v(1), \ldots, v(n-s-1)$ in order of deletion, $(*)$
facets F_0 and G_0 of $\mathscr{C}_0$

let $v(t_1)$ be the first vertex that belongs to $F_0 \cup G_0$. If $v(t_1) \in F_0 \sim G_0$ $\langle$resp. $G_0 \sim F_0\rangle$, let $\mathscr{C}_1 = \mathscr{C}_0 \setminus v(1) \setminus \cdots \setminus v(t_1)$ and let $F_1 \langle G_1 \rangle$ be a facet whose intersection with $F_0 \langle G_0 \rangle$ is a ridge that omits $v(t_1)$. (Such a facet exists because $\mathscr{C} \setminus v(t_1)$ is pure s-dimensional.) Then carry on, with $\mathscr{C}_1, v(t_1+1), \ldots, v(n-s-1)$, F_1 and G_0 $\langle F_0$ and $G_1\rangle$ in place of $(*)$. The procedure eventually yields a

$$\mathscr{C}_m = \mathscr{C}_{m-1} \setminus v(t_{m-1}+1) \setminus \cdots \setminus v(t_m)$$

and facets F_i and G_j with $i + j = m$ such that $F_i \cap G_j$ contains the first shedding vertex $v(t_{m+1})$ that is in $F_i \cup G_j$. This vertex is to belong to all subsequent F_i's and G_j's, and with that understanding the ridge-path formation may be continued in dimension $s-1$, replacing $(*)$ by

$\operatorname{link}(v(t_{m+1}), \mathscr{C}_m)$
the remaining shedding vertices in the shedding for this link
the facets of F_i and G_j that omit $v(t_{m+1})$.

Proceeding in this manner, always shedding vertices so that no vertex is revisited, and reducing the dimension whenever the next shedding vertex in $F_i \cup G_j$ belongs to $F_i \cap G_j$, we build sequences of facets $F_0, F_1, \ldots$ and $G_0, G_1, \ldots$ which eventually meet —say $F_p = Q_q$ with no earlier equality—and the combined sequence $F_0, \ldots, F_{p-1}$, $F_p = G_q, G_{q-1}, \ldots, G_0$ is a nonrevisiting ridge-path from F_0 to G_0. That proves 5.6.

In the case of weak vertex-decomposability, information on links is absent. However, when it happens that $v(t_{m+1}) \in F_i \cap G_i$ the sequences $F_0, \ldots, F_i$ and $G_0, \ldots, G_j$ can *both* be extended by letting F_{i+1} $\langle$resp. $G_{j+1}\rangle$ be a facet whose intersection with F_i $\langle G_j \rangle$ is a ridge that omits $v(t_{m+1})$. Then carry on with F_{i+1} and G_{j+1} in the complex

$$\mathscr{C}_{m+1} = \mathscr{C}_m \setminus v(t_m+1) \setminus \cdots \setminus v(t_{m+1}).$$

That leads to 5.5's bound $2(n-s-1)$. ■

Let us say that a simplicial polytope is *k-decomposable* or *weakly k-decomposable* if its boundary complex has that property. §6 describes a simplicial 4-polytope that is not vertex-decomposable, and hence 5.5 cannot be used directly to prove the Hirsch conjecture. However, for each fixed $k \geqslant 1$ $\langle$resp. $k \geqslant 0\rangle$ it is unknown whether all simplicial polytopes are k-decomposable $\langle$weakly k-decomposable$\rangle$. This is of interest, for in view of 5.6 weak k-decomposability implies a polynomial upper bound on the diameter function Δ.

The behavior of decomposability seems to lie at the heart of the difficulty of constructing a counterexample to the Hirsch conjecture, for as stated in 5.8 below, the most readily accessible simplicial polytopes are all vertex-decomposable (see also §8), and as stated in 5.8, decomposability is preserved under several of the few operations known to preserve polytopality of complexes.

5.7. *Suppose that $\mathscr{C}$ is a simplicial s-complex with n vertices. If $\mathscr{C}$ is a triangulated ball and $s \leqslant 2$ or $n \leqslant s+3$ then $\mathscr{C}$ is vertex-decomposable. If $\mathscr{C}$ is a triangulated sphere and $s \leqslant 2$ or $n \leqslant s+4$ then $\mathscr{C}$ is polytopal and vertex-decomposable.*

PROOF. Polytopality of s-spheres is due to [SR] (see also [Gr1]) when $s = 2$, to [Ma], [Kl'1] when $n \leqslant s+4$. Vertex-decomposability of triangulated 2-balls and 2-spheres is proved by [PB1], thus providing (in view of 5.3) another proof that

$R(3, n) = 0$. Each triangulated s-ball with $\leqslant s + 2$ vertices is a subcomplex of the boundary complex of an $(s + 1)$-simplex and is easily seen to be vertex-decomposable. If a triangulated s-sphere has $\leqslant s + 3$ vertices the antistar of any vertex is a ball of the sort just mentioned, and vertex-decomposability follows.

To complete the proof of 5.7 we show by induction on s that if a simplicial complex $\mathscr{C}$ has $s + 4$ vertices and is the boundary complex of an $(s + 1)$-polytope P (or, equivalently for this number of vertices, is a triangulated s-sphere), then each vertex x of $\mathscr{C}$ is a shedding vertex; that is, both link$(X, \mathscr{C})$ and $\mathscr{C} \setminus x$ are vertex-decomposable. The vertex-decomposability of link$(x, \mathscr{C})$ follows by induction, since, by P's convexity, ast$(x, \mathscr{C})$ is a triangulated s-ball with $s + 3$ vertices and link$(x, \mathscr{C})$ is a triangulated $(s - 1)$-sphere with at most $s + 3$ vertices. To prove that $\mathscr{C} \setminus x$ is vertex-decomposable, we use the Gale diagrams discussed in [Gr1].

Let $\overline{P}$ be P's distended Gale diagram, a subset of the unit circle U of $\mathbb{R}^2$. Let $\bar{x}$ be the image of x in $\overline{P}$, $\bar{y}$ a point of $\overline{P}$ that is closest to $-\bar{x}$, and A the minor arc of U that joins $-\bar{x}$ and $-\bar{y}$. Since the minor arc joining $-\bar{x}$ and $\bar{y}$ misses $\overline{P}$, A must include a point $\bar{z} \in \overline{P}$ and the set $\{x, y, z\}$ is therefore a coface of P. This property of cofaces implies that each face of P which misses $\{x, y\}$ lies in a facet of P which misses $\{x, y\}$, whence $\mathscr{C} \setminus x \setminus y$ is an s-ball that has only $s + 2$ vertices and hence is vertex-decomposable. But then $\mathscr{C} \setminus x$ is vertex-decomposable with the shedding vertex y and $\mathscr{C}$ is vertex-decomposable with the shedding vertex s. This argument also proves that every s-ball with $s + 3$ vertices is vertex-decomposable for it is always the antistar of some vertex x in an s-sphere with $s + 4$ vertices and we have just seen that x can serve as a shedding vertex. ∎

We do not know just how far the bounds $s + 3$ and $s + 4$ can be raised without losing vertex-decomposability, but some limitations are implied by the examples of §6.

The following observations are due to [PB1].

5.8. *If complexes $\mathscr{C}$ and $\mathscr{D}$ are k-decomposable then so is their join $\mathscr{C} \cdot \mathscr{D}$.*
A stellar subdivision of a k-decomposable complex is k-decomposable.
A dual wedge of a complex $\mathscr{C}$ is vertex-decomposable if and only if $\mathscr{C}$ is.

A *shelling* of a pure s-complex $\mathscr{C}$ is a permutation $F_1, \dots, F_k$ of $\mathscr{C}$'s facets such that for $1 < j \leqslant k$ the intersection $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is topologically an $(s - 1)$-ball or $(s - 1)$-sphere (and hence in the latter case is the entire boundary of F_j). When $\mathscr{C}$ is simplicial this amounts to saying $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is a nonempty union of $(s - 1)$-faces of F_j. A complex is *shellable* if it admits a shelling. Results on shellability have been surveyed by [DK2] and [EKPS].

Bruggesser and Mani [BM] prove that each polytopal complex is shellable, and [PB1] observe that a pure simplicial s-complex is s-decomposable if and only if it is shellable. Thus, in view of 5.5, the notion of k-decomposability relates shellability to the Hirsch conjecture and other polynomial bounds on diameters of polytopes. If $\mathfrak{D}(s, k)$ is the class of all k-decomposable pure simplicial s-complexes, then of course

$$\mathfrak{D}(s, s) \supset \mathfrak{D}(s, s - 1) \supset \cdots \supset \mathfrak{D}(s, 1) \supset \mathfrak{D}(s, 0).$$

For the boundary complex $\mathscr{C}$ of a simplicial $(s + 1)$-polytope P, we have $\mathscr{C} \in \mathfrak{D}(s, s)$ by [BM], [PB1], and in fact $\mathscr{C} \in \mathfrak{D}(s, s - 1)$ by a shelling result of [DK1]. It is unknown whether $\mathscr{C} \in \mathfrak{D}(s, 1)$ in general. By 5.6, P's dual is a Hirsch polytope if $\mathscr{C} \in \mathfrak{D}(s, 0)$, but for $s \geqslant 3$ there are examples of polytopes P for which $\mathscr{C} \notin \mathfrak{D}(s, 0)$. More information about decomposability and shellability appears in the next section.

6. More counterexamples to stronger statements. The main purpose of this section is to discuss the examples mentioned in (viii), (ix) and (x) of 4.2. However, we want first to record the structure of the (combinatorially) unique simplicial 4-polytope that has 9 vertices and ridge-diameter 5. With vertices $a, \ldots, i$, it has 15 facets not incident to i—$ab\underline{cd}$, $ac\underline{d}f$, $ade\underline{h}$, $a\underline{d}fg$, $adgh$, $aeg\underline{h}$, $b\underline{c}de$, $b\underline{c}eh$, $bcf\underline{g}$, $bcgh$, $bf\underline{g}h$, $cdeh$, $cdfg$, $cdgh$, $ef\underline{gh}$—and 12 facets incident to i, obtained by replacing any one of the 12 underlined letters with i. This complex $\mathscr{C}$ is of interest because a computer search reveals that of the 1297 combinatorial types of triangulated 3-manifolds with 9 vertices (one nonsphere, 154 nonpolytopal spheres, and 1142 polytopes [AS′], [ABS]), all but $\mathscr{C}$ are of ridge-diameter < 5. The complex $\mathscr{C}$ $\langle$resp. $\mathscr{C} \setminus \{i\}\rangle$ is dual to the boundary complex of [KW]'s simple 4-polytope Q with 9 facets $\langle$unbounded polyhedron Q' with 8 facets$\rangle$ and edge-diameter 5, and $\mathscr{C} \setminus \{i\}$ is a triangulated 3-ball that is by 5.5 not vertex-decomposable. For further discussion of Q', see [LW1, 2].

Since the polyhedra Q and Q' played an essential role in 4.1 and in (ii)–(vii) of 4.2, it seems that $\mathscr{C}$ or a close relative might provide a good building block for constructing a counterexample to the Hirsch conjecture. However, $\mathscr{C}$ is vertex-decomposable. The first non-vertex-decomposable polytopal 3-sphere known to us is the boundary complex of a certain simplicial 4-polytope with 12 vertices, described below. It was constructed by Lockeberg [Lo] for another purpose, and we learned of it through P. Mani and P. McMullen. To show it is not vertex-decomposable, let us first review some elementary notions from piecewise-linear topology.

A *subdivision* of a simplicial complex $\mathscr{C}$ is a simplicial complex $\mathscr{S}$ such that $\cup\mathscr{S} = \cup\mathscr{C}$ and each face of $\mathscr{C}$ is a union of faces of $\mathscr{S}$. A *combinatorial s-ball* $\langle$resp. *combinatorial s-sphere*$\rangle$ is a simplicial s-complex $\mathscr{C}$ such that some subdivision of $\mathscr{C}$ is isomorphic with (that is, admits an incidence-preserving one-to-one correspondence with) a simplicial subdivision of the complex $\mathscr{T}^s$ that consists of an s-simplex T^s and its faces $\langle$resp. of $\mathscr{B}(T^{s+1})\rangle$. Equivalently [G1], $\mathscr{C}$ is obtainable from $\mathscr{T}^s$ $\langle\mathscr{B}(T^{s+1})\rangle$ by a sequence of isomorphisms, stellar subdivisions and inverses of stellar subdivisions. All triangulated s-balls and s-spheres are combinatorial for $s \leqslant 4$ [Mo] but not for $s \geqslant 5$ [Ca]. All shellable triangulated balls are combinatorial [DK1], and this applies in particular to boundary complexes of simplicial polytopes. For two complexes $\mathscr{C}$ and $\mathscr{D}$, $\mathscr{C} \sim \mathscr{D}$ denotes the set-theoretic difference—i.e., the collection of all members of $\mathscr{C}$ that are not members of $\mathscr{D}$.

6.1. *For an edge xy of a combinatorial s-sphere $\mathscr{C}$, the following three conditions are equivalent*: (i) xy *is* shrinkable *in the sense that the complex $\mathscr{C}'$ obtained from $\mathscr{C}$ by identifying x and y is a combinatorial sphere*; (ii) *the complex* $(\mathscr{C} \setminus x) \setminus y$ $(= \mathscr{C} \setminus y) \setminus x)$ *is a combinatorial s-ball*; (iii) $\mathrm{st}(x, \mathscr{C}) \cap \mathrm{st}(y, \mathscr{C}) = \mathrm{st}(xy, \mathscr{C})$.

PROOF. The work "combinatorial" is omitted in what follows, for all the relevant balls and spheres are combinatorial. Dimensions are not specified when they are obviously the right ones. The boundary complex of a ball $\mathscr{B}$ is denoted by $\partial\mathscr{B}$, and $\mathscr{B}^0 := \mathscr{B} \sim \partial\mathscr{B}$. The proof is merely sketched, for it uses only such standard results [Gl] as the following: $\mathscr{A} \cup \mathscr{B}$ is an s-ball if $\mathscr{A}$ and $\mathscr{B}$ are s-balls and $\mathscr{A} \cap \mathscr{B}$ is an $(s-1)$-ball in $(\partial\mathscr{A}) \cap (\partial\mathscr{B})$; $\mathscr{S} \sim \mathscr{B}^0$ is an s-ball if $\mathscr{B}$ is an s-ball in the s-sphere $\mathscr{S}$.

To carry out the identification in (i), replace each face of $\mathscr{C}$ of the form $x \cdot y \cdot F$ or $y \cdot F$ by $x \cdot F$. If the resulting $\mathscr{C}'$ is a sphere then $\mathrm{ast}(x, \mathscr{C}')$ is a ball, and since $\mathrm{ast}(x, \mathscr{C}) = (\mathscr{C} \setminus x) \setminus y$ this shows (i) implies (ii).

In (iii), $\subset$ is obvious, and if the set $(\mathrm{st}(x, \mathscr{C}) \cap \mathrm{st}(y, \mathscr{C})) \setminus \mathrm{st}(xy, \mathscr{C})$ is nonempty it includes a face F such that $x, y \notin F$. Since link $(F, \mathscr{C})$ is a sphere and link $(F, (\mathscr{C} \setminus x) \setminus y)$ is a ball, F is at least $(s-2)$-dimensional or there is a vertex z such that

$$z \in \mathrm{link}(x, \mathrm{link}(F, \mathscr{C})) \cap \mathrm{link}(y, \mathrm{link}(F, \mathscr{C})),$$

$$z \cdot F \in \mathrm{link}(x, \mathscr{C}) \cap \mathrm{link}(y, \mathscr{C}) \quad \text{and} \quad z \cdot F \notin \mathrm{link}(xy, \mathscr{C}).$$

By induction there is an $(s-1)$- or $(s-2)$-face G such that

$$G \in (\operatorname{link}(x, \mathscr{C}) \cap \operatorname{link}(y, \mathscr{C})) \setminus \operatorname{link}(xy, \mathscr{C}).$$

If $\dim G = s-1$ then the 0-dimensional ball $\operatorname{link}(G, (\mathscr{C} \setminus x) \setminus y)$ is empty, and the contradiction shows that (ii) implies (iii). If $\dim G = s-2$, then $\operatorname{link}(G, \mathscr{C} \setminus x) \setminus y)$ is a 1-dimensional ball and $\operatorname{link}(G, \mathscr{C})$ is a 1-sphere. As both x and y are in $\operatorname{link}(G, \mathscr{C})$, the edge xy must be in $\operatorname{link}(G, \mathscr{C})$ too, for $\operatorname{link}(G, \mathscr{C})$ can only be generated from $\operatorname{link}(G, (\mathscr{C} \setminus x) \setminus y)$ by the addition of x, y, xy, and two more edges. This contradicts the fact that $G \notin \operatorname{link}(xy, \mathscr{C})$, and shows again that (ii) implies (iii).

Now let

$$\mathscr{A} = \operatorname{st}(y, \operatorname{link}(x, \mathscr{C})), \quad \mathscr{B} = \operatorname{st}(x, \mathscr{C}), \quad \mathscr{D} = \operatorname{st}(y, \mathscr{C}) \setminus \operatorname{st}^0(x, \mathscr{C}),$$

so that plainly $\mathscr{A} \subset \mathscr{B} \cap \mathscr{D}$. If (iii) holds then for each $F \in \mathscr{B} \cap \mathscr{D}$ it is true that $x \notin F$ and

$$F \in \operatorname{st}(x, \mathscr{C}) \cap \operatorname{st}(y, \mathscr{C}) = \operatorname{st}(xy, \mathscr{C}),$$

whence $x \cdot y \cdot F \in \mathscr{C}$ or $(x \cdot F \in \mathscr{C}$ and $y \in F)$ and consequently $F \in \mathscr{A}$. Thus (iii) implies that $\mathscr{A} = \mathscr{B} \cap \mathscr{D}$ and $\mathscr{B} \cup D$ is a ball $\mathscr{E}$. But

$$\partial \mathscr{E} = (\partial \mathscr{B} \setminus A^0) \cup (\partial \mathscr{D} \setminus A^0) \quad \text{and}$$
$$F \in \partial \mathscr{E} \Leftrightarrow F \in \mathscr{E} \quad \text{and} \quad x, y \notin F,$$

so $\partial \mathscr{E} = \operatorname{link}(x, \mathscr{C}')$ with $\mathscr{C}'$ as in (i). It follows that

$$\mathscr{C}' = (x \cdot \partial \mathscr{E}) \cup (\mathscr{C} \setminus \mathscr{D}^0),$$

and since this is a sphere we conclude (iii) implies (i). ∎

6.2. *If a triangulated s-sphere $\mathscr{C}$ is vertex-decomposable and has more than $s + 2$ vertices then $\mathscr{C}$ has a shrinkable edge.*

PROOF. Since each shellable ball is combinatorial, it is clear that if a vertex-decomposable s-complex $\mathscr{B}$ is a proper subcomplex of a combinatorial s-sphere then $\mathscr{B}$ is a combinatorial s-ball. Hence each vertex-decomposable s-sphere $\mathscr{C}$ with more than $s + 2$ vertices has an edge xy such that $(\mathscr{C} \setminus x) \setminus y$ is an s-ball, and such an edge is shrinkable by 6.1. ∎

We are now ready for

6.3. *There is a simplicial 4-polytope with 12 vertices that has no shrinkable edge and hence is not vertex-decomposable.*

PROOF. With vertices $a, \ldots, l$, the 48 facets of [Lo]'s polytope P are as follows:

bcde	bfij	cdgk	adil
cdef	bgij	dghk	adel
defg	fgij	cghk	deil
degh	dfgj	cdik	agil
afgh	cdgj	adik	aehl
efgh	cdfj	acik	afhl
aghi	cghj	abhk	efhl
bghi	bchj	abck	befl
abhi	bghj	bchk	bfil
bdei	bcej	adek	beil
bcdi	cefj	aejk	afgl
abci	befj	dehk	fgil

In constructing P, [Lo]'s aim was to find a polytope which cannot be obtained from a simplex by successive facet-splittings (cf. §8), and these correspond dually to the inverses of edge-shrinkings. To see that no edge xy of $\mathscr{B}(P)$ is shrinkable, note there is always a vertex z such that yz and zz are edges but xyz is not a face of $\mathscr{B}(P)$; then use 6.1 (iii). ■

To obtain [Lo]'s polytope from the cyclic polytope $C(9,4)$, for each facet F of $C(9,4)$ let $\mathscr{G}(F)$ consist of the five facets of $C(9,4)$ that have at least one 2-face in common with F. Let F_1, F_2 and F_3 be facets of $C(9,4)$ for which the collections $\mathscr{G}(F_i)$ are pairwise disjoint, and for $i = 1,2,3$ let v_i be a point of $\mathbb{R}^4$ that is beyond all members of $\mathscr{G}(F_i)$ but beneath all other facets of $C(9,4)$. (Each facet F of a polytope P determines a hyperplane H that bound two open halfspaces, one containing $P \sim H$ and the other missing P. Points of the former open halfspace are *beneath* F (relative to P). See [Gr'1] for further discussion of cyclic polytopes and the beneath-beyond terminology.) Then v_1, v_2, v_3 and the 9 vertices of $C(9,4)$ determine a simplicial 4-polytope whose boundary complex is isomorphic to the one described above. This complex is weakly vertex-decomposable, for a, k, l, i, b, h, e, g is a shedding order.

In [AS2]'s list of neighborly 3-spheres with 10 vertices, there is one (called N_{416}^{10}) that is weakly vertex-decomposable but not vertex-decomposable (by the same reasoning as before). For some time it was unknown whether this sphere was polytopal, but Bokowski and Sturmfels [BS] recently showed that it is. This shows that the number 12 in 6.3 can be replaced by 10. By using a dual wedge construction for this polytope and applying 5.9, 6.3 can be generalized as follows:

6.4. *For each $d \geqslant 4$ there is a simplicial d-polytope that has $d + 6$ vertices and is not vertex-decomposable.*

As mentioned earlier, we do not know whether all simplicial polytopes (or even all combinatorial spheres) are weakly vertex-decomposable. It follows from 5.4 that $d + 6$ in 6.4 cannot be replaced by $d + 3$, but we don't know about $d + 4$ or $d + 5$.

We turn now to 4.2(ix). To show this fails for $d = 28$, Walkup [Wa2] constructs a triangulated 27-sphere, with 56 vertices and more than 8000 facets, such that each ridge-path between a specified pair of facets revisits a vertex of the sphere. Mani and Walkup [MW] have a smaller sphere of this sort, based on a nonspherical and nonsimplicial 2-complex constructed by Larman [La].

TABLE 1

abcd	cgor	bfen	gmst	bfps
abcr	dgor	cfeo	hmst	cgps
acdr	dhpr	cgfo	hnst	cgms
abdt	agpt	dgfp	enst	dhms
bcdt	agmt	dhgp	eost	dhns
abmr	bhmt	ahqr	fost	anoq
bcnr	bhnt	aeqr	fpst	bopq
cdor	cent	beqr	ahnq	cpmq
adpr	ceot	bfqr	aeoq	dmnq
abmt	dfot	cfqr	beoq	anos
bcnt	dfpt	cgqr	bfpq	bops
cdot	aghp	dgqr	cfpq	cpms
adpt	behm	dhqr	cgmq	dmns
ahpr	cfen	aehn	dgmq	mnoq
aemr	dgfo	bfeo	dhnq	mopq
bemr	ahgm	cgfp	aens	mnps
bfnr	aehm	dhgm	aeos	nops
cfnr	behn	gpst	bfos	mnop

6.5. *There is a triangulated 3-sphere $\mathscr{D}$, with 16 vertices and 90 facets, such that each ridge-path between a specified pair of facets revisits a vertex of the sphere. Hence* $R_\sigma(4,16) \geqslant 1$.

The following, related to 4.2(x), is a consequence of dual wedging applied to $\mathscr{D}$:

6.6. $\Delta_\sigma(12,24) \geqslant 13$.

Table 1 lists all the facets of $\mathscr{D}$, the crucial two facets being *abcd* and *mnop*.

In view of 5.4, the [MW] sphere $\mathscr{D}$ is not vertex-decomposable, and [Al] shows it's not polytopal. We don't know whether it's shellable. If not, it would be the first example of a nonshellable 3-sphere and the first example of a nonshellable combinatorial sphere, though nonshellable 3-balls are known and for $s \geqslant 5$ there are triangulated spheres that are not combinatorial and hence not shellable. (Shellability is surveyed by [DK2] and [EKPS].)

[MW] also describe a shellable 3-sphere that has 20 vertices and has two facets between which there is no nonrevisiting ridge-path. The above example $\mathscr{D}$ is actually a "contraction" of this sphere, which explains why some letters are missing in the alphabetical description of $\mathscr{D}$.

[La′] extends the reasoning of [KW] to show the equivalence of several forms of the Hirsch conjecture for abstract polytopes. However, by 6.5–6.6 these all fail for general triangulated spheres and hence for abstract polytopes.

7. General lower and upper bounds. A small mystery concerning the Hirsch conjecture is that although no one has come close to proving $\Delta(d,n) \leqslant n-d$ in general, the only (d,n) for which it is known that $\Delta(d,n) \geqslant n-d$ are the pairs $(d, 2d+1)$ with $d \geqslant 4$ and the pairs (d,n) with $n \leqslant 2d$. The best general lower bounds known for Δ and Δ_u are as follows:

7.1.

$$\Delta(d,n) \geqslant \left\lceil (n-d) - \frac{n-d}{\lfloor 5d/4 \rfloor} \right\rceil - 1,$$

$$\Delta_u(d,n) \geqslant n-d+\min(\lfloor d/4 \rfloor, \lfloor (n-d)/4 \rfloor).$$

These bounds are due to Adler [Ad] and [KW] respectively. A single class of polytopes constructed in [Kl1] shows

$$\Delta(d,n) \geqslant (d-1)\lfloor n/d \rfloor - d + 2.$$

The sharper bound in 7.1 is based on the recursions of 2.2 and the following additional recursion:

7.2.

$$\Delta(d, m+n-d) \geqslant \Delta(d,m) + \Delta(d,n) - 1.$$

Inequality 7.2 comes from properties of the "sum" $P \oplus Q$ of two simple d-polytopes P and Q, first used by [Ba′2]. Truncate vertices v and w of P and Q respectively, creating new facets F and G which are $(d-1)$-simplices. Transform P and Q projectively to make F and G coincide, to place P and Q on opposite sides of the hyperplane H containing the new $F = G$, and to make the additional facets adjacent to F or G all perpendicular to H. Then "glue" P and Q together at F and G, creating a new simple d-polytope $P \oplus Q$ whose number of facets is $f_{d-1}(P) + f_{d-1}(Q) - d$. To establish 7.2, note that for any two vertices u of $P \sim \{v\}$ and x of $Q \sim \{w\}$, the distance $\delta_{P\oplus Q}(u,x)$ is equal to the sum $\delta_P(u,v) + \delta_Q(w,x)$ or to this sum minus 1.

A large mystery concerning the Hirsch conjecture is that although no one has shown $\Delta(d,2d) > d$ for large d, the best known upper bound on Δ is provided by 7.3 below, due to Larman [La]. ([Ba′7] claims a better bound, but its proof is in error.) The remainder of this section reviews Larman's proof, showing that it applies not only to Δ_u but in a much wider context.

7.3. *For $d \geqslant 3$, $\Delta_u(d, n) \leqslant 2^{d-1}n$.*

It follows from 7.1 and 7.3 that for each *fixed* d, $\Delta(d, n)$ and $\Delta_u(d, n)$ both grow linearly in n, but any upper bound that is polynomial in both d and n would be of great interest.

If an edge-path Π in a complex $\mathscr{C}$ is regarded as the union of its edges, then a *visit* of Π to a face F of $\mathscr{C}$ may be defined as a connected component of $\Pi \cap F$. Let us say that Π is a *V_k-path* if it does not visit any face more than k times, that $\mathscr{C}$ is *V_k-connected* if between any two of its vertices there is a V_k-path, and a polyhedron is V_k-connected if its boundary complex has that property. The reasoning of [La] proves

7.4. *If the graph of a complex $\mathscr{C}$ is connected and each face of $\mathscr{C}$ is V_k-connected, then $\mathscr{C}$ itself is V_{2k} connected.*

Since each 3-polyhedron is V_1-connected, it follows from 7.4 and a straightforward induction that

7.5. *Between any two vertices of an s-complex whose graph is connected $\langle$resp. of a d-polyhedron for $d \geqslant 3\rangle$ there is a path that visits each face at most 2^{s-1} $\langle$at most $2^{d-3}\rangle$ times.*

That 7.5 is best possible for $s \leqslant 2$ $\langle$resp. $d \leqslant 4\rangle$ follows from the properties of a 2-complex described in [La] $\langle$of the [KW] polyhedron showing $\Delta_u(4, 8) = 5\rangle$. For $s \geqslant 3$ and $d \geqslant 5$, we do not know whether 7.5 is best possible. As was seen in §§1–2, the Hirsch conjecture is equivalent to the assertion that 2^{d-3} may replaced by 1 in the case of simple polytopes.

When a path Π traverses an edge of a polyhedron, it ends a visit to at least one facet and starts a visit to at least one (to exactly one in each case if the polyhedron is simple). Hence if Π uses more than $2^{d-3}n$ edges in a polyhedron with only n facets, it visits some facet more than 2^{d-3} times. Thus 7.3 follows from 7.5 and we may concentrate on the basic result 7.4 which will be proved in a more general setting.

Consider a graph G and a family $\mathscr{F}$ of subgraphs (called *faces*) of G, and define visits, V_k-paths and V_k-connectedness as before. Then 7.4 may be extended as follows:

7.6. *Suppose that each face of a graph G is V_k-connected, and u and v are vertices of G that are joined by a sequence of successively intersecting V_k-paths. Then u and v are joined by a V_{2k}-path.*

PROOF. By hypothesis, there is a sequence

$$\Sigma\colon\ u = u_0, \Pi_1, u_1, \Pi_2, u_2, \ldots, \Pi_m, u_m = v$$

such that for $1 \leqslant k \leqslant m$, Π_i is a V_k-path that starts at the vertex u_{i-1} and ends at the vertex u_i. Among all such sequences, consider only those for which m is minimum; among these, consider only those for which the length of Π_1 is minimum; among these, consider only those for which the length of Π_2 is minimum; $\cdots$; among these, consider only those for which the length of Π_m is minimum. We claim that after this pruning the path Π described by any of the remaining sequences Σ is a V_{2k}-path from u to v.

Suppose that Π visits some face F of G more than $2k$ times, let x $\langle$resp. $y\rangle$ be the first $\langle$last$\rangle$ vertex of Π in F, and let r be such that $x \in \Pi_r \sim \{u_r\}$. Since each Π_i is a V_k-path, there do not exist indices i and j such that each vertex of $\Pi \cap F$ belong to Π_i or Π_j. This implies that (α) $y \in \Pi_s$ for some $s > r + 2$ or (β) $y \in \Pi_{r+2} \sim \{u_{r+1}\}$. Now consider the path Π' that results from replacing the portion of Π between x and y by a V_k-path from x to y in the V_k-connected face F. If (α) holds then Π' bypasses

Π_{r+1} and Π_{r+2}, while if (β) holds and $x = u_{r-1}$ then Π' bypasses Π_r and Π_{r+1}, in each case contradicting the minimality of m. If (β) holds and $x \neq u_{r-1}$ then Π' is still formed from m V_k-paths, but since $x \neq u_r$ the minimality of the length of Π_r is contradicted. ■

With respect to set inclusion, each complex is a partially ordered set $(\mathscr{P}, \leqslant)$, such that

(i) there is a least element 0 and $0 \leqslant x$ for each $x \in \mathscr{P}$;

(ii) for each $x \in \mathscr{P}$, all maximal chains from 0 to x have the same number $r(x)$ of elements.

For a poset $(\mathscr{P}, \leqslant)$ satisfying (i) and (ii), define the *dimension* of an element $x \in \mathscr{P}$ as $r(x) - 2$ (for example, $\dim 0 = -1$) and the dimension of $\mathscr{P}$ as the maximum of $\dim x$ for $x \in \mathscr{P}$. The *graph* of $\mathscr{P}$ has as its vertices the 0-dimensional elements of $\mathscr{P}$, and as its edges the pairs $\{u, v\}$ of distinct vertices such that there exists $e \in \mathscr{P}$ with $\dim e = 1$, $u < e$ and $v < e$. In this graph and its subgraphs, *paths* and *diameters* are defined in the usual way, and an *m-face* is defined as a subgraph spanned, for some $x \in \mathscr{P}$ with $\dim x = m$, by the set $\{u \in \mathscr{P}\colon u \leqslant x \text{ and } \dim u = 0\}$. Having defined faces, we may define *visits* and *V_k-paths* as before. The following extension of 7.5 is a consequence of 7.6.

7.7. *If $\mathscr{P}$ is an s-dimensional poset that satisfies (i) and (ii), and $\mathscr{P}$'s graph and its faces are all connected, then between any two vertices of $\mathscr{P}$ there is a path that visits each face at most 2^{s-1} times.*

If $\mathscr{P}$ is as in 7.7 and, in addition, each 3-face of $\mathscr{P}$ is isomorphic with the graph of a 3-polyhedron or the dual graph of a triangulated torus or projective plane (see 5.1), then 2^{s-1} may be replaced by 2^{s-2}. And 7.3 may be extended to $\mathscr{P}$ if for each edge $\{x, y\}$ of $\mathscr{P}$ there are s-faces E and F such that $x \in E \sim F$ and $y \in F \sim E$.

The class of posets $\mathscr{P}$ subject to 7.7 is much more general than the class of complexes. It includes many of the polystromas considered in [Gr'3] where often the poset-dimension of an element is different from the geometric dimension of its realization. It also includes several generalized manifolds as considered in [Ba'5] and projective arrangements of hyperplanes where $\leqslant$ is given by the relation of inclusion between intersections of hyperplanes. In the poset associated with such an arrangement, the elements of dimension 1 are the lines formed by intersecting hyperplanes and the elements of dimension 0 are the points of these lines. It turns out that the diameter of an arrangement in projective d-space is at most d.

8. Bounds for special classes of polyhedra. This section discusses the special classes of polyhedra, restricted in structure but unrestricted in dimension and number of facets, for which the Hirsch conjecture or bounds close to it have been proved. Some of these polyhedra arise as the feasible regions of important sorts of LP problems. For several of the classes, proofs of the Hirsch conjecture really consist of arguments for vertex-decomposability, and we suspect vertex-decomposability is also hidden in some proofs where its use is less evident. Thus, in view of the counterexample to vertex-decomposability described in §6, it seems possible that all proofs for special classes use a property not shared by all polytopes and hence have little chance of providing a key to the general Hirsch conjecture.

The following classes are discussed in separate subsections:

Matroid complexes [PB1], [PB2];

Leontief substitution polyhedra [Gr], [PB2];

Polytopes arising from the shortest path problem [Sa];

Dual transportation polyhedra [Ba3], [BR3];

Special transportation polytopes [Ba2], [BG], [BR2], [EKK], [PR];

Billera-Lee polytopes [BL], [Le].

A class of neighborly polytopes [Ba'9].
The references are papers containing the original proofs or further analysis of the class.

Matroid complexes. A *matroid complex* is a simplicial complex $\mathscr{M}$ such that for each subset A of $\mathscr{M}$'s vertex-set, all faces that are maximal with respect to having all vertices in A are of the same dimension. Since clearly the complexes $\mathscr{M} \setminus v$ and $\operatorname{link}(v, \mathscr{M})$ are also matroid complexes for each vertex v of $\mathscr{M}$, it is evident that

8.1. *Each matroid complex is vertex-decomposable and hence satisfies the dual Hirsch conjecture.*

A pure simplicial s-complex $\mathscr{C}$ with n vertices is *balanced* [St2] if it is possible to assign to each vertex a unique label from the set $\{0, \dots, n-s\}$ in such a way that for each $i \in \{0, \dots, n-s\}$ all facets have the same number $c_i \geqslant 1$ of vertices labeled i. (This implies $n \leqslant 2s$.) It is *fully balanced* if, in addition, each set of $s+1$ vertices with the designated set of labels (c_i labels i for $0 \leqslant i \leqslant n-s$) is the vertex-set of a facet. For an arbitrary set A of vertices of a fully balanced complex, let $\mu_i(A)$ denote the minimum of c_i and the number of vertices in A that have label i. Then every face whose vertex-set lies in A is contained in such a face of dimension $\Sigma_0^{n-s}\mu_i(A) - 1$, and hence:

8.2. *Each fully balanced complex is a matroid complex and hence vertex-decomposable.*

It would be of interest to find other classes of (not necessarily fully) balanced complexes that are vertex-decomposable. An example is the barycentric subdivision [Gl] $\mathscr{C}'$ of a shellable complex $\mathscr{C}$. Shellability of $\mathscr{C}$ implies vertex-decomposability of $\mathscr{C}'$, and a balance may be obtained by labeling each vertex of $\mathscr{C}'$ with the dimension of the face of C of which it is a barycenter.

Leontief substitution polyhedra. A (pre-)*Leontief substitution polyhedron* is the set of P of all solutions of a system $Ax = b$, $x \geqslant 0$ where b is a nonnegative column vector and A is an $m \times n$ matrix of rank m that has at most one positive entry in each column. Grinold [Gr] shows that each such polyhedron is a Hirsch polyhedron. The boundary structure of Leontief polyhedra is studied in more detail by [PB2], who establish the following result for the bounded case:

8.3. *If P is a nondegenerate (simple) Leontief substitution polytope then P is a Hirsch polytope because the boundary complex $\mathscr{B}(P^*)$ of P's dual P^* is fully balanced, hence a matroid complex and hence vertex-decomposable.*

Polyhedra arising from the shortest-path problem. As described by Saigal [Sa], these polyhedra are defined by the following system of constraints:

$$x = (x_{11}, \dots, x_{ij}, \dots, x_{mm}) \geqslant 0, \qquad i, j = 1, \dots, m,$$

$$\sum_{j=1}^{m} x_{1j} - \sum_{j=1}^{m} x_{j1} = 1, \quad \sum_{j=1}^{m} x_{ij} - \sum_{j=1}^{m} x_{ji} = 0, \qquad i = 2, \dots, m-1.$$

Plainly these are Leontief substitution polyhedra and hence Hirsch polyhedra. ([Sa]'s proof is graph-theoretic in nature.)

[Sa] conjectures that whenever a d-polytope in $\mathbb{R}^d$ is the intersection $P \cap C$ of a Hirsch polyhedron P with a cube C, then $P \cap C$ is a Hirsch polytope. This, it is observed, would imply that the polyhedra arising from network-type LP problems are all Hirsch polyhedra. However, 8.6 below shows that [Sa]'s conjecture for bounded P is actually equivalent to the Hirsch conjecture.

For a facet F of a d-polytope P in $\mathbb{R}^d$, let H_F denote the hyperplane that contains F and let $H_F(P)$ denote the closed halfspace that contains P and is bounded by H_F. The facet F is said to be *removable from* P if the polyhedron $\bigcap_{\text{facet } G \neq F} H(G, P)$ is bounded, and to be *projectively removable* from P if there is a projective transformation that carries P onto a polytope P' such that the image F' of F is removable from P'.

8.4. *A facet F is projectively removable from a polytope P if and only if there is an edge of P disjoint from F.*

PROOF. The "only if" part is obvious. For the "if" part, let $[x, y]$ be an edge of P disjoint from F and let the facets $G_0, \ldots, G_d$ of P be such that G_0 includes x but not y, G_d includes y but not x, and $\bigcap_1^{d-1} G_i = [x, y]$. The polyhedron $\bigcap_0^d H_{G_i}(P)$ may of course be unbounded, but there is a projective transformation that carries it onto a d-simplex. Since the simplex is bounded and F is not among the G_i's, F is projectively removable from P.

8.5. *A polytope is a simplex if and only if none of its facets is projectively removable.*

PROOF. For the "if" part, use 8.4 and a characterization of simplices provided by 2.3 of [Kl2].

8.6. *For each fixed d-polyhedron Q in $\mathbb{R}^d$, with $Q \neq \mathbb{R}^d$, the following two assertions are equivalent*:

whenever a d-polytope in $\mathbb{R}^d$ is the intersection $P \cap Q$ of a Hirsch polytope P with Q, then $P \cap Q$ is a Hirsch polytope;

every d-polytope is a Hirsch polytope.

PROOF. Assuming the first assertion to be correct, the second can be proved by induction on the number of facets. Consider a d-polytope X that is not a simplex. By 8.5 there is a facet F of X and there is a projective transformation that carries X onto a polytope X' such that the image F' of F is removable from X', leaving a polytope P'. Let G be any facet of Q, and let X'' be the image of X' under an affine transformation (for example, a suitable contraction followed by a rigid motion) such that $X'' \subset Q$ and $F'' \subset G$. Let P be the image of P' under this transformation. Then P has one less facet than X'', and $X'' = P \cap Q$. Thus P is a Hirsch polytope by the inductive hypothesis, whence X' is a Hirsch polytope by the first assertion of 8.6 and X is a Hirsch polytope because it is combinatorially equivalent to X'. ■

Dual transportation polyhedra. The dual formulation of transportation problems yields the following system of inequalities on $(u_1, \ldots, u_m, v_1, \ldots, v_n) \in \mathbb{R}^{m+n}$, where $c_{ij} \geqslant 0$ for all i and j;

$$u_i + v_j \leqslant c_{ij} \quad \text{for } 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n.$$

A polyhedron defined in this way contains a line, but Balinski [Ba3] "factors out" the line by adding the constraint $u_1 = 0$ and thus obtains an unbounded pointed polyhedron that is of the dimension $m + n - 1$ and has mn facets. He shows that the Hirsch upper bound of $mn - (m + n - 1) = (m - 1)(n - 1)$ applies to the diameters of these polyhedra, and is attained when $c_{ij} = (m - i)(j - 1)$.

The partition of feasible vectors into "u-components" and "v-components" naturally gives rise to bipartite graphs that have m labeled nodes in one part and n in the other. The bound on the diameter is obtained by studying the polyhedra in terms of these graphs and a suitable "pivoting operations". A more detailed study of this operation might yield a proof that the dual transportation polyhedra are vertex-decomposable, for the operation corresponds to removal of facets; this in turn might yield another algorithm for the dual transportation problem. Additional combinatorial properties of the dual transportation polyhedra, especially the numbers of faces of various dimensions, are studied in [BR3].

Special transportation polytopes. For positive real numbers $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ with $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, there is defined the transportation polytope

$$P_{m,n}((a_1, \ldots, a_m), (b_1, \ldots, b_n))$$
$$:= \left\{ x = (x_{ij}) \colon x_{ij} \geqslant 0 \textit{ for all } i \textit{ and } j, \textstyle\sum_{j=1}^n x_{ij} = a_i \textit{ for all } i, \sum_{i=1}^m x_{ij} = b_j \textit{ for all } j \right\}.$$

Since $P_{m,n}$ is of dimension $(m-1)(n-1)$ and has mn facets, the upper bound on $\delta(P_{m,n})$ provided by the Hirsch conjecture is $m+n-1$. Using certain trees to describe vertices as in the case of the dual program, Balinski [Ba2] proves:

8.7. *Suppose that the $k_i k_1$; k; k_m are nonnegative integers. Then for $n = 1 + \sum_{i=1}^m k_i$,*

$$\delta\left(P_{m,n}(\{k_i m + 1\}; \{m\})\right) = \begin{cases} m+n-1 & \textit{for } m, n > 2 \\ n & \textit{for } m = 2, \\ m & \textit{for } n = 2, \end{cases}$$

and for $n = m - 1 + \sum_{i=1}^{m} k_i$,

$$\delta\left(P_{m,n}(\{(k_i + i)m - 1\}, \{m\})\right) = \begin{cases} m+n-1 & \textit{for } m > 2, \\ n & \textit{for } m = 2. \end{cases}$$

The special polytopes $P_n := P_{n,n}(\{1\}; \{1\})$ are called *assignment polytopes* [BR2], and also *polytopes of doubly stochastic matrices* [*BG*]. The following result (i) is established in both papers, (ii) in [BR2] and (iii) in [BG].

8.8. (i) $\delta(P_n) = 2$ *for $n > 3$, and $\delta(P_3) = 1$;*

(ii) *the LP form of the Hirsch conjecture holds for P_n; i.e., any pair of feasible bases of P_n is connected by a path of successively adjacent feasible bases of length $\leqslant 2n - 1$;*

(iii) *whenever F is a d-face of P_n with m facets, $\delta(F) \leqslant m - d$ with equality if and only if $m \leqslant 2d$.*

Padberg and Rao [PR] generalize 8.8(ii) to a class of polytopes that includes the polytopes P_n and the polytopes arising from the weighted matching problem on complete graphs. They also show that the polytopes arising from the asymmetric travelling salesman problem are of diameter $\leqslant 2$, and prove for these polytopes a weaker version of 8.8(ii): for each feasible basis B and each vertex v, B is connected by a path of successively adjacent feasible bases to some basis associated with v. Similar results are obtained by [BP] for the convex hull of integer solutions to the set partitioning problem.

Additional bounds on the diameters of special transportation polytopes appear in [EKK].

Since transportation polytopes are generally not simple, it would be interesting to know whether the Hirsch conjecture still holds for the nearby simple polytopes obtained from transportation polytopes in the described classes by slightly perturbing the hyperplanes determining their facets. The faces of P_n seem to be quite "stable" with respect to the Hirsch conjecture because [BG]'s proof of 8.8(ii) relies heavily on the existence of certain simple faces (boxes) in the boundary complex, and the combinatorial structure of these is unchanged by small perturbations in the constraints.

Billera-Lee polytopes. As was mentioned in the Introduction [St3] shows that the f-vectors of simplicial polytopes always satisfy the conditions proposed by [Mc2], and

Billera and Lee [BL] show that each sequence satisfying these conditions is in fact the f-vector of a simplicial polytope. The boundary complexes of [BL]'s polytopes are obtained as certain subcomplexes of the boundary complexes of cyclic polytopes. Using this embedding, Lee [Le] shows that each of the [BL] polytopes has a vertex whose antistar is vertex-decomposable. The vertex-decomposability of the link of this vertex (and hence of the [BL] polytopes) is unsettled, but the result on the antistar yields a bound on the ridge-diameter that is only one greater than the conjectured Hirsch bound. As restated in terms of simple polytopes, the result is as follows:

8.9. *For each sequence $f = (f_0, \ldots, f_{d-1})$ that is the f-vector of a simple d-polytope, there exists a simple d-polytope P such that P's f-vector is f and $\delta(P) \leqslant f_{d-1} - d + 1$.*

Hence the existence of short paths cannot be excluded by properties of f-vectors alone.

A class of neighborly polytopes. [Kl4] shows the duals of cyclic polytopes are Hirsch polytopes; that is, the ridge-diameter $\varphi(d, n)$ of a cyclic d-polytope with n vertices is $\leqslant n - d$. (It is proved that $\varphi(d, n) = n - d$ for $n \leqslant 2d$, conjectured that $\varphi(d, n) = \lfloor n/2 \rfloor$ for $n \geqslant 2d$.) This result is strengthened by [Pr] who shows the cyclic polytopes are actually vertex-decomposable. At least for $d = 4$, the cyclic d-polytopes are contained in a larger class of neighborly polytopes discussed by [Ba′9] (see also [Sh]). [Ba′9] claims the duals of his polytopes are Hirsch polytopes, but we are unable to fill a gap in the proof of his Theorem 5. We shall, however, show that his polytopes are weakly vertex-decomposable (his argument implicitly claims vertex-decomposability) and hence obtain the weaker bound $2(n - d)$ for their ridge-diameters.

The polytopes considered in [Ba′9] are simple d-polytopes obtained from the d-simplex by successive facet-splitting. The polar P of such a polytope is of course simplicial, and from $\mathscr{B}(P)$ one can return to $\mathscr{B}(T^d)$ by successive edge-shrinking as described in §6. In the special construction of [Ba′9], a simple path $x_1, \ldots, x_k$ is formed by the edges that are shrunk, and each complex $\mathscr{B}(P) \setminus x_1 \cdots \setminus x_i$ $(1 \leqslant i \leqslant k)$ is a $(d-1)$-ball by 6.1 and hence certainly a pure $(d-1)$-complex. Since $\mathscr{B}(P) \setminus x_1 \setminus \cdots \setminus x_k$ is a $(d-1)$-simplex it follows that $\mathscr{B}(P)$ is weakly vertex-decomposable. In general we don't know enough about the boundaries of the balls $\mathscr{B}(P) \setminus x_1 \setminus \cdots \setminus x_i$ to claim that Barnette's polytopes are vertex-decomposable (and hence Hirsch polytopes), but for $d = 4$ this does follow from 5.6.

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