MANIFOLDS IN THE SKELETONS OF CONVEX POLYTOPES, TIGHTNESS, AND GENERALIZED HEAWOOD INEQUALITIES

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Abstract. If a 2-dimensional manifold in the 2-dimensional skeleton of a convex d-polytope P contains the 1-skeleton of P then d is bounded in terms of the genus of the surface: this is essentially Heawood's inequality. In this paper we prove a higher dimensional analogue about $2k$-dimensional manifolds containing the k-skeleton of a simplicial convex polytope. Related conjectures are formulated for tight polyhedral submanifolds and generalized Heawood inequalities, including an Upper Bound Conjecture for combinatorial manifolds.

A *polyhedral n-manifold* M^n is a finite cell complex whose i-dimensional cells are convex i-polytopes, such that the intersection of any two cells is either empty or a lower dimensional cell, and such that the vertex figure of each vertex is a polyhedral $(n-1)$-sphere. The latter condition guarantees that a typical neighbourhood of a vertex is a topological ball. A *polyhedral submanifold* of a convex d-polytope P is by definition a subcomplex of the boundary complex of P which is a polyhedral manifold.

Examples of 2-dimensional submanifolds of polytopes are the following:

1. Triangulated surfaces in the 2-skeleton of a higher dimensional simplex, in particular such with a complete edge graph [Ri2],
2. Coxeter's regular skew polyhedra $\{4, d \mid 4^{[d/2]-1}\}$ [Cox], regarded as surfaces of type $\{4, d\}$ in the 2-skeleton of the d-cube containing the 1-skeleton of the d-cube.

These examples were of importance also for the investigation of tight submanifolds of Euclidean space. Tightness is a generalization of convexity — for a definition in general and basic facts compare [Kui1] or [Kui2]. In the special case of 2-manifolds M without boundary, we can define $M \hookrightarrow E^d$ to be *tight* if and only if every hyperplane cuts it into at most two pieces (Banchoff's *Two-Piece-Property*). For smooth tight surfaces in E^d, the substantial codimension is always at most 3. In the polyhedral case there are tight surfaces with arbitrarily high codimension. These are just the examples 1 and 2 [Ba1], [Ba2] according to the following lemma:

Lemma 1: *(i) If $M^2 \hookrightarrow E^d$ is a tight polyhedral surface then $\overset{*}{M}$ contains the 1-skeleton of the convex hull $\mathcal{H}M$ of M.*

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(ii) Moreover, if M^2 is a subcomplex of the boundary of its convex hull then the converse is also true:

$$M \text{ tight} \quad \Leftrightarrow \quad Sk_1(\mathcal{H}M) \subseteq M\ .$$

The proof is more or less obvious from the definition. Observe that the 1-skeleton of a convex polytope certainly has the Two-Piece-Property. This is preserved if we add two-dimensional faces.

A subcomplex of the boundary complex of a polytope P may be called k-*Hamiltonian* if it contains the k-dimensional skeleton $Sk_k(P)$. In particular, Lemma 1 says that any 1-Hamiltonian 2-dimensional submanifold of a convex polytope is tight.

Theorem 1 [Kü4]: *Let M^2 be a 2-dimensional submanifold of a convex d-polytope P which is 1-Hamiltonian. Then the following holds:*

(i) $\binom{d-2}{2} \le 3(2-\chi(M))$.

(ii) For $d \ge 4$, equality in (i) holds if and only if P is a simplex.

We shall not repeat the proof here, but just remark that (i) is essentially *Heawood's inequality*

$$\binom{d-2}{2} \le 3(2-\chi) \quad \Leftrightarrow \quad d+1 \le \frac{1}{2}\left(7+\sqrt{49-24\chi}\right)$$

where the integer part of the right hand side of the last inequality is known as the *Heawood colouring number* [Ri2]. Theorem 1 remains true under the weaker assumption that $M \hookrightarrow E^d$ is a tight polyhedral surface, not contained in any hyperplane [Ba1], [Ba2].

For the discussion of higher dimensional submanifolds, we remark that a k-Hamiltonian $2k$-dimensional submanifold of a convex polytope is necessarily $(k-1)$-connected in the sense of homotopy theory. On the other hand, for such a manifold the tightness condition is easy to formulate: a $(k-1)$-connected $2k$-manifold embedded in E^d is called *tight* if every hyperplane cuts it into at most two pieces such that each piece is again $(k-1)$-connected.

Lemma 1 remains true for $(k-1)$-connected $2k$-manifolds of we just replace the 1-skeleton by the k-skeleton. In particular, any k-Hamiltonian $2k$-submanifold of a convex polytope is tight. Higher dimensional examples in the cube have been studied in the earlier paper [KS]. A particular consequence is that, for arbitrary $d \ge 2k+1$ there is a tight $(k-1)$-connected polyhedral $2k$-manifold in E^d.

Theorem 2: *Let M^{2k} be a $2k$-dimensional submanifold of a simplicial convex d-polytope P which is k-Hamiltonian. Then the following holds:*

(i) $\binom{d-k-1}{k+1} \le (-1)^k\binom{2k+1}{k+1}(\chi(M)-2)$.

(ii) For $d \ge 2k+2$, equality in (i) holds if and only if P is a simplex.

We suggest calling the inequality in (i) a *generalized Heawood inequality*. Note that by assumption M is $(k-1)$-connected, and thus the right hand side of (i) is nonnegative:

$$(-1)^k(\chi(M)-2) = rk\ H_k(M;\mathbb{Z})\ .$$

In the case of equality in (ii), the submanifold must be a $(k+1)$-neighbourly triangulation. Examples exist in dimension $2k = 2,4,8$; see example 1 for $k=1$, [KB] or [KL] for $k=2$, and [BK2] for $k=4$.

The assumption that P is simplicial is more of a technical nature. We conjecture that Theorem 2 is true for arbitrary convex polytopes. Note that (ii) does not hold for $d = 2k+1$ because the boundary of any d-polytope is an example of such a case.

Proof: The idea is to compare the h-vector of P with the h-vector of M. Recall that the f-vector $(f_{-1}, f_0, f_1, \ldots)$ consists of the numbers f_i of i-dimensional simplices, where formally $f_{-1} := 1$. We write $f(P)$ for the f-vector of P, $f(M)$ for the f-vector of M. By assumption, $f_i(M) = f_i(P)$ for $i = -1, 0, \ldots, k$. The h-vector $(h_0, h_1, \ldots)$ is defined by

$$h_j(P) = \sum_{i=-1}^{j-1}(-1)^{j-i-1}\binom{d-i-1}{j-i-1}f_i(P)$$

and

$$h_j(M) = \sum_{i=-1}^{j-1}(-1)^{j-i-1}\binom{2k-i}{j-i-1}f_i(M)\ .$$

The *Dehn-Sommerville equations* [Kl1] say that

$$h_j(P) - h_{d-j}(P) = 0 \qquad \text{for} \quad 0 \le j \le \frac{1}{2}(d-1)$$

and

$$h_j(M) - h_{2k+1-j}(M) = (-1)^{2k+1-j}\binom{2k+1}{j}(\chi(M)-2) \quad \text{for} \quad 0 \le j \le k\ .$$

In particular,

$$h_{k+1}(M) - h_k(M) = (-1)^k\binom{2k+1}{k+1}(\chi(M)-2)\ . \tag{1}$$

The most important ingredient of our proof is the *Generalized Lower Bound Theorem* [MW], [St]:

$$h_{j+1}(P) - h_j(P) \ge 0 \qquad \text{for} \quad 0 \le j \le \frac{1}{2}(d-1)\ .$$

Another way of expressing this is

$$f_j(P) \ge \sum_{i=-1}^{j-1}(-1)^{j-i-1}\binom{d-i}{j-i}\cdot f_i(P)\ .$$

In order to prove the inequality in (i), we start with the equation (1) and then put in successively the inequalities of the Generalized Lower Bound Theorem for $j = k, k-1, \ldots, 0$. At each step, we get certain new coefficients $c^i_{j,d}$ for the f_i :

$$(-1)^k\binom{2k+1}{k+1}(\chi(M)-2) \;=\; h_{k+1}(M) - h_k(M)$$

$$= \sum_{i=-1}^{k} (-1)^{k-i} \binom{2k+1-i}{k+1} \cdot f_i$$

$$\geq \sum_{i=-1}^{k-1} (-1)^{k-i-1} c^i_{k-1,d} \cdot f_i$$

$$\vdots$$

$$\geq \sum_{i=-1}^{j} (-1)^{j-i} c^i_{j,d} \cdot f_i$$

$$\vdots$$

$$\geq c^1_{1,d} \cdot f_1 - c^0_{1,d} \cdot f_0 + c^{-1}_{1,d}$$

$$\geq c^0_{0,d} \cdot f_0 - c^{-1}_{0,d}$$

$$\geq c^0_{0,d} \cdot (d+1) - c^{-1}_{0,d}.$$

$$= c^{-1}_{-1,d}.$$

We still have to justify these inequalities by showing that all the coefficients $c^i_{j,d}$ are nonnegative.

The coefficients $c^i_{j,d}$ obey the following recursion formula:

$$c^i_{k,d} = \binom{2k+1-i}{k+1}$$

and

$$c^i_{j-1,d} = c^i_{j,d} \cdot \binom{d-i}{j-i} - c^i_{j,d} \quad \text{for} \quad i < j .$$

By induction, we show that $c^i_{i,d} = \binom{d-k-i-2}{k-i} \geq 0$ for $i = -1, 0, \ldots, k$.

This is trivial for $i = k$ because $c^k_{k,d} = 1$.

Now we assume that the assertion holds for $j = i+1, i+2, \ldots, k$. By repeated applications of the recursion formula, we obtain

$$c^i_{i,d} = c^{i+1}_{i+1,d} \binom{d-i}{1} - c^i_{i+1,d}$$

$$= c^{i+1}_{i+1,d} \binom{d-i}{1} - c^{i+2}_{i+2,d} \binom{d-i}{2} + c^i_{i+2,d}$$

$$\vdots$$

$$= \sum_{j=i+1}^{k} (-1)^{j-i-1} c^j_{j,d} \binom{d-i}{j-i} + (-1)^{k-i} \binom{2k+1-i}{k+1}$$

$$= \sum_{j=i+1}^{k} (-1)^{j-i-1} \binom{d-k-j-2}{k-j} \binom{d-i}{j-i} + (-1)^{k-i} \binom{2k+1-i}{k+1}$$

$$= \binom{d-k-i-2}{k-1} + \sum_{j=i}^{k} (-1)^{j-i-1} \binom{d-k-j-2}{k-j} \binom{d-i}{j-i} + (-1)^{k-i} \binom{2k+1-i}{k+1}$$

$$= \binom{d-k-i-2}{k-1} + (-1)^{k-i-1} \sum_{j=i}^{k} \binom{-d+2k+1}{k-j} \binom{d-i}{j-i} + (-1)^{k-i} \binom{2k+1-i}{k+1}$$

$$= \binom{d-k-i-2}{k-1} + (-1)^{k-i-1} \binom{2k-i+1}{k-i} + (-1)^{k-i} \binom{2k+1-i}{k+1} \qquad (2)$$

$$= \binom{d-k-i-2}{k-i}.$$

or the equation (2), see [Rn, 1.3] or [Kl1].

Now part (i) of Theorem 2 follows from $c^{-1}_{-1,d} = \binom{d-k-1}{k+1}$.

For part (ii), we look at the inequality if it becomes an equality at each step. In articular, the last step

$$c^0_{0,d} \cdot f_0 - c^{-1}_{0,d} \geq c^0_{0,d} \cdot (d+1) - c^{-1}_{0,d}$$

lls us that equality implies $f_0 = d+1$ if $c^0_{0,d} = \binom{d-k-2}{k} > 0$.

This is true for $d \geq 2k+2$. Observe that this argument breaks down if $d = 2k+1$.

onjecture A: *The assertion of Theorem 2 holds under the assumption that M^{2k} $(k-1)$-connected and that $M \hookrightarrow E^d$ is a tight polyhedral embedding not lying in hyperplane.*

Conjecture A is true for $k = 1$ and the bound is essentially sharp; see [Ba1], ü1]. For arbitrary k, Theorem 2 is a special case of Conjecture A.

onjecture B: *The assertion of Theorem 2 holds under the assumption that M^{2k} $(k-1)$-connected and admits an embedding of the k-skeleton of the d-dimensional mplex.*

Again this is true for $k = 1$ by the discussion of the genus of the complete aph [Ri2]. Compare the theorem of van Kampen and Flores which says that e sphere S^{2k} does not admit an embedding of the k-skeleton of the d-simplex if $\geq 2k+2$ [Grü, 11.1]. The complex projective plane does admit an embedding of e 2-skeleton of the 8-dimensional simplex [KB].

onjecture C: *For any triangulation of a manifold M^{2k} with n vertices, the folowing inequality holds:*

$$\binom{n-k-2}{k+1} \geq (-1)^k \binom{2k+1}{k+1} (\chi(M) - 2)$$

with equality if and only if the triangulation is (k+1)*-neighbourly; that is, if* $f_k(M) = \binom{n}{k+1}$.

Conjecture C is a weak form of an *Upper Bound Conjecture for combinatorial manifolds*; cf. [Kl2]. It is a consequence of a more general conjecture made by G. Kalai. Conjecture C is true if

- $k = 1$ or $k = 2$ ([Ri1], [JR], [Kü3]),
- M is a sphere (trivial),
- $n \leq 3k + 3$ ([BK1]),
- M has the homology of $S^j \times S^{2k-j}$, $j < k$ ([BK1]),
- M is a manifold like a projective plane in the sense of [EK] ([BK1]),
- $n > k^2 + 4k + 2$ (this holds by the same argument as in the case of the classical UBC, [Grü]).

In particular, any triangulation of a K3-surface must have at least 16 vertices. (There is a 16-vertex triangulation of the Kummer variety with 16 nodes in [Kü2].) Furthermore, any triangulation of the Cayley projective plane must have at least 27 vertices. It does not seem to be known whether these bounds are attained or not.

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