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QUADRISECANTS OF KNOTS AND LINKS

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ABSTRACT

We show that every non-trivial tame knot or link in $\mathbb{R}^3$ has a quadrisecant, i.e. four collinear points. The quadrisecant must be topologically non-trivial in a precise sense. As an application, we show that a nonsingular, algebraic surface in $\mathbb{R}^3$ which is a knotted torus must have degree at least eight.

1 Introduction

An elementary count of degrees of freedom suggests that a randomly-chosen curve in $\mathbb{R}^3$, if sufficiently complicated, should contain four collinear points. One precise interpretation of this intuition is the following two theorems:

Theorem 1 (Pannwitz,Morton,Mond) *Every non-trivial piecewise linear or smooth knot in $\mathbb{R}^3$ in general position has four collinear points.*

Theorem 2 (Pannwitz,Morton,Mond) *If two smooth or PL circles A and B in $\mathbb{R}^3$ in general position have a non-zero linking number, then there is a line in $\mathbb{R}^3$ which intersects A, then B, then A again, and then B again.*

These theorems are presented by Pannwitz [6] and Morton and Mond [5]. (They are also mentioned by Burde and Zieschang [2].) The arguments of Pannwitz yield a lower bound on the number of collinearities and a generalization of the second theorem to the case of two circles which are linked in the sense that each represents a non-trivial homotopy class in the complement of the other. The main theorem of this paper is a different generalization of Theorems 1 and 2:

Theorem 3 *Every non-trivial tame link in $\mathbb{R}^3$ has four collinear points.*

Since Theorem 3 resembles Theorems 1 and 2, we describe the extra cases covered by the new result. A **non-trivial link** is a collection of disjoint circles embedded in $\mathbb{R}^3$ which is not the boundary of a collection of disjoint, embedded disks. The Whitehead link and the Borromean rings are two examples of non-trivial links which do not satisfy the hypothesis of Theorem 2. A **tame link** is a collection of disjoint circles which are collared by solid tori. Equivalently, a link is tame if it is topologically equivalent to a smooth link in $\mathbb{R}^3$. However, a tame link may have a very different geometry from a smooth link; for example, its Hausdorff dimension may be greater than 1. Moreover, Theorem 3 is not restricted to links which have any particular transversality properties or are in general position in any sense.

To eliminate the general position hypothesis, we first prove a stronger theorem about (smooth) links in general position: Such a link has a line which intersects it four times in a topologically non-trivial way. The stronger conclusion is used in a limiting argument to pass from links in general position to arbitrary tame links.

Theorem 3 has an interesting corollary about the topology of real algebraic surfaces. The question of the topology of real algebraic surfaces led the author to the topic of this paper.//

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Corollary 4 *If an algebraic surface in $\mathbb{R}^3$ contains the boundary of a knotted solid torus or linked solid tori, the surface has degree at least 8.*

The theorem also suggests a definition:

Definition 5 *If L is a link in $\mathbb{R}^3$, a **secant** of L is a line segment whose endpoints lie in L, a **trisecant** of L is a secant of L and a point p, the **middle point**, which lies in both L and the interior of the secant, and a **quadrisecant** is a secant with two middle points.*

Equivalently, a quadrisecant is a pair of distinct trisecants with the same underlying line segment. A **degenerate secant** is a single point. The set of secants has a natural topology, as does the set of trisecants. For a sequence of trisecants to converge, we insist that the middle points converge as well.

As motivation for the main theorem, we present a simple proof of a weaker result:

Theorem 6 *Every non-trivial knot K in $\mathbb{R}^3$ has a trisecant.*

Proof: Suppose that there exists a point p in K such that no points q and r in K are collinear with p. Then the union of the chords $\overline{pq}$ for all q in K is evidently an embedded disk with boundary K. Therefore K is trivial. □

The proof of Theorem 3 is an extended elaboration of this argument.

I would like to thank my advisor, Andrew Casson, for his encouragement and helpful comments. I would also like to thank George Francis for the hand-drawn figures, which greatly aid the exposition of the geometric constructions presented in the paper.

2 General position

There is a general theory of general position, presented by Wall [9] and used by Morton and Mond [5]. We review some elements of this theory:

Definition 7 *If X is a topological space with a measure, a property P of members of X is **generic** if it is true on a set with full measure, and a member of X is in **general position** with respect to P if it satisfies P. A member of X is in **general position** if it is in general position with respect to all applicable generic properties mentioned in this paper.*

Usually X is a space of functions. A **polynomial function** from the unit circle S^1 in $\mathbb{R}^2$ to $\mathbb{R}^3$ is a function given by polynomials of some degree d in the standard coordinates in $\mathbb{R}^2$. The set of all such functions forms a finite-dimensional vector space P_d, and we will consider all generic properties relative to P_d for some $d > 0$ with the usual Cartesian topology and measure. A function $K : S^1 \to \mathbb{R}^3$ is a **knot** if it is injective. This is a generic property; polynomial functions in general position are polynomial knots.

More generally, define kS^1 to be the disjoint union of k unit circles, consider the vector space $P_{d,k}$ of k-tuples of polynomial functions, and define a **link** to be an injective function from $kS^1 \to \mathbb{R}^3$ for some k.

The concept of a polynomial link is not an essential one in this paper, but the following lemmas, whose proofs are easy, make it useful:

Lemma 8 *Given an arbitrary smooth function $f : kS^1 \to \mathbb{R}^3$, there is a sequence of polynomial links (of varying degree) whose values and first derivatives converge uniformly to those of f. We can choose the sequence to be in general position.*

Lemma 9 *A property P of members of a finite-dimensional vector space is a **polynomial property** or an **algebraically generic** property if there exists some non-trivial polynomial p on the vector space such that P is true at all points for which p is non-zero. All polynomial properties are generic.*

If L is a polynomial link with k components, we define a projection function $\pi_L : kS^1 \times kS^1 - \Delta \to S^2$, where Δ is the diagonal, by:

$$\pi_L(a,b) = (L(a) - L(b))/|L(a) - L(b)|.$$

We view π_L as a family of maps $\pi_L(\cdot, b)$ parameterized by the second variable.

The main result of this section is the following lemma. Neither the lemma nor the proof have more mathematical content than equivalent lemmas in [5] and [6], and the key idea is originally due to Reidemeister [7], so the proof here is sketched to some extent.

Lemma 10 *With L and π_L defined as above, it is a polynomial property for L to be a* ***smooth embedding****, i.e. its derivative does not vanish anywhere. It is also a polynomial property of L for there to exist a finite set of points of kS^1, called the set of* ***special points****, whose complement is the set of* ***generic points****, such that for a generic point a and a special point b:*

I. *$\pi_L(\cdot, a)$ is a smooth immersion of a 1-manifold with ends, where the ends correspond to the tangent directions of L at a.*

II. *$\pi_L(\cdot, a)$ does not pass through the two tangent directions.*

III. *$\pi_L(\cdot, a)$ is everywhere one-to-one or two-to-one.*

IV. *If $\pi_L(\cdot, a)$ is two-to-one at a point of S^2, it is self-transverse at that point.*

V. *$\pi_L(\cdot, b)$ has all of the previous properties at all but one point of S^2 and has three of the previous properties at the remaining point p. In this case, as a varies from one side of b to the other, the structure of $\pi_L(\cdot, a)$ near p is characterized by one of the corresponding diagrams in Figure 1.*

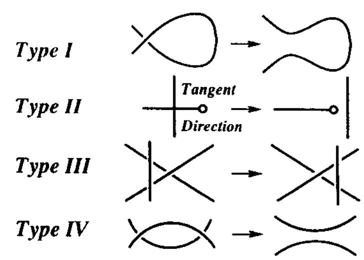

Fig. 1

Proof: The **algebraic dimension** of a subset S of a vector space V is the Krull dimension of the ring of polynomial functions restricted to S. (The Krull dimension of a commutative ring is the maximum length of an ascending chain of prime ideals [3].) We will need two basic facts about algebraic dimension: The algebraic dimension image of a set S under a projection (or more generally a polynomial map) is less than or equal to the algebraic dimension of S, and the complement of a set of algebraic codimension 1 or more is a polynomial property. In the following discussion we will also use codimension to mean the difference of the dimension of a pair of nested sets.

For simplicity, we consider only the case of knots. Observe that in the vector space of ordered quadruplets of points in $\mathbb{R}^3$, the set of collinear quadruplets has algebraic codimension 4. Given four points a, b, c, and d on the unit circle, the space of knots K of degree d (for $d \geq 2$) projects onto the space of quadruplets of points in $\mathbb{R}^3$. Therefore the set of knots K of degree d such that $K(a)$, $K(b)$, $K(c)$, and $K(d)$ are collinear has codimension 4 as well, as does the analagous set in the space of quintuples (K, a, b, c, d), where a, b, c, and d are four distinct points on the circle. By projection, the set of pairs (K, a) for which there exists b, c, and d such that $K(a)$, $K(b)$, $K(c)$, and $K(d)$ are collinear has codimension at least 1. Except for an algebraic subset of the set of knots, the set of a for a knot K for which b, c, and d can be found with this property is polynomial, i.e. finite. Such a b, c, and d would have to exist in order for $\pi_L(\cdot, a)$ to be three-to-one. Thus, part III of the lemma is proved for knots.

The rest of the lemma can be proved in the same fashion, namely by keeping track of the codimension of certain sets. Informally, a set of algebraic codimension n is an n-**fold coincidence**. Parts I and II of the lemma hold because, given points a and b on a link L, it would take a 2-fold coincidence for the tangent to L at b to contain a, and allowing a to vary, it would take a 1-fold coincidence in the choice of a, or allowing a to vary, a 1-fold coincidence for the choice of b. Part IV of the lemma holds because, given a, b, and c on a link L, it would take a 3-fold coincidence for a, b, and c to be collinear and for the tangent lines at b and c to be coplanar.

For part V, the case when condition IV of the lemma fails typefies the method of proof. Informally, at a special point a for which $\pi_L(\cdot, a)$ is somewhere 3-to-1, three arms of the projection of the link meet at a point and it would take a coincidence for there to be a fourth arm at the point or for two of the arms to have the same slope. Near a the front two arms cross at a point and it would take a coincidence for that crossing to travel parallel to the third arm instead of passing through it.

Geometrically, it would take a 6-fold coincidence for five given points on a link L to be collinear, and it would take a 5-fold coincidence for four given points on L to be collinear and for two of the tangent lines to be coplanar. In either case, it would take a 1-fold coincidence in the choice of L for such a set of points to exist. Finally, consider collinear four points a, b, c, and d on L and let l_a, l_b, l_c, and l_d be the tangent lines at these points. The set of lines that intersect l_a, l_b, and l_c sweeps out a surface, and it is a 1-fold coincidence in the choice of l_d for it to be tangent to that surface. If it is not tangent, then $\pi_L(\cdot, a)$ will look as it does in case IV of Figure 1. □

3 Knots in general position

The arguments in this section follow arguments of Pannwitz [6] and Morton and Mond [5]. The only new feature is the notion of topological non-trivial quadrisecants, which we will need to generalize Theorem 1 to arbitrary knots.

We begin with a simple lemma and a definition:

Lemma 11 *Let C be a compact set in $\mathbb{R}^n$. Then not every point of C lies between two other points of C.*

Proof: If p is any point in $\mathbb{R}^n$, then a point $q \in C$ which is farthest from p has this property, because if q lay between two other points, one of them would be still farther away. □

Definition 12 *A secant of a link L with no extra interior intersections with L is* ***topologically trivial*** *if its endpoints lie on the same component of L, and if it, together with one of the two arcs of this component, bounds a disk whose interior does not intersect L. The disk may intersect itself and the secant. A quadrisecant $\overline{ad}$ with middle points b and c is topologically trivial if any of the secants $\overline{ab}$, $\overline{bc}$, and $\overline{cd}$ are. Similarly for a trisecant.*

Lemma 13 *A knot in general position has a topologically non-trivial quadrisecant.*

Proof: Let K be a polynomial knot in general position. Let M be the set of unordered pairs of points of S^1, or equivalently the set of secants of K. M is topologically a Möbius strip. We define O to be the subset M consisting of those pairs of points (a, b) with the property that at least one point of K lies between $K(a)$ and $K(b)$. Lemma 10 has implications about the local structure of O. For fixed b, the set L_b of all (a, b) in M is a line segment which wraps around M as in Figure 2a. The intersection $O \cap L_b$ is a finite set. If b is a generic point, the topology of $\pi_K(\cdot, b)$, and therefore the topology of $O \cap L_b$, cannot change as we vary b slightly. But if b is a special point, the topology of $O \cap L_{b-\epsilon}$ differs from that of $O \cap L_{b+\epsilon}$, as illustrated in Figure 2b. For example, if a is a special point at which condition IV of Lemma 10 for $\pi_K(\cdot, a)$ fails, then there exist three points b, c, and d so that a, b, c, and d, in that order, make a quadrisecant of K. The trisecants a, c, d and a, b, d represent the same point of O, and if condition V of Lemma 10 holds, they represent arms of O that cross. Meanwhile the trisecant a, b, c represents a point of O that lies elsewhere along L_a.

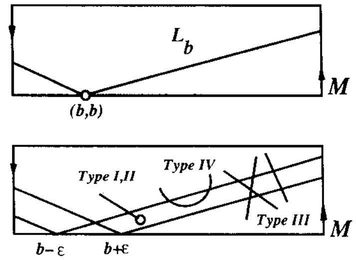

Fig. 2

It follows that O is the image of a self-transverse smooth immersion of a 1-manifold, and a self-crossing corresponds to a quadrisecant. If C is a curve of points in O which does not "make turns" at the self-crossings, then C is a continuous curve of trisecants.

The significance of O is that it is an obstruction to the following construction: Recall that on a Möbius strip, there are two kinds of properly embedded arcs, non-separating arcs and separating arcs. Suppose that A is a non-separating arc of M which avoids O. Then A corresponds to a family of secants whose interiors do not intersect K. This family of secants induces a map D from the unit disk to $\mathbb{R}^3$ whose boundary is K and whose interior does not intersect K. By Dehn's lemma, K is trivial.

Suppose that K has no quadrisecants. Then O is an embedded 1-manifold. By elementary homology theory, if O obstructs all non-separating arcs, there is a circular component C of O which winds around M either one or two times. The curve C is a continuous family of trisecants. We consider the corresponding families of points $\{a,b\}_t$ and m_t, with $t \in S^1$, such that $K(m_t)$ lies between $K(a_t)$ and $K(b_t)$. If C winds once around M, the endpoints travel half way around S^1 and then switch places, and since m_t is trapped between them, it must jump discontinuously, a contradiction. If C winds twice around M, the endpoints each wind once around S^1, and therefore so does m_t. Thus, every point of K lies between two other points, which contradicts Lemma 11.

Topological non-triviality is achieved by a modification of this construction. Let O' be the subset of O consisting of topologically non-trivial trisecants and quadrisecants which are non-trivial at the middle secant. Observe that O' is also the image of a smooth immersion: If O' contains a self-intersection point of O but does not contain all four arms of the self-intersection, then it must contain exactly two arms, and they must be opposite rather than adjacent. In this case the self-intersection point is a quadrisecant which is topologically trivial on one side. Therefore if O' has a self-crossing, it corresponds to a topologically non-trivial quadrisecant. If there are no such quadrisecants, O' must also have a circular component C with all of the properties mentioned above, provided that O' is also an obstruction to all non-separating arcs A.

Let A be a non-separating arc which avoids O'. Choose A to be transverse to O. As before, we construct the disk D_A from the secants of A, but this time D_A does not avoid K. Consider a point in $A \cap O$ corresponding to a quadrisecant Q which is topologically trivial in the middle. The secants of A make a disk which intersects K in two points as shown in Figure 3a. By hypothesis, there exists a disk D_Q which bounds the middle secant of Q and an arc of K. Using D_Q and a tubular neighborhood of K, we can alter D_A to obtain a disk D'_A which avoids K in the vicinity of T, as shown in Figure 3b. Similarly, consider a point in $A \cap O$ corresponding to a topologically trivial trisecant. The geometry of the secants of A is shown in Figure 3c. As before, we attach a tubular neighborhood of an arc of K and two parallel copies of a disk bounding this arc and a secant of T, as indicated in Figure 3d. However, the geometry of the resulting disk D'_A is tricky in the neighborhood of the endpoint of T, particularly since D'_A must intersect itself, if not K. This geometry is illustrated in detail in Figures 3e and 3f. In this fashion, we can repeatedly modify D_A to obtain a disk D whose interior avoids K as before, and Dehn's lemma applies. □

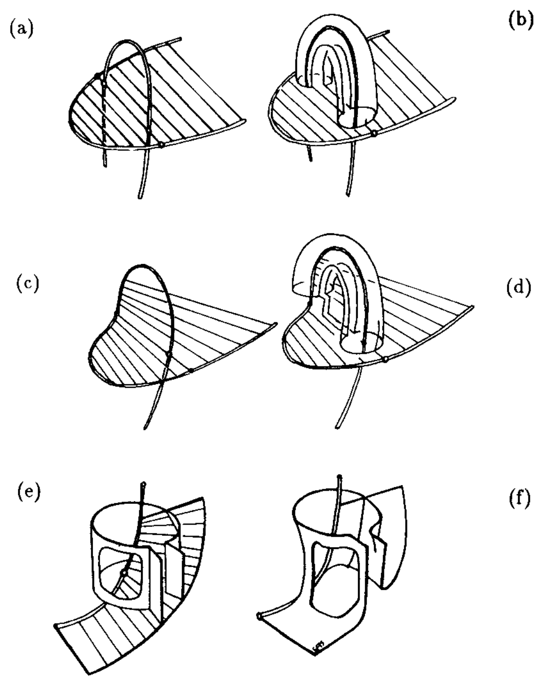

Fig. 3

4 Links in general position

The result of this section is a completion of analagous results in [5] and [6]. The arguments there roughly correspond to the $\omega_1 \neq 0$ case of the proof of Lemma 14, although the argument in [6] is somewhat more general than this special case.

Lemma 14 *Every non-trivial link L in general position has a topologically non-trivial quadrisecant.*

Proof: We may assume without loss of generality that no component of L bounds a disk whose interior avoids L.

Let K be a component of L. Let M_K be the Möbius strip of secants of K, and let O'_K be the corresponding set of topologically non-trivial trisecants and quadrisecants which are non-trivial in the middle. As before, O'_K must be an obstruction to non-separating arcs A, and we obtain a circle C which winds around M. If the middle points of C also lie on K, we may apply the proof of the previous lemma. But the middle points may lie on some other component H of L. In this case, the secants of C induce a map f from a surface E to $\mathbb{R}^3$, where E is either an annulus or a Möbius strip, depending on whether C winds once or twice around K. We may choose f so that the median of E maps to the middle points of the trisecants of C.

The set of lines l perpendicular to H at a given point p is homeomorphic to a circle, and the corresponding set T of all ordered pairs (l,p) is homeomorphic to a torus. We may orthogonally project each trisecant $t \in C$ to a line perpendicular to H, i.e. a member of T, thereby obtaining a map f from C to T. Since C is a circle, this map has an ordered pair of winding numbers (ω_1, ω_2) which are well-defined up to an orientation of C. There are three cases to consider, depending on the values of the winding numbers.

Suppose that $\omega_1 = \omega_2 = 0$. We construct a disk whose boundary is K and whose interior avoids L. The map f intersects H at the median, but it may also intersect K at some other points, because C may include some quadrisecants which are topologically trivial on one side. In this case we can modify f according the prescription in Figure 3a to obtain a map f' which avoids K in the interior and which agrees with f in a neighborhood of the median. Since both winding numbers are zero, we may now homotop f' in a neighborhood of H to obtain a map f'' which avoids H and is constant on the median of E. Finally, we identify the median of E to a point to obtain a space E' and a map f'''. If E is a Möbius strip, E' is a disk, but if E is an annulus, E' is two disks identified at a point. Either way, we obtain the desired spanning disk, which we may convert to an embedded disk by Dehn's Lemma.

Suppose instead that $\omega_2 = 0$ but $\omega_1 \neq 0$. Then we extend E to a line bundle E' and extend f linearly to a map $f' : E' \to \mathbb{R}^3$. We can homotop f' in a neighborhood of H without changing its values in $E' \backslash E$ to a map f'' which has constant value p on the zero section of E', but we cannot make f'' avoid H. Let $p \in H$. As before, we identify the zero section of E' to a point and obtain a space E'', and correspondingly alter f'' to obtain a map $f''' : E'' \to \mathbb{R}^3$. This time the intersection number between H and f''' at p is ω_1. But since f''' is a closed map from the pseudo-manifold E'' to $\mathbb{R}^3$, it induces a well-defined homology class in the infinite homology of $\mathbb{R}^3$. H induces another such homology class, and by elementary homology theory, the total intersection number between f''' and H must be zero. The map f''' must intersect H at another point, and therefore f'' does also. Suppose that $f''(x)$ is this point, with $x \in E'$. The point x cannot be in E, therefore $f'(x) = f''(x)$. Since f' is linear on the fibers of E', the image under f' of the fiber containing x yields a quadrisecant Q. The quadrisecant Q is necessarily topologically non-trivial, because if the intersection points of Q are labeled in order as a, b, c, and x, then $b, x \in H$ and $a, c \in K$.

The only remaining possibility is that $\omega_2 \neq 0$. In this case, every point of H lies between two points of K. We may repeat the whole argument with each component of L playing the role of K, thereby obtaining a function f from components of L to components of L such that every point of $f(K)$ lies between two points of K. The map f must have at least one circular orbit, and we may set C to be the set in $\mathbb{R}^3$ which is the union of all components of L in this orbit. Evidently, C is a compact set and every point of C lies between two other points of C, a contradiction by Lemma 11. □

5 Arbitrary tame knots and links

Definition 15 *A link L in $\mathbb{R}^3$ is **tame** if there exists a homeomorphism h of $\mathbb{R}^3$ which carries L to a polynomial link, or equivalently a piecewise linear or smooth link.*

Lemma 16 *If L is a tame link, there exists a homeomorphism h of $\mathbb{R}^3$ which maps L to a smooth link with h smooth on $\mathbb{R}^3 - L$.*

Proof: Let K be a tame knot and let h be an arbitrary homeomorphism such that $h(K)$ is smooth. Using a tubular neighborhood of $h(K)$, we can choose $T_1, T_2, T_3, \ldots$ to be a sequence of nested, parallel tori converging to $h(K)$. Let $T_i' = h^{-1}(T_i)$. By the theory of triangulations and smoothings of 3-manifolds (see [4, p. 217]), there exists a sequence of smooth tori T_i'', with each T_i'' lying between T_i' and T_{i+1}', and a sequence of diffeomorphisms $h_i' : T_i'' \to T_i$ such that ${h'}_i^{-1}$ and $h^{-1}|_{T_i}$ are isotopic as maps from T_i to $\mathbb{R}^3 - K$. Furthermore, we can arrange that the distance between ${h'}_i^{-1}$ and h^{-1} goes to zero as $i \to \infty$. By the isotopy condition, the h_i''s may be extended smoothly to each region between T_i' and T_{i+1}' and the region outside T_1 to obtain a diffeomorphism $h' : \mathbb{R}^3 - K \to \mathbb{R}^3 - h(K)$. Because of the distance condition, we can continuously extend h' to K by setting it equal to h on K. This continuous extension is the desired map.

The proof in the case of links is similar. □

We are now in a position to prove Theorem 3. In fact, we can prove something slightly stronger:

Theorem 17 *If L is a non-trivial tame link in $\mathbb{R}^3$, then L has a quadrisecant, none of whose component secants are subsets of L.*

Proof: Let L be a non-trivial, tame link and let h be a homeomorphism given by Lemma 16. Let N be a tubular neighborhood of $h(L)$, let N' be the normal bundle of $h(L)$, and choose a diffeomorphism $n : N \to N'$. Consider a sequence of links L_i such that $h(L_i)$ is disjoint from L and $n(h(L_i))$ is a smooth section. Choose the sequence so that $h(L_i)$ converges smoothly to $h(L)$, i.e. $n(h(L_i))$ converges smoothly to the zero section. Since h is a diffeomorphism outside of L, we may choose each L_i to be a polynomial link in general position.

By hypothesis, each L_i has the same isotopy type as L, and in particular each L_i is non-trivial. Therefore each L_i has a topologically non-trivial quadrisecant Q_i. By compactness, $\{Q_i\}$ has a convergent subsequence in the space of line segments in $\mathbb{R}^3$; we may suppose without loss of generality that the original sequence converges. The resulting limit is a secant of L. We must show that the endpoints and middle points of the quadrisecants do not converge together.

For each i, let S_i be a topologically non-trivial secant of L_i and suppose that the S_i's converge to a point p on L. Let B be a round, open ball in N' centered at $n(h(p))$. Then there exists an i such that S_i and an arc A of L_i with the same endpoints as S_i are both contained in $h^{-1}(n^{-1}(B))$. For each point $s \in n(h(S_i))$, we consider the line segment from s to t, where t is the point in $n(h(L_i))$ which lies in the same fiber of N' as s, as illustrated in Figure 4. Since $n(h(L_i))$ is a section, t is unique. The union of these line segments is the image of a spanning disk of $n(h(A \cup S_i))$ which does not intersect $n(h(S_i))$. Therefore S_i is topologically trivial, a contradiction.

The proof that the limit of the S_i's is not a subset of L is similar. □

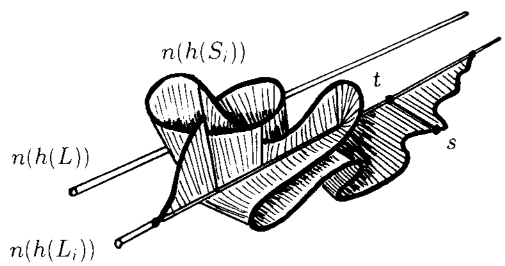

Fig. 4

Corollary 4 follows from Theorem 17:

Proof: Let $\{T_i\}$ be a non-trivially linked collection of solid tori. For each i and each $n > 0$, let $l_{i,n}$ be the shortest non-contractible loop in T_i which is homotopically n times the core of T_i. Let l_i be a shortest member of the set $\{l_{i,n}\}$. If we let D and D' be two disjoint, non-separating disks in T_i for some i, then we see that the length of $l_{i,n}$ is bounded below by n times the distance between D and D'. Therefore l_i exists, although it may not be unique.

Suppose that for some $a, b \in S^1$, $l_i(a) = l_i(b)$. Then we can divide l_i into two loops from $l_i(a)$ to itself. At least one of these loops must be non-contractible and both loops are shorter, which is a contradiction. Thus, each l_i is an embedding. If we let L be the union of the images of the l_i's, then L is a satellite link of the T_i's. By a theorem in knot theory [8, p. 113], L must be a non-trivial link if the T_i's are.

Since a geodesic in a smooth manifold with smooth boundary must be C^1 (see Reference [1]; a proof was also suggested to the author by Tom Ilmanen), L must be a tame link. By the preceding theorem, L must have a quadrisecant Q such that no component secant of Q is contained in L. Suppose that a component secant S of Q were contained entirely inside some T_i. Let p be a path which goes from one endpoint of S to the other. Then we can divide l_i into two paths q_1 and q_2 to make two loops $q_1 p$ and $q_2 p$ whose composition is homotopic to l_i. At least one of these loops must be non-contractible, therefore they cannot both be shorter. Therefore each component secant of Q must have one point which lies outside the T_i's.

Finally, suppose that $P(x, y, z)$ is a non-trivial polynomial whose zero set contains ∂T_i for all i. Then the restriction of P to the line containing Q must be non-trivial and must have at least 8 real roots, counting multiplicity. Therefore P has degree at least 8. □

The author once believed that the loop in a solid torus which is the shortest non-zero multiple of the core is necessarily homotopic to the core. However, this is false by an example of Doug Jungreis. Consider the region S in $\mathbb{R}^3$ which consists of the set of points (x, y, z) such that:

$$|x - \sin(L_1^2 y)/L_1 - \sin(L_2^2 z)/L_2| < \epsilon,$$

where L_1 is very large, L_2 is much larger still, and ϵ is much smaller than $1/L_2$. The region S could be described as a corrugated sheet, and it has the property that if $a, b \in S$ and the straight-line distance from a to b is greater than 1, then this distance is much less than the length of the shortest path in S from a to b. If M is a smooth Möbius strip in $\mathbb{R}^3$ whose tangent plane varies slowly, we can approximate M with a solid torus T which is topologically a tubular neighborhood of M but which is geometrically quite different: T is the union of a thick tube centered around the boundary of M and a corrugated sheet which approximates the interior of M, as shown in Figure 5. Clearly the shortest non-trivial loop in T stays close to the boundary of M and is therefore homotopically twice the core.

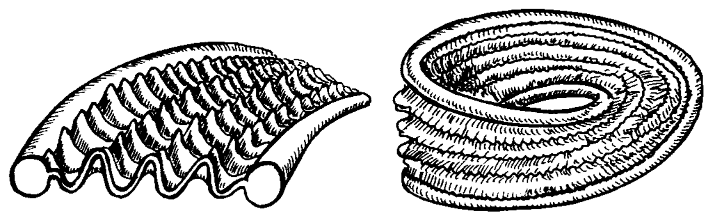

Fig. 5

It is easy to show that the bound in Corollary 4 is the best possible: If we choose two numbers $r_1 > r_2$, then the surface given by:

$$(x^2 + y^2 + z^2 - r_1^2 - r_2^2)^2 - 4(x^2 + y^2)r_1^2 = 0$$

is a torus. If $r_1 > 2r_2$, we can multiply two such equations together to obtain two linked tori.

6 Questions open to the author

The most serious restriction of Corollary 4 is the fact that it only applies to closed surfaces in $\mathbb{R}^3$, while the usual context for studying degrees of real algebraic surfaces is $\mathbb{R}P^3$. We view a subset of $\mathbb{R}^3$ as a subset of $\mathbb{R}P^3$ which is disjoint from the "plane at infinity", which is an $\mathbb{R}P^2$. Define a **flat** plane in $\mathbb{R}P^3$ to be the image of the plane at infinity under a projective transformation of $\mathbb{R}P^3$, and a **topological** plane to be the image of the plane at infinity under a homeomorphism of $\mathbb{R}P^3$.

Conjecture 18 *If a non-trivial link in $\mathbb{R}P^3$ is disjoint from some topological plane, then it has four collinear points.*

Conjecture 19 *If an algebraic surface in $\mathbb{R}P^3$ is disjoint from some topological plane and bounds a collection of non-trivially linked solid tori, then the surface has degree at least 8.*

The following questions have also eluded the author:

Conjecture 20 *An algebraic surface in $\mathbb{R}^3$ which is a smooth torus which knotted on the inside has degree at least eight.*

Conjecture 21 *Every wild knot in $\mathbb{R}^3$ has infinitely many quadrisecants.*

Question 22 *What is the lowest possible degree of a polynomial surface in $\mathbb{R}^3$ which is the boundary of the tubular neighborhood of a trefoil knot?*

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