CONVEX HULLS OF f- AND β-VECTORS

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ABSTRACT. In this paper we describe the convex hulls of the sets of f- and β-vectors of different classes of simplicial complexes on n vertices. These include flag complexes, order complexes of posets, matroid complexes and general abstract simplicial complexes. As a result of this investigation, standard linear programming problems on these sets can be solved, including maximization of the Euler characteristics or of the sum of the Betti numbers.

1. INTRODUCTION

In this paper we investigate extremal questions concerning f- and β-vectors for different classes of abstract simplicial complexes. The classes treated here are:

(1) flag (also known as clique) complexes on n vertices;
(2) general simplicial complexes on n vertices.

Consider for example a flag complex on n vertices (see Definition 3.1). Its f-vector (or β-vector) is a point on the integer grid $\mathbb{Z}^n$ in $\mathbb{R}^n$. One could ask questions like: what is the maximum (or minimum) of the Euler characteristic on such complexes, what is the maximum of the sum of the Betti numbers or just of a single Betti number, say β_{11}? Extremal problems of this type have been treated thoroughly in the literature. For example [BK, Theorem 1.4] answers the question: *what is the maximum of the Euler characteristic and the sum of the Betti numbers of an abstract simplicial complex on n vertices?* Some other papers where similiar questions have been treated are [Koz], [Mar], [Re], [SYZ], [Z]. Relations between f- and β-vectors were studied, see for example [BK], [May].

Another direction of research has been to investigate convex hulls associated to Sperner families with different conditions imposed. Some of this work can be found in [DGH], [EE], [EFK1], [EFK2], [EFK3], [En], [KS]. This research is relevant here, because β-vectors of simplicial complexes with at most $n+1$ vertices form exactly the same set as f-vectors of Sperner families on the set $\{1, 2, \ldots, n\}$.

One unifying approach to this kind of problems would be to consider them all as standard problems of linear programming. Namely, given some set of points in $\mathbb{R}^n$, optimize (i.e. maximize or minimize) some linear function over this set.

Obviously finding the convex hull of this set of points would settle all questions of that type in one step. Because once we know the extreme points of this convex hull, then in order to prove some linear inequality it suffices to check it for these points only. In this way we, for example, can get a different proof for the result of Björner and Kalai cited above.

Note that the f-vectors of simplicial complexes on n vertices are completely characterized by the Kruskal-Katona theorem, see [Kr], [Ka]. Also the characterization of the set of β-vectors can be obtained through [BK, Theorem 1.3] and the characterization of the f-vectors of Sperner families, independently discovered by [C] and [DGH]. Unfortunately, these results give little insight into the general optimization problems which we study here.

The surprising part of this investigation is that these convex hulls turn out to have a very easy description. For the case of flag complexes the extreme points are given by Turán graphs and for the case of general simplicial complexes they are given by complete skeleta of different dimensions. The analagous simplicity seems to be far from given for other classes of simplicial complexes, for example for those associated with graphic matroids.

In sections 3 and 4 we will find the convex hulls of the sets of f- and β-vectors for the classes of complexes described above. The corresponding questions for matroid complexes and complexes associated with finite posets are answered at once by our results.

At the end we propose a conjecture concerning convex hulls of f- and β-vectors for r-colorable complexes.

2. Basic notations and definitions

Let G be a simple graph on n vertices. We denote by $E(G)$ the set of its edges and by $V(G)$ the set of its vertices. We say that $S \subseteq V(G)$ forms a **clique** if the corresponding induced subgraph is complete, if $|S| = k$ then we say that S forms a k-**clique**. Let $c_k(G)$ be the number of k-cliques of G, then the vector $(c_1, c_2, \dots, c_n)$ is called the **clique vector** of G.

If G_1 and G_2 are two graphs with disjoint vertex sets, then $G = G_1 \oplus G_2$ will denote the graph defined by

$$V(G) = V(G_1) \cup V(G_2)$$

and

$$E(G) = E(G_1) \cup E(G_2) \cup \{(x,y) | x \in V(G_1),\, y \in V(G_2)\}$$

We say that G is an r-**partite complete graph** (of type $(k_1, \dots, k_r)$) if

$$G = A_1 \oplus A_2 \oplus \cdots \oplus A_r$$

where $E(A_i) = \emptyset$ and $|V(A_i)| = k_i$ for all $i = 1, \dots, r$.

We say that a poset P is of **level type** $(p_1, \dots, p_r)$ if $P \simeq \hat{0} \oplus p_1 1 \oplus \cdots \oplus p_r 1 \oplus \hat{1}$, where $\oplus$ denotes ordered sum, and $k1$ denotes an antichain consisting of k elements. Sometimes we just say that a poset has level type. For any inquiries concerning posets we refer to [S, Chapter 3].

Finally a few words about our terminology concerning algebraic topology. Let C be an abstract simplicial complex. Let f_k denote the number of faces of dimension k and β_k - the kth Betti number of C (in this paper we will consider reduced homology only). Then the vector $(f_0, \dots, f_{n-1})$ is called **f-vector** of C and the vector $(\beta_0, \dots, \beta_{n-1})$ - **β-vector** of C.

We say that a complex C is a **complete k-skeleton** if C has all possible faces up to cardinality k and no faces of cardinality $k+1$ and more.

If x is a vertex of C, then $\mathrm{st}(x)$ and $\mathrm{lk}(x)$ are simplicial subcomplexes of C defined by

$$\mathrm{st}(x) = \{X \in C | X \cup \{x\} \in C\},\ \mathrm{lk}(x) = \{X \in C | x \notin X,\ X \cup \{x\} \in C\}$$

From now on, n will always denote the fixed number of vertices in our graph or complex.

3. The case of flag complexes

Definition 3.1. *Let G be a simple graph on n vertices. Define an abstract simplicial complex C associated to this graph in the following way: we take the set of vertices of the graph as the set on which we define our simplicial complex and we say that a collection of vertices forms a face if and only if the corresponding collection of vertices of G forms a complete subgraph (clique). The abstract simplicial complexes obtained in this way are called* ***flag (clique) complexes****. We will denote such complexes by $C(G)$.*

An example of flag complexes is provided by the complexes associated with posets (given a finite poset P, we take its elements as vertices of a complex and the sets forming chains as faces).

Definition 3.2. *We call a graph G an* ***rth Turán graph on n vertices*** *if G is a complete r-partite graph with sizes of the maximal independent sets as equal as possible. We will denote this graph by $T_r(n)$ or just T_r.*

The Turán graphs come up in different contexts all over extremal graph theory and are optimal in many senses (one can find a nice survey in [B]). In our case they turn out to determine the extreme points in the convex hulls of the sets of f-vectors and β-vectors of flag complexes.

Let us denote the f-vector of $T_r(n)$ by $F_r(n)$ (or just F_r) and the β-vector by $B_r(n)$ (or just B_r). Then

$$F_r(n)_{i-1} = \sum_{1 \le j_1 < \cdots < j_i \le r} k_{j_1} \dots k_{j_i},\ \text{where } k_s = \left[\frac{n+s-1}{r}\right]$$

and

$$B_r(n)_{r-1} = \prod_{i=1}^{r} (k_i - 1), \quad B_r(n)_i = 0,\ i \neq r-1.$$

We will need an operation on graphs, which we call compression. Its poset version has previously been used in [Koz], [Z]. In the context of graphs it appears in for example [MM]. It is different from a similiar operation on simplicial complexes also called compression in [GK], [F], [CL]. Let x and y be vertices in a graph G not joined by an edge and let $\{x_1, \dots, x_m\}$ be the set of neighbours of x, and let $\{y_1, \dots, y_k\}$ be the set of neighbours of y, then we define an **(x,y)-compression** (or an **y to x compression**) of G as a graph G^* given by:

$$E(G^*) = (E(G) \setminus \{(y, y_1), \dots, (y, y_k)\}) \cup \{(y, x_1), \dots, (y, x_m)\}.$$

Let G_x and G_y be the subgraphs of G induced by $\{x, x_1, \dots, x_m\}$, resp. $\{y, y_1, \dots, y_k\}$. We write $C = C(G)$, $C^* = C(G^*)$, $C_x = C(G_x)$ and $C_y = C(G_y)$. Then (x, y)-compression changes the f-vector linearly, namely

$$f_k(C^*) = f_k(C) + f_{k-1}(C_x) - f_{k-1}(C_y). \tag{3.1}$$

So, if l is a linear function on f-vectors and l' is obtained from l by shifting the arguments by one, then

$$l(f(C^*)) = l(f(C)) + l'(f(C_x)) - l'(f(C_y)). \tag{3.2}$$

If G^{**} is the (y, x)-compression of G and $C^{**} = C(G^{**})$ then either $l(f(C^*)) \geq l(f(C))$ or $l(f(C^{**})) \geq l(f(C))$; thus if $l(f(C))$ is maximal then $l(f(C)) = l(f(C^*)) = l(f(C^{**}))$ and we can compress however we want to, preserving the value of the function l. We use this observation in the proof of the next theorem.

Theorem 3.3. *The convex hull of the set of f-vectors of flag complexes on n vertices is given by conv$\{F_1, F_2, \dots, F_n\}$.*

Proof. Choose a linear function l and a complex C where l achieves its maximal value. Pick a vertex $x \in C$ and let $\{y_1, \dots, y_m\}$ be the set of vertices not adjacent to x. Then the argument above allows us to compress these vertices to x and thus obtain a graph G^*, such that

$$G^* = G_1 \oplus \{x, y_1, \dots, y_m\}$$

where $G_1 = G \setminus \{x, y_1, \dots, y_m\}$. Now we can continue compressing G_1 and so on. When the process terminates, we end up with a complete k-partite graph where l achieves its maximum.

We prove now that we actually can obtain a Turán graph. Assume that our k-partite graph is not a Turán graph, then we can find two maximal independent sets of sizes a and b such that $a - b \geq 2$. Let $p_1, \dots, p_a$ and $q_1, \dots, q_b$ be the vertices of these independent sets. Form a new graph T by reconnecting the vertex p_a: erasing its old connections and connecting it to $p_1, \dots, p_b$ and all other vertices in the graph which are outside the two independent sets. Then the clique vector of T is the same as that of G, since the links of p_a are the same in both graphs and $T \setminus \{p_a\}$ is equal to $G \setminus \{p_a\}$. Now we can shift p_a to q_1 and obtain a new k-partite graph with independent sets of more equal sizes and the value of l the same as that of G. Continuing in that way we eventually end up with a Turán graph. □

Theorem 3.4. *The convex hull of the set of β-vectors of flag complexes on n vertices is given by conv$\{B_1, B_2, \dots, B_n\}$.*

Proof. Let again l be a linear function (though this time in the space of β-vectors) and let C be a flag complex where l achieves its maximum. We denote the underlying graph by G. To apply the same kind of argument as before we need to know how our compression operation influences l. Observe that the simplex conv$\{B_1, B_2, \dots, B_n\}$ is obtained by cutting the positive cone (i.e. the cone defined by $x_i \geq 0$, $i = 1, \dots, n$) in $\mathbb{R}^n$ by a hyperplane with a positive normal vector. To prove the statement of the theorem it is enough to show that the B_i's maximize all linear functions with non-negative coefficients, we give a short argument for that. Let the linear function be $l = l_+ - l_-$, where both l_+ and l_- have only positive coefficients (and non-intersecting support). Let B be one of the points $B_1, \dots, B_n$ that optimizes l_+ and let X be some point other than the B that optimizes l. Then $l_+(B) \geq l_+(X)$ and $l_-(B) = 0$ (since B is on an axis in the support of l_+) and $l_-(X) \geq 0$ (since X is in the positive octant and l_- is positive); thus $l(B) = l_+(B) - l_-(B) \geq l_+(X) - l_-(X) = l(X)$ and hence B optimizes l as well. So we can assume that

$$l(\beta_0, \dots, \beta_{n-1}) = \sum_{k=0}^{n-1} \alpha_k \beta_k, \text{ where } \alpha_k \geq 0.$$

Let x and y be vertices of G. We introduce the following notion: $G_1 = G \setminus \{y\}$, $C_1 = C \setminus \{y\}$ (i.e. we take away the vertex y and all the sets of C containing y) and let G^* (resp. C^*) be the result of the y to x compression of G (resp. C).

First let us note that C is the union of the two simplicial complexes C_1 and $\mathrm{st}(y)$. The intersection of these two simplicial complexes is obviously $\mathrm{lk}(y)$, hence we get a Mayer-Vietoris sequence

$$\dots \to H_k(C_1) \to H_k(C) \to H_{k-1}(\mathrm{lk}(y)) \to \dots \to H_0(C_1) \to H_0(C) \to 0.$$

Note that we used the fact that $\mathrm{st}(y)$ is a cone and hence has trivial homology groups. The sequence above is an exact sequence, hence looking at its short subsequences of the type

$$\dots \to H_k(C_1) \to H_k(C) \to H_{k-1}(\mathrm{lk}(y)) \to \dots$$

we can conclude that

$$\beta_k(C) \leq \beta_k(C_1) + \beta_{k-1}(\mathrm{lk}(y)).$$

Summing up with coefficients α_i (which are non-negative!) we obtain

$$l(\beta(C)) \leq l(\beta(C_1)) + l'(\beta(\mathrm{lk}(y))) \tag{3.3}$$

where l' is obtained from l by shifting the arguments by one.

On the other hand we can apply the same argument to the complex C^*, which is the union of C_1 and a cone (namely $\mathrm{st}(x)$, where x is exchanged to y) with the intersection $\mathrm{lk}(x)$. The only difference will be that all the mappings

$$i_k : H_k(\mathrm{lk}(x)) \longrightarrow H_k(C_1)$$

are zero mappings, because $\mathrm{lk}(x)$ is mapped into a cone in C_1. Hence instead of inequalities as above we get exact equalities for the Betti numbers of C^*, namely

$$l(\beta(C^*)) = l(\beta(C_1)) + l'(\beta(\mathrm{lk}(x))). \tag{3.4}$$

If $l'(\beta(\mathrm{lk}(x))) > l'(\beta(\mathrm{lk}(y)))$ then according to equations 3.3 and 3.4 we get

$$l(\beta(C^*)) > l(\beta(C))$$

which is a contradiction to our assumption of optimality of C. If $l'(\beta(\mathrm{lk}(x))) < l'(\beta(\mathrm{lk}(y)))$, then an x to y compression increases the value of l, which again yields a contradiction. Hence $l'(\beta(\mathrm{lk}(x))) = l'(\beta(\mathrm{lk}(y)))$ and using equations 3.3 and 3.4 again we see that we are free to shift whichever way we want. So we are back in the situation of the previous proof.

Shifting in the same way as above we end up with a complete k-partite graph, say of type $(t_1, \dots, t_k)$. Then all the Betti numbers are equal to 0, except for the

$(k-1)$th one which is equal to $\prod_{i=1}^{k}(t_i-1)$. Hence the corresponding β-vector gives a point on the kth coordinate axis which lies in our simplex since the maximum of such a product is achieved when the t_i's are as equal as possible, i.e. for Turán-graphs. □

Corollary 3.5. *The convex hull of the set of f- and β-vectors of complexes on n vertices associated with posets are given by conv$\{F_1, F_2, \dots, F_n\}$ and conv$\{B_1, B_2, \dots, B_n\}$ respectively.*

Proof. The clique complex of a Turán graph is the same as the order complex of a level type poset. On the other hand if we have a poset P we can associate with it a graph G, by taking the elements of P as vertices of G and connecting two vertices by edge whenever the corresponding elements are comparable. Then the order complex of P obviously translates into the clique complex of G. We can conclude that the set of order complexes is a subset of the set of all flag complexes and it contains the complexes associated to Turán graphs, hence the convex hulls of f- and β-vectors must be the same. □

Note. Corollary 3.5 generalizes [Koz, Theorem 4.4] and [Z, Theorem 2.5].

4. The case of simplicial complexes

In this section we consider the same problem as above for the class of simplicial complexes on n vertices.

First we introduce some notation. Let S_k be the complete k-skeleton complex on n vertices (S_1 will denote a complex with no faces except for vertices). We denote the f- and β-vectors of S_k by $\tilde{F}_k(n)$ and $\tilde{B}_k(n)$ respectively. Then

$$\tilde{F}_k(n) = (\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{k}, 0, \dots, 0)$$

and

$$\tilde{B}_k(n)_{k-1} = \binom{n-1}{k}, \ \tilde{B}_k(n)_i = 0, \ i \neq k-1.$$

Let us define an operation on complexes which we call a **generalized compression**. Let C be a simplicial complex on n vertices and x and y two of its vertices. Let furthermore

$$\{X_1, \dots, X_k\} = \{X \in C | x \in X, \, y \notin X\},$$

$$\{Y_1, \dots, Y_m\} = \{Y \in C | x \notin Y, \, y \in Y\}.$$

Then we define an **(x, y)-compression** of C (or a **y to x compression**) as the complex C^* given by:

$$C^* = (C \setminus \{Y_1, \dots, Y_m\}) \cup \{(X_i \setminus \{x\}) \cup \{y\} | i = 1, \dots, k\}.$$

Let us see that C^* is again a simplicial complex. It is enough to verify that if $S \subseteq (X_1 \setminus \{x\}) \cup \{y\}$ then $S \in C^*$. We consider two cases. First, if $y \notin S$ then $S \subseteq X_1$, hence $S \in C$, but $S \notin \{Y_1, \dots, Y_m\}$, which gives $S \in C^*$. If on the contrary $y \in S$, let us write $S' = (S \setminus \{y\}) \cup \{x\}$. Then $S' \subseteq X_1$, hence $S' \in C$, so $S' \in \{X_1, \dots, X_k\}$. Say $S' = X_i$, then $S = (X_i \setminus \{x\}) \cup \{y\}$ and so $S \in C^*$.

Observe that an (x, y)-compression is different from another similiar and frequently used operation on simplicial complexes called an (i, j)-shift (see Chapter 4 in [F] for a description). An (i, j)-shift preserves the f-vector, while our compression operation changes the f-vector linearly:

$$f(C^*) = f(C) + f(X_1, \ldots, X_k) - f(Y_1, \ldots, Y_m). \tag{4.1}$$

So if l is a linear function on f-vectors then

$$l(f(C^*)) = l(f(C)) + l(f(X_1, \ldots, X_k)) - l(f(Y_1, \ldots, Y_m)). \tag{4.2}$$

Theorem 4.1. *The convex hull of the set of f-vectors of simplicial complexes on n vertices is given by conv$\{\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_n\}$.*

Proof 1. Take l a linear function on f-vectors and let C be a simplicial complex where l achieves its maximum. In the same way as in the proof of Theorem 3.3, but using formula (4.2) instead of (3.2), we observe that we can perform (x, y)-compressions without changing the value of l. Though now we have a simpler situation, as we do not need to bother if x and y are adjacent or not. So eventually we end up with a complex C^* which cannot be compressed anymore (just pick from the beginning some special x and compress everything to it). This means that for any two vertices of C, x and y, we have

$$\{X \setminus \{x\} \,|\, x \in X,\, y \notin X\} = \{Y \setminus \{y\} \,|\, x \notin Y,\, y \in Y\}.$$

It is now a routine argument to see that this property implies that C^* is a complete k-skeleton, as we have that

$$x \in X, y \notin X, X \in C^* \Rightarrow (X \setminus \{x\}) \cup \{y\} \in C^*.$$

So we see that for any linear function, one of the points where it attains its maximum is the f-vector of a k-skeleton. This proves the result. □

It is also possible to derive this result using a different method involving some linear algebra. We give this proof in the next section.

Theorem 4.2. *The convex hull of the set of β-vectors of simplicial complexes on n vertices is given by conv$\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n\}$.*

Proof. The proof of this theorem is very similiar to that of the Theorem 3.4. For that reason we will only show how to adapt the above argument to this particular case. The only difference is that we use generalized compression instead of the usual one. Let again l be a linear function with nonnegative coefficients, C an optimal complex, x, y vertices of C, $C_1 = C \setminus \{y\}$ and C^* the result of the y to x compression. Then just as above we can write both C and C^* as unions of two simplicial complexes and (using the Mayer-Vietoris exact sequences) estimate the Betti numbers. Inequality (3.3) remains valid here. The only special thing about compression was that we actually obtained equality in (3.4). The argument was that in a compressed complex we have $\text{lk}(x) = \text{lk}(y)$ and hence the mapping induced by inclusion:

$$H_k(\text{lk}(y)) \to H_k(C^* \setminus \{y\})$$

was trivial. But this is true even for the generalized compression. Though links of x and y are not equal any more, we still have that $\text{lk}(y)$ is contained in $\text{st}(x)$ and hence is mapped into a cone. So the mapping above is trivial even in this case and all the arguments used in the proof of Theorem 3.4 go through. □

Corollary 4.3. *The convex hull of the set of f- and β-vectors of matroid complexes on n vertices are given by conv$\{\tilde{F}_1,\dots,\tilde{F}_n\}$ and conv$\{\tilde{B}_1,\dots,\tilde{B}_n\}$ respectively.*

Proof. Matroid complexes are just a special case of simplicial complexes, on the other hand, the complete k-skeletons correspond to uniform matroids (see e.g. [W] for the definition), hence the result follows. □

Note. It would be interesting to determine the corresponding convex hulls for more restricted classes of matroids, for example for the class of graphic matroids.

5. A direct proof of Theorem 4.1

Here we describe another method for finding the convex hull of a set of points. It provides another proof of Theorem 4.1.

Proof 2. We know that conv$\{\tilde{F}_1,\dots,\tilde{F}_n\}$ is a simplex and that $\tilde{F}_k = (\binom{n}{1},\dots,\binom{n}{k},0,\dots,0)$. Let us compress our space along each coordinate by dividing the ith coordinates by $\binom{n}{i}$. Let v_k denote the vector with first k coordinates equal to one and the rest equal to zero for $k=0,\dots,n$, $v_0=0$. Then we are left with proving that whenever we have an f-vector $(f_0,\dots,f_{n-1})$ of a simplicial complex on n vertices, the vector $g=(g_0,\dots,g_{n-1})$, given by $g_i = f_i/\binom{n}{i+1}$, lies in the simplex spanned by $v_0,\dots,v_n$. Then the vector g is inside this simplex if and only if

$$g=\sum_{k=1}^{n}\alpha_k v_k, \text{ where } \sum_{k=1}^{n}\alpha_k \le 1 \text{ and } \alpha_k \ge 0,\ k=1,\dots,n.$$

Let M be the matrix with column vectors v_k. M is an upper triangular matrix and if α is the vector $(\alpha_1,\dots,\alpha_n)$, then we have a matrix identity:

$$g = M\cdot\alpha \text{ , or } \alpha = M^{-1}\cdot g.$$

But since

$$M^{-1}=\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

we get

$$\alpha = M^{-1}\cdot g = (g_0-g_1, g_1-g_2,\dots,g_{n-1})$$

and the conditions on the vector α translate into the following inequalities:

$$1\ge g_0\ge g_1\ge\dots\ge g_{n-1}.$$

Since it is obvious that $1\ge g_0$ we only need to prove that $g_{k-1}\ge g_k$ for $k=1,\dots,n-1$. This means that we have to prove that

$$f_{k-1}/\binom{n}{k}\ge f_k/\binom{n}{k+1}$$

or that

$$f_{k-1}\binom{n}{k+1}\ge f_k\binom{n}{k}$$

which after cancellation of common factors from factorials transforms into

$$(n-k)f_{k-1}\ge(k+1)f_k \tag{5.1}$$

This inequality is well known and can be found in, for example, Section 8 of [GK]. We give here a short simple argument.

Let C to be a simplicial complex with f-vector $(f_0, \ldots, f_{n-1})$ and let us count pairs (A, B), where $A, B \in C$, $|A| = k+1$, $|B| = k$ and $B \subset A$. On one hand this number is equal to $(k+1)f_k$ since every face with $k+1$ elements contains exactly $k+1$ faces of cardinality k. On the other hand every face with k elements is contained in at most $n-k$ faces of cardinality $k+1$, hence the number of pairs that we count is at most $(n-k)f_{k-1}$. This yields the inequality 5.1. □

6. Open problems

We finish this paper with a conjecture concerning the class of r-colorable simplicial complexes.

Definition 6.1. *A simplicial complex is called* ***r-colorable*** *if its 2-skeleton is r-colorable in the graph-theoretical sense.*

Conjecture 6.2. *The convex hull of the set of f-vectors (resp. β-vectors) of r-colorable complexes on n vertices is equal to* $conv\{F_1, \ldots, F_n\}$ *(resp.* $conv\{B_1, \ldots, B_n\}$*), where F_i (B_i) denotes the f-vector (β-vector) of the complete i-skeleton of the flag complex associated with $T_r(n)$.*

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